The algebra of essential relations on a finite set

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joint work with

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Peking University
Let $X$ and $Y$ be finite sets. A correspondence from $X$ to $Y$ is a subset of $Y \times X$. Let $C(Y, X)$ denote the set of correspondences from $X$ to $Y$. A correspondence from $X$ to $X$ is called a relation on $X$. Correspondences can be composed: if $S \subseteq Z \times Y$ and $R \subseteq Y \times X$, then $S \circ R = \{(z, x) \in Z \times X | \exists y \in Y, (z, y) \in S, (y, x) \in R\}$.

In particular $C(X, X)$ is a monoid, with identity element $\Delta_X = \{(x, x) | x \in X\} \subseteq X \times X$.

When $k$ is a commutative ring, let $R^X = kC(X, X)$ denote the algebra of relations on $X$. Serge Bouc (CNRS-LAMFA)
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Correspondences, Relations

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Serge Bouc (CNRS-LAMFA)

The algebra of essential relations

Peking University, 14/06/2013
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When $k$ is a commutative ring, let $\mathcal{R}_X = k\mathcal{C}(X, X)$ denote the algebra of relations on $X$. 

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Serge Bouc (CNRS-LAMFA)
A relation $R \in C(X, X)$ is called inessential if there exists $Y$ with $|Y| < |X|$, and correspondences $S \in C(X, Y)$ and $T \in C(Y, X)$ such that $R = S \circ T$, i.e. $X T R Y S > > >$

A relation $R \in C(X, X)$ is called essential if it is not inessential.

Example: Suppose $|X| \geq 2$, and $R = U \times V$, for $U, V \subseteq X$. Then $Y = \{y\}$, $S = U \times \{y\}$, and $T = \{y\} \times V$. Then $R = S \circ T$ is inessential.

Let $I_X \subseteq R_X = k C(X, X)$ denote the set of linear combinations of inessential relations on $X$. Then $I_X$ is a two sided ideal of $R_X$, and the quotient $E_X = R_X / I_X$ is called the algebra of essential relations on $X$. 
A relation \( R \in \mathcal{C}(X, X) \) is called inessential if there exists \( Y \) with \(|Y| < |X|\), and correspondences \( S \in \mathcal{C}(X, Y) \) and \( T \in \mathcal{C}(Y, X) \) such that \( R = S \circ T \), i.e. \( X T R / / X Y S > > > > > > > > A \)

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\[
\begin{array}{ccc}
X & \xrightarrow{R} & X \\
\downarrow{T} & & \downarrow{S} \\
Y & & 
\end{array}
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Essential relations

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  ![Diagram with relations]

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![Diagram](attachment:image.png)

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\begin{equation}
\begin{array}{c}
X \\
\downarrow \quad R \\
X
\end{array} \quad \begin{array}{c}
Y \\
\downarrow T \\
\uparrow S
\end{array}
\end{equation}

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![Diagram](https://via.placeholder.com/150)

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Let $\mathcal{I}_X \subseteq \mathcal{R}_X = k\mathcal{C}(X, X)$ denote the set of linear combinations of inessential relations on $X$. Then $\mathcal{I}_X$ is a **two sided ideal** of $\mathcal{R}_X$. 
Essential relations

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$$
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X & \xrightarrow{R} X \\
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- Let $\mathcal{I}_X \subseteq \mathcal{R}_X = k\mathcal{C}(X, X)$ denote the set of linear combinations of inessential relations on $X$. Then $\mathcal{I}_X$ is a two sided ideal of $\mathcal{R}_X$, and the quotient $\mathcal{E}_X = \mathcal{R}_X / \mathcal{I}_X$ is called the algebra of essential relations on $X$. 

Serge Bouc (CNRS-LAMFA)

The algebra of essential relations

Peking University, 14/06/2013
From now on, the set $X$ is fixed (and understood).

$\text{Set } n = |X|, \quad E = E_X, \quad \Delta = \Delta_X, \ldots$

The algebra $E$ has a $k$-basis consisting of the essential relations on $X$.

In $E$, the product of two essential relations $R$ and $S$ is equal to $R \circ S$ if $R \circ S$ is essential, and to 0 otherwise.

Classical definitions:

- If $R$ is a relation, set $R^{\text{op}} = \{(x, y) \mid (y, x) \in R\}$.
- $R$ is reflexive $\iff \Delta \subseteq R$.
- $R$ is transitive $\iff R^2 \subseteq R$.
- $R$ is a preorder $\iff \Delta \subseteq R = R^2$.
- $R$ is symmetric $\iff R = R^{\text{op}}$.
- $R$ is an equivalence relation $\iff \Delta \subseteq R = R^{\text{op}} = R^2$.
- $R$ is antisymmetric $\iff R \cap R^{\text{op}} \subseteq \Delta$.
- $R$ is an order $\iff R = R^2$ and $R \cap R^{\text{op}} \subseteq \Delta$. 

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**Example:** Let $X = \{1, 2\}$.
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The algebra $\mathcal{E}$ has a $k$-basis consisting of the essential relations on $X$. In $\mathcal{E}$, the product of two essential relations $R$ and $S$ is equal to $R \circ S$ if $R \circ S$ is essential, and to 0 otherwise.

**Example:** Let $X = \{1, 2\}$.

If $R = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, then $R \circ S = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.
• From now on, the set $X$ is fixed (and understood). Set $n = |X|$, $\mathcal{E} = \mathcal{E}_X$, $\Delta = \Delta_X$, . . .

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• **Example:** Let $X = \{1, 2\}$.
  If $R = 1 \xrightarrow{1} 1$, then $R^2 = 1 \xrightarrow{1} 1$

    $2 \xrightarrow{2} 2$  $2 \xrightarrow{2} 2$
From now on, the set $X$ is fixed (and understood). Set $n = |X|$, $\mathcal{E} = \mathcal{E}_X$, $\Delta = \Delta_X$, ...

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If $R = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, then $R^2 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = X \times X = 0$. 

![Diagram](http://example.com/diagram.png)
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Classical definitions:
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Classical definitions: if $R$ is a relation, set $R^{op} = \{(x, y) \mid (y, x) \in R\}$.
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**Classical definitions:** if $R$ is a relation, set $R^{op} = \{(x, y) \mid (y, x) \in R\}$.

- $R$ is reflexive $\iff \Delta \subseteq R$.
- $R$ is transitive

\[ \Delta = \Delta_X \subseteq \mathcal{E} = \mathcal{E}_X \]
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- $R$ is **reflexive** $\iff$ $\Delta \subseteq R$.
- $R$ is **transitive** $\iff$ $R^2 \subseteq R$.
- $R$ is a **preorder** $\iff$ $\Delta \subseteq R = R^2$.
- $R$ is **symmetric**
From now on, the set $X$ is fixed (and understood). Set $n = |X|$, $\mathcal{E} = \mathcal{E}_X$, $\Delta = \Delta_X$, …

The algebra $\mathcal{E}$ has a $k$-basis consisting of the essential relations on $X$. In $\mathcal{E}$, the product of two essential relations $R$ and $S$ is equal to $R \circ S$ if $R \circ S$ is essential, and to 0 otherwise.

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Characterization

Recall that $X$ is a finite set of cardinality $n$. 

Lemma A relation $R$ on $X$ is inessential $\iff \exists U_i, V_i \subseteq X, 1 \leq i \leq n - 1$ such that $R = \bigcup_{i=1}^{n-1} (U_i \times V_i)$.

If $R$ is a preorder, and not an order, then $R$ is inessential.

If $R$ is an order, and if $\Delta \subseteq Q \subseteq R$, then $Q$ is essential.

Let $\Sigma$ the group of permutations of $X$. Then $\sigma \in \Sigma \mapsto \Delta_\sigma = \{ (\sigma(x), x) | x \in X \} \in C(X, X)$ is a monoid homomorphism. Moreover $\Delta_\sigma$ is essential.

Theorem Let $R$ be an essential relation on $X$. Then there exists $\sigma \in \Sigma$ such that $R \supseteq \Delta_\sigma$, i.e. $R = S \Delta_\sigma$, where $S$ is reflexive.

Proof: One direct proof, another one using a theorem of P. Hall (1935).
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Serge Bouc (CNRS-LAMFA)

The algebra of essential relations

Peking University, 14/06/2013
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Let $R$ be an essential relation on $X$. Then there exists $\sigma \in \Sigma$ such that $R \supseteq \Delta \sigma$, i.e. $R = S \Delta \sigma$, where $S$ is reflexive.

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A nilpotent ideal

If $S$ is reflexive, then $\Delta \subseteq S \subseteq S^2 \subseteq \ldots \subseteq S^m = S^{m+1}$. This limit is the transitive closure of $S$, denoted by $S_*$. It is a preorder.

There are two cases:

- either $S$ is not an order. Then $S = 0$ in $E$.
- or $S$ is an order. Then $\Delta \subseteq S \subseteq S = \Rightarrow S$ is essential.

Proposition

Let $N$ be the $k$-submodule of $E$ generated by the element of the form $(S - S)\Delta_\sigma$, for $\Delta \subseteq S$ and $\sigma \in \Sigma$.

Then $N$ is a two sided nilpotent ideal of $E$.

Proof (sketch):

Let $S \supseteq \Delta$, and $m \in N$ such that $S^m = S$. Let $Q \supseteq \Delta$. Then $Q(S - S) = QS - QS = (QS - QS) - (Q(S - S))$ since $QS = QS$. Hence $QN \subseteq N$.

$(S - S)^m = m \sum_{i=0}^{\infty} (-1)^i (m_i S^m - i S^i) = (m \sum_{i=0}^{\infty} (-1)^i (m_i S^m)) = 0$. 

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A nilpotent ideal

- If $S$ is reflexive
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A nilpotent ideal

- If \( S \) is reflexive, then \( \Delta \subseteq S \subseteq S^2 \subseteq \ldots \subseteq S^m = S^{m+1} \). This limit is the transitive closure of \( S \), denoted by \( \overline{S} \). It is a preorder.
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Let \( \mathcal{N} \) be the \( k \)-submodule of \( \mathcal{E} \) generated by the element of the form \((S - \overline{S})\Delta_{\sigma}\), for \( \Delta \subseteq S \) and \( \sigma \in \Sigma \).
A nilpotent ideal

- If $S$ is reflexive, then $\Delta \subseteq S \subseteq S^2 \subseteq \ldots \subseteq S^m = S^{m+1}$. This limit is the transitive closure of $S$, denoted by $\bar{S}$. It is a preorder.
- There are two cases:
  - either $\bar{S}$ is not an order. Then $\bar{S} = 0$ in $\mathcal{E}$.
  - or $\bar{S}$ is an order. Then $\Delta \subseteq S \subseteq \bar{S} \implies S$ is essential.

**Proposition**

Let $\mathcal{N}$ be the $k$-submodule of $\mathcal{E}$ generated by the element of the form $(S - \bar{S})\Delta_\sigma$, for $\Delta \subseteq S$ and $\sigma \in \Sigma$.

Then $\mathcal{N}$ is a two sided nilpotent ideal of $\mathcal{E}$. 
A nilpotent ideal

- If \( S \) is reflexive, then \( \Delta \subseteq S \subseteq S^2 \subseteq \ldots \subseteq S^m = S^{m+1} \). This limit is the transitive closure of \( S \), denoted by \( S \). It is a preorder.
- There are two cases:
  - either \( S \) is not an order. Then \( S = 0 \) in \( E \).
  - or \( S \) is an order. Then \( \Delta \subseteq S \subseteq S \implies S \) is essential.

**Proposition**

Let \( \mathcal{N} \) be the \( k \)-submodule of \( E \) generated by the element of the form \( (S - S)\Delta_\sigma \), for \( \Delta \subseteq S \) and \( \sigma \in \Sigma \).

Then \( \mathcal{N} \) is a two sided nilpotent ideal of \( E \).

**Proof (sketch):** Let \( S \supseteq \Delta \)
A nilpotent ideal

- If $S$ is reflexive, then $\Delta \subseteq S \subseteq S^2 \subseteq \ldots \subseteq S^m = S^{m+1}$. This limit is the transitive closure of $S$, denoted by $\overline{S}$. It is a preorder.
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Let $\mathcal{N}$ be the $k$-submodule of $\mathcal{E}$ generated by the element of the form $(S - \overline{S})\Delta_\sigma$, for $\Delta \subseteq S$ and $\sigma \in \Sigma$.

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Proof (sketch): Let $S \supseteq \Delta$, and $m \in \mathbb{N}$ such that $S^m = \overline{S}$. 
A nilpotent ideal

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**Proof (sketch):** Let $S \supseteq \Delta$, and $m \in \mathbb{N}$ such that $S^m = \overline{S}$.

- Let $Q \supseteq \Delta$. 
A nilpotent ideal

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- Let $Q \supseteq \Delta$. Then $Q(S - \overline{S}) = QS - Q\overline{S} =$
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- Let $Q \supseteq \Delta$. Then $Q(S - \overline{S}) = QS - Q\overline{S} = (QS - Q\overline{S}) - (Q\overline{S} - Q\overline{S})$ since $Q\overline{S} = Q\overline{S}$. 

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A nilpotent ideal

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A nilpotent ideal

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**Proof (sketch):** Let $S \supseteq \Delta$, and $m \in \mathbb{N}$ such that $S^m = \overline{S}$.

- Let $Q \supseteq \Delta$. Then $Q(S - \overline{S}) = QS - Q\overline{S} = (QS - Q\overline{S}) - (Q\overline{S} - \overline{Q\overline{S}})$ since $\overline{Q\overline{S}} = Q\overline{S}$. Hence $QN \subseteq \mathcal{N}$.

- $(S - \overline{S})^m = \ldots$
A nilpotent ideal

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  since $Q\overline{S} = \overline{QS}$. Hence $Q\mathcal{N} \subseteq \mathcal{N}$.

- $(S - \overline{S})^m = \sum_{i=0}^{m} (-1)^i \binom{m}{i} S^{m-i} \overline{S}^i = \ldots$
A nilpotent ideal

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- $(S - \overline{S})^m = \sum_{i=0}^{m} (-1)^i \binom{m}{i} S^{m-i} \overline{S}^i = \overline{S} + \ldots$
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- $(S - \overline{S})^m = \sum_{i=0}^{m} (-1)^i \binom{m}{i} S^{m-i} \overline{S}^i = \overline{S} + \sum_{i=1}^{m} (-1)^i \binom{m}{i} \underbrace{S^{m-i} \overline{S}}_{\overline{S}}$
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- $(S - \overline{S})^m = \sum_{i=0}^{m}(-1)^i \binom{m}{i} S^{m-i} \overline{S}^i = \left(\sum_{i=0}^{m}(-1)^i \binom{m}{i}\right) \overline{S} = 0$.
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A nilpotent ideal

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Permuted orders

Let $P = E / N$, called the algebra of permuted orders on $X$. It has a $k$-basis consisting of relations $S \Delta \sigma$, where $S$ is an order and $\sigma \in \Sigma$.

The product of $S \Delta \sigma \cdot T \Delta \tau$ in $P$ is equal to $S \cdot \sigma T \Delta \sigma \tau$ if $S \cdot \sigma T$ is an order, and to 0 otherwise, where $\sigma T = \Delta \sigma \cdot \Delta^{-1}$.

The algebra $P$ is $\Sigma$-graded: for $\sigma \in \Sigma$, the degree $\sigma$ part $P_\sigma$ of $P$ is the $k$-submodule generated by the elements $S \Delta \sigma$, where $S$ is an order.

The subalgebra $P_1$ has a $k$-basis consisting of the set $O$ of orders on $X$. For $S, T \in O$, the product $ST$ in $P_1$ is equal to $ST = S \cup T$.

Hence $P_1$ is commutative.

The group $\Sigma$ acts on $P_1$ by conjugation, and $P$ is the semidirect product $P_1 \rtimes \Sigma$. 

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Permuted orders

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Serge Bouc (CNRS-LAMFA)
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Permuted orders

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Let $\mathcal{P} = \mathcal{E}/\mathcal{N}$, called the algebra of permuted orders on $X$. It has a $k$-basis consisting of relations $S \Delta_\sigma$, where $S$ is an order and $\sigma \in \Sigma$. The product of $S \Delta_\sigma.T \Delta_\tau$ in $\mathcal{P}$ is equal to $S.\sigma^{-1} T \Delta_{\sigma \tau}$.
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Permutated orders

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The algebra $\mathcal{P}$ is $\Sigma$-graded: for $\sigma \in \Sigma$, the degree $\sigma$ part $\mathcal{P}_\sigma$ of $\mathcal{P}$ is the $k$-submodule generated by the elements $S\Delta_\sigma$, where $S$ is an order.
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The algebra of essential relations

Peking University, 14/06/2013 

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Let \( \mathcal{P} = \mathcal{E}/\mathcal{N} \), called the algebra of permuted orders on \( X \). It has a \( k \)-basis consisting of relations \( S\Delta_\sigma \), where \( S \) is an order and \( \sigma \in \Sigma \). The product of \( S\Delta_\sigma \cdot T\Delta_\tau \) in \( \mathcal{P} \) is equal to \( \overline{S.\sigma T}\Delta_{\sigma\tau} \) if \( \overline{S.\sigma T} \) is an order, and to 0 otherwise, where \( \sigma T = \Delta_\sigma T\Delta_{\sigma^{-1}} \).

The algebra \( \mathcal{P} \) is \( \Sigma \)-graded: for \( \sigma \in \Sigma \), the degree \( \sigma \) part \( \mathcal{P}_\sigma \) of \( \mathcal{P} \) is the \( k \)-submodule generated by the elements \( S\Delta_\sigma \), where \( S \) is an order.

The subalgebra \( \mathcal{P}_1 \) has a \( k \)-basis consisting of the set \( \mathcal{O} \) of orders on \( X \). For \( S, T \in \mathcal{O} \), the product \( ST \) in \( \mathcal{P}_1 \)
Permutated orders

Let $\mathcal{P} = \mathcal{E}/\mathcal{N}$, called the algebra of permuted orders on $X$. It has a $k$-basis consisting of relations $S\Delta_\sigma$, where $S$ is an order and $\sigma \in \Sigma$. The product of $S\Delta_\sigma . T\Delta_\tau$ in $\mathcal{P}$ is equal to $\overline{S.\sigma T}\Delta_{\sigma\tau}$ if $\overline{S.\sigma T}$ is an order, and to 0 otherwise, where $\sigma T = \Delta_\sigma T\Delta_{\sigma^{-1}}$.

The algebra $\mathcal{P}$ is $\Sigma$-graded: for $\sigma \in \Sigma$, the degree $\sigma$ part $\mathcal{P}_\sigma$ of $\mathcal{P}$ is the $k$-submodule generated by the elements $S\Delta_\sigma$, where $S$ is an order.

The subalgebra $\mathcal{P}_1$ has a $k$-basis consisting of the set $\mathcal{O}$ of orders on $X$. For $S, T \in \mathcal{O}$, the product $ST$ in $\mathcal{P}_1$ is equal to $\overline{ST} = \overline{S} \cup \overline{T}$.
Let $P = \mathcal{E}/\mathcal{N}$, called the algebra of permuted orders on $X$. It has a $k$-basis consisting of relations $S\Delta_\sigma$, where $S$ is an order and $\sigma \in \Sigma$. The product of $S\Delta_\sigma . T\Delta_\tau$ in $P$ is equal to $\overline{S.\sigma T}\Delta_{\sigma \tau}$ if $\overline{S.\sigma T}$ is an order, and to 0 otherwise, where $\sigma T = \Delta_\sigma T\Delta_{\sigma^{-1}}$.

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The subalgebra $P_1$ has a $k$-basis consisting of the set $\mathcal{O}$ of orders on $X$. For $S, T \in \mathcal{O}$, the product $ST$ in $P_1$ is equal to $\overline{ST} = \overline{S} \cup \overline{T}$. Hence $P_1$ is commutative.
Let $\mathcal{P} = \mathcal{E}/\mathcal{N}$, called the algebra of \textit{permuted orders} on $X$. It has a $k$-basis consisting of relations $S\Delta_\sigma$, where $S$ is an order and $\sigma \in \Sigma$. The product of $S\Delta_\sigma.T\Delta_\tau$ in $\mathcal{P}$ is equal to $S.\sigma T \Delta_{\sigma \tau}$ if $S.\sigma T$ is an order, and to 0 otherwise, where $\sigma T = \Delta_\sigma T \Delta_{\sigma^{-1}}$.

The algebra $\mathcal{P}$ is $\Sigma$-graded: for $\sigma \in \Sigma$, the degree $\sigma$ part $\mathcal{P}_\sigma$ of $\mathcal{P}$ is the $k$-submodule generated by the elements $S\Delta_\sigma$, where $S$ is an order.

The subalgebra $\mathcal{P}_1$ has a $k$-basis consisting of the set $\mathcal{O}$ of orders on $X$. For $S, T \in \mathcal{O}$, the product $ST$ in $\mathcal{P}_1$ is equal to $ST = S \cup T$. Hence $\mathcal{P}_1$ is commutative.

The group $\Sigma$ acts on $\mathcal{P}_1$ by conjugation.
Permuted orders

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- The group $\Sigma$ acts on $\mathcal{P}_1$ by conjugation, and $\mathcal{P}$ is the semidirect product $\mathcal{P}_1 \rtimes \Sigma$. 
The algebra of permuted orders

If $R \in \mathcal{O}$, then $R^2 = R$.

If $R, S \in \mathcal{O}$, then $RS = R \cup S = \text{Sup}_\mathcal{O}(R, S)$ or 0.

Notation
For $R \in \mathcal{O}$, let $f_R \in \mathcal{P}_1$ defined by $f_R = \sum_{R \subseteq S \in \mathcal{O}} \mu_{\mathcal{O}}(R, S) S$, where $\mu_{\mathcal{O}}$ is the Möbius function of the poset $\mathcal{O}$, ordered by inclusion.

Theorem 1
The elements $f_R$, for $R \in \mathcal{O}$, are orthogonal idempotents of $\mathcal{P}_1$, and $\sum_{R \in \mathcal{O}} f_R = 1$.

Moreover $\mathcal{P}_1 f_R = kf_R$, for $R \in \mathcal{O}$.

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The algebra of permuted orders

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The algebra of permuted orders

Notation
For $R \in O$, set $\Sigma_R = \{ \sigma \in \Sigma | \sigma R = R \}$, and $e_R = \sum_{\sigma \in \Sigma / \Sigma_R} f_{\sigma R}$.

Theorem
1. The elements $e_R$, for $R \in [\Sigma O]$, are orthogonal central idempotents of $P$, and $\sum_{R \in [\Sigma O]} e_R = 1$.
2. The algebra $P$ is isomorphic to $\prod_{R \in [\Sigma O]} P e_R$.
3. For $R \in O$, the algebra $P e_R$ is isomorphic to $\text{Mat}_{\Sigma: \Sigma_R} (k \Sigma_R)$. 

Serge Bouc (CNRS-LAMFA)
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Theorem

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For $R \in \mathcal{O}$, set $\Sigma_R = \{ \sigma \in \Sigma \mid \sigma R = R \}$, and $e_R = \sum_{\sigma \in [\Sigma/\Sigma_R]} f_{\sigma R}$.

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1. The elements $e_R$, for $R \in [\Sigma \setminus \mathcal{O}]$, are orthogonal central idempotents of $\mathcal{P}$, and $\sum_{R \in [\Sigma \setminus \mathcal{O}]} e_R = 1$.
2. The algebra $\mathcal{P}$ is isomorphic to $\prod_{R \in [\Sigma \setminus \mathcal{O}]} \mathcal{P} e_R$.
3. For $R \in \mathcal{O}$, the algebra $\mathcal{P} e_R$ is isomorphic to $\text{Mat}_{|\Sigma : \Sigma R|}(k\Sigma_R)$. 
The simple $\mathcal{E}$-modules

Assume that $k$ is a field. Recall that $P = \mathcal{E}/N$, where $N$ is nilpotent, and that $P \cong \prod_{\Sigma : \Sigma} R \otimes k$.

Theorem 1
The surjection $\mathcal{E}/P$ induces a one to one correspondence between the simple $\mathcal{E}$-modules and the simple $P$-modules.

Theorem 2
The simple $P$-modules (up to isomorphism) are the modules of the form $P f R \otimes k$, where $R \in \Sigma O$, and $V$ is a simple $k$-module (up to isomorphism).

If $\text{char}(k) > n$, then $P$ is semisimple, and $N = J(E)$. 

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The simple $\mathcal{E}$-modules

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Theorem 1

The surjection $\mathcal{E} \to \mathcal{P}$ induces a one to one correspondence between the simple $\mathcal{E}$-modules and the simple $\mathcal{P}$-modules.

The simple $\mathcal{P}$-modules (up to isomorphism) are the modules of the form $\mathcal{P}f_R \otimes k\Sigma_R V$, where $R \in [\Sigma \setminus \emptyset]$, and $V$ is a simple $k\Sigma_R$-module (up to isomorphism).

If $\text{char}(k) > n$, then $\mathcal{P}$ is semisimple, and $\mathcal{N} = J(E)$.
The simple $\mathcal{E}$-modules

Assume that $k$ is a field. Recall that $\mathcal{P} = \mathcal{E}/\mathcal{N}$, where $\mathcal{N}$ is nilpotent, and that

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Assume that $k$ is a field. Recall that $\mathcal{P} = \mathcal{E}/\mathcal{N}$, where $\mathcal{N}$ is nilpotent, and that $\mathcal{P} \cong \prod_{R \in [\Sigma \backslash \emptyset]} \text{Mat}_{\Sigma: \Sigma_R}(k\Sigma_R)$.

**Theorem**

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The simple $\mathcal{E}$-modules

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Serge Bouc (CNRS-LAMFA)
The simple $\mathcal{E}$-modules

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The simple $\mathcal{E}$-modules

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The simple $\mathcal{E}$-modules

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3. If $\text{char}(k) > n$, then $\mathcal{P}$ is semisimple.
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Serge Bouc (CNRS-LAMFA)
The simple $\mathcal{E}$-modules

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Some simple $\mathcal{R}_X$-modules

Proposition

Let $\mathcal{R}$ be an order on $X$. If $\mathcal{S} \in \mathcal{C}(X, X)$, define a $k$-endomorphism $\beta(\mathcal{S})$ of $k\Sigma$ by

$$
\beta(\mathcal{S}) : \sigma \in \Sigma \mapsto \begin{cases} 
\tau \sigma & \text{if } \tau \in \Sigma, \Delta \subseteq \tau - 1 \mathcal{S} \subseteq \sigma, \\
0 & \text{if no such } \tau.
\end{cases}
$$

The map $\beta(\mathcal{S})$ is well defined, and its image is contained in $\text{End}_{k\Sigma}(k\Sigma)$.

The map $\mathcal{S} \mapsto \beta(\mathcal{S})$ extends to an algebra homomorphism $k\mathcal{C}(X, X) = \mathcal{R}_X \to \text{End}_{k\Sigma}(k\Sigma)$, which endows $k\Sigma$ with a structure of $(\mathcal{R}_X, k\Sigma\mathcal{R})$-bimodule.

If $V$ is a simple $k\Sigma\mathcal{R}$-module, then $k\Sigma \otimes k\Sigma\mathcal{R}V$ is a simple $\mathcal{R}_X$-module $\Lambda(\mathcal{R}, V)$.

If $(\mathcal{R}', V')$ is another pair consisting of an order $\mathcal{R}'$ on $X$ and a simple $k\Sigma\mathcal{R}'$-module, then the $\mathcal{R}_X$-modules $\Lambda(\mathcal{R}, V)$ and $\Lambda(\mathcal{R}', V')$ are isomorphic if and only if the pairs $(\mathcal{R}, V)$ and $(\mathcal{R}', V')$ are conjugate by $\Sigma$. 

Serge Bouc (CNRS-LAMFA)
Proposition

Let $R$ be an order on $X$. 

The map $S \mapsto \beta_R(S)$ extends to an algebra homomorphism $kC(X, X) \to \text{End}_{k\Sigma R}(k\Sigma)$, which endows $k\Sigma$ with a structure of $(\mathcal{R}_X, k\Sigma R)$-bimodule.

If $V$ is a simple $k\Sigma R$-module, then $k\Sigma \otimes k\Sigma R V$ is a simple $\mathcal{R}_X$-module $\Lambda R, V$.

If $(\mathcal{R}_X', V')$ is another pair consisting of an order $\mathcal{R}_X'$ on $X$ and a simple $k\Sigma R'$-module, then the $\mathcal{R}_X$-modules $\Lambda R, V$ and $\Lambda R', V'$ are isomorphic if and only if the pairs $(\mathcal{R}_X, V)$ and $(\mathcal{R}_X', V')$ are conjugate by $\Sigma$. 

Serge Bouc (CNRS-LAMFA)
Let $R$ be an order on $X$. If $S \in C(X, X)$, define a $k$-endomorphism $\beta_R(S)$ of $k\Sigma$ by

$$\beta_R(S) : \sigma \in \Sigma \mapsto$$

The map $\beta_R(S)$ is well defined, and its image is contained in $\text{End}_k k\Sigma_R(k\Sigma)$. This endows $k\Sigma$ with a structure of $(R_X, k\Sigma_R)$-bimodule.

If $V$ is a simple $k\Sigma_R$-module, then $k\Sigma \otimes k\Sigma_R V$ is a simple $R_X$-module $\Lambda_R$, $V$.

If $(R', V')$ is another pair consisting of an order $R'$ on $X$ and a simple $k\Sigma_{R'}$-module, then the $R_X$-modules $\Lambda_R$, $V$ and $\Lambda_{R'}$, $V'$ are isomorphic if and only if the pairs $(R, V)$ and $(R', V')$ are conjugate by $\Sigma$. 

Serge Bouc (CNRS-LAMFA)
Some simple $\mathcal{R}_X$-modules

**Proposition**

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Some simple $R_X$-modules

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Some simple $\mathcal{R}_X$-modules

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2. The map $S \mapsto \beta_R(S)$ extends to an algebra homomorphism $k\mathcal{C}(X, X) = \mathcal{R}_X \rightarrow \text{End}_{k\Sigma_R}(k\Sigma)$.
Some simple $\mathcal{R}_X$-modules

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Let $R$ be an order on $X$. If $S \in \mathcal{C}(X, X)$, define a $k$-endomorphism $\beta_R(S)$ of $k\Sigma$ by

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1. The map $\beta_R(S)$ is well defined, and its image is contained in $\text{End}_{k\Sigma_R}(k\Sigma)$.

2. The map $S \mapsto \beta_R(S)$ extends to an algebra homomorphism $k\mathcal{C}(X, X) = \mathcal{R}_X \to \text{End}_{k\Sigma_R}(k\Sigma)$, which endows $k\Sigma$ with a structure of $(\mathcal{R}_X, k\Sigma_R)$-bimodule.
Let $R$ be an order on $X$. If $S \in C(X, X)$, define a $k$-endomorphism $\beta_R(S)$ of $k\Sigma$ by

$$\beta_R(S) : \sigma \in \Sigma \mapsto \begin{cases} \tau \sigma & \text{if } \tau \in \Sigma, \Delta \subseteq \tau^{-1}S \subseteq \sigma R \\ 0 & \text{if no such } \tau. \end{cases}$$

1. The map $\beta_R(S)$ is well defined, and its image is contained in $\text{End}_{k\Sigma_R}(k\Sigma)$.
2. The map $S \mapsto \beta_R(S)$ extends to an algebra homomorphism $kC(X, X) = \mathcal{R}_X \to \text{End}_{k\Sigma_R}(k\Sigma)$, which endows $k\Sigma$ with a structure of $(\mathcal{R}_X, k\Sigma_R)$-bimodule.
3. If $V$ is a simple $k\Sigma_R$-module

Some simple $\mathcal{R}_X$-modules
Some simple $\mathcal{R}_X$-modules

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3. If $V$ is a simple $k\Sigma_R$-module, then $k\Sigma \otimes_{k\Sigma_R} V$ is a simple $\mathcal{R}_X$-module $\Lambda_{R,V}$.
Some simple $\mathcal{R}_X$-modules

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4. If $(R', V')$ is another pair consisting of an order $R'$ on $X$ and a simple $k\Sigma_{R'}$-module
Some simple $\mathcal{R}_X$-modules

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Let $R$ be an order on $X$. If $S \in \mathcal{C}(X, X)$, define a $k$-endomorphism $\beta_R(S)$ of $k\Sigma$ by

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4. If $(R', V')$ is another pair consisting of an order $R'$ on $X$ and a simple $k\Sigma_{R'}$-module, then the $\mathcal{R}_X$-modules $\Lambda_{R,V}$ and $\Lambda_{R',V'}$ are isomorphic.
Some simple $\mathcal{R}_X$-modules

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3. If $V$ is a simple $k\Sigma_R$-module, then $k\Sigma \otimes_{k\Sigma_R} V$ is a simple $\mathcal{R}_X$-module $\Lambda_R, V$.
4. If $(R', V')$ is another pair consisting of an order $R'$ on $X$ and a simple $k\Sigma_{R'}$-module, then the $\mathcal{R}_X$-modules $\Lambda_R, V$ and $\Lambda_{R'}, V'$ are isomorphic if and only if the pairs $(R, V)$ and $(R', V')$ are conjugate by $\Sigma$. 
Examples

$R = \Delta$, then $\Sigma$, and $R \times$ maps surjectively to $k$, by $S \mapsto \sigma$ if $S = \Delta \sigma$, and $S \mapsto 0$ if there is no such $\sigma \in \Sigma$.

If $R$ is a total order, then $\Sigma = \{1\}$, and $P \sim = \text{Mat}_{n!}(k)$.

In this case $k \Sigma$ becomes a simple $R \times$-module.

Remark: Which finite groups can occur as $\Sigma$? Answer: all! (Birkhoff 1946, Thornton 1972, Barmak-Minian 2009).
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Examples:

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The poset of posets

Question:

Compute \( \mu_O(R, S) \), for \( R, S \in O \) with \( R \subseteq S \)?

Proposition 1

\[ \mu_O(R, S) \neq 0 \iff \forall (x, y) \in S - R, x \text{ is maximal in } y \text{ (for } S) \, i.e. \]

\[ \left( (x, z) \in S \text{ and } (z, y) \in S \right) \Rightarrow (x = z \text{ or } z = y) \].

In this case \( \mu_O(R, S) = (-1)^{|S - R|} \).

Proposition

Let \( O \) denote the poset of preorders on \( X \).

Let \( \Gamma \) denote the largest element \( O \) (the coarse preorder).

Then \( O - \{ \Delta, \Gamma \} \) is homotopy equivalent to a wedge of \( (n - 1)! \) spheres of dimension \( 2n - 4 \).

Hence \( \mu_O(\Delta, \Gamma) = (n - 1)! \).
Question:

\[ \mu_O(R, S) \neq 0 \iff \forall (x, y) \in S - R, x \text{ is maximal in } y \text{ (for } S) \, \exists (x, z) \in S \text{ and } (z, y) \in S \implies (x = z \text{ or } z = y) \]
**Question:** Compute $\mu_\mathcal{O}(R, S)$, for $R, S \in \mathcal{O}$ with $R \subseteq S$?
The poset of posets

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**Proposition**

1. $\mu_{\mathcal{O}}(R, S) \neq 0 \iff \forall (x, y) \in S - R, x$ is maximal in $y$ (for $S$) i.e. $(x, z) \in S$ and $(z, y) \in S \Rightarrow (x = z$ or $z = y)$. 

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The poset of posets

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2. *The poset* $\mathcal{O} - \{\Delta\}$
The poset of preorders

**Question:** Compute $\mu_O(R, S)$, for $R, S \in O$ with $R \subseteq S$?

**Proposition**

1. $\mu_O(R, S) \neq 0 \iff \forall (x, y) \in S - R, x \text{ is maximal in } y \text{ (for } S)$
   
   i.e. $((x, z) \in S \text{ and } (z, y) \in S) \Rightarrow (x = z \text{ or } z = y)$.
   
   *In this case* $\mu_O(R, S) = (-1)^{|S-R|}$.

2. The poset $O - \{\Delta\}$ is homotopy equivalent to the sphere $S^{n-2}$.
The poset of preorders

**Question:** Compute \( \mu_\mathcal{O}(R, S) \), for \( R, S \in \mathcal{O} \) with \( R \subseteq S \)?

**Proposition**

1. \( \mu_\mathcal{O}(R, S) \neq 0 \iff \forall (x, y) \in S - R, \ x \text{ is maximal in } y \ (\text{for } S) \)
   
   i.e. \( ((x, z) \in S \text{ and } (z, y) \in S) \Rightarrow (x = z \text{ or } z = y) \).

   In this case \( \mu_\mathcal{O}(R, S) = (-1)^{|S - R|} \).

2. The poset \( \mathcal{O} - \{\Delta\} \) is homotopy equivalent to the sphere \( S^{n-2} \).

**Proposition**

Let \( \overline{\mathcal{O}} \) denote the poset of preorders on \( X \).
The poset of preorders

**Question:** Compute \( \mu_{\mathcal{O}}(R, S) \), for \( R, S \in \mathcal{O} \) with \( R \subseteq S \)?

**Proposition**

1. \( \mu_{\mathcal{O}}(R, S) \neq 0 \iff \forall (x, y) \in S - R, \ x \text{ is maximal in } y \ (\text{for } S) \)
   
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   *In this case* \( \mu_{\mathcal{O}}(R, S) = (-1)^{|S - R|} \).

2. The poset \( \mathcal{O} - \{\Delta\} \) is homotopy equivalent to the sphere \( S^{n-2} \).

**Proposition**

Let \( \overline{\mathcal{O}} \) denote the poset of preorders on \( X \). Let \( \Gamma \) denote the largest element \( \overline{\mathcal{O}} \) (the coarse preorder).
The poset of preorders

**Question:** Compute $\mu_{\mathcal{O}}(R, S)$, for $R, S \in \mathcal{O}$ with $R \subseteq S$?

**Proposition**

1. $\mu_{\mathcal{O}}(R, S) \neq 0 \iff \forall (x, y) \in S - R, \ x \text{ is maximal in } y \ (\text{for } S) \ \text{i.e.} \ ((x, z) \in S \ \text{and} \ (z, y) \in S) \ \Rightarrow \ (x = z \ \text{or} \ z = y).$

   *In this case* $\mu_{\mathcal{O}}(R, S) = (-1)^{|S - R|}.$

2. The poset $\mathcal{O} - \{\Delta\}$ is homotopy equivalent to the sphere $S^{n-2}$.

**Proposition**

Let $\overline{\mathcal{O}}$ denote the poset of preorders on $X$. Let $\Gamma$ denote the largest element $\overline{\mathcal{O}}$ (the coarse preorder).

*Then* $\overline{\mathcal{O}} - \{\Delta, \Gamma\}$
The poset of preorders

**Question:** Compute $\mu(O)(R, S)$, for $R, S \in O$ with $R \subseteq S$?

**Proposition 1**

$\mu(O)(R, S) \neq 0 \iff \forall (x, y) \in S - R, x \text{ is maximal in } y \text{ (for } S) \implies (x = z \text{ or } z = y)$.

In this case $\mu(O)(R, S) = (-1)^{|S - R|}$.

**Proposition 2**

The poset $O - \{\Delta\}$ is homotopy equivalent to the sphere $S^{n-2}$.

**Proposition**

Let $\overline{O}$ denote the poset of preorders on $X$. Let $\Gamma$ denote the largest element $\overline{O}$ (the coarse preorder).

Then $\overline{O} - \{\Delta, \Gamma\}$ is homotopy equivalent to a wedge of $(n - 1)!$ spheres of dimension $2n - 4$. 
The poset of preorders

**Question:** Compute $\mu_{\mathcal{O}}(R, S)$, for $R, S \in \mathcal{O}$ with $R \subseteq S$?

**Proposition**

1. $\mu_{\mathcal{O}}(R, S) \neq 0 \iff \forall (x, y) \in S - R, \ x \text{ is maximal in } y \ (\text{for } S)$
   i.e. $((x, z) \in S \text{ and } (z, y) \in S) \Rightarrow (x = z \text{ or } z = y)$.  
   In this case $\mu_{\mathcal{O}}(R, S) = (-1)^{|S - R|}$.

2. The poset $\mathcal{O} - \{\Delta\}$ is homotopy equivalent to the sphere $S^{n-2}$.

**Proposition**

Let $\overline{\mathcal{O}}$ denote the poset of preorders on $X$. Let $\Gamma$ denote the largest element $\overline{\mathcal{O}}$ (the coarse preorder).

Then $\overline{\mathcal{O}} - \{\Delta, \Gamma\}$ is homotopy equivalent to a wedge of $(n - 1)!$ spheres of dimension $2n - 4$. Hence $\mu_{\overline{\mathcal{O}}}(\Delta, \Gamma) = (n - 1)!$. 