Representations of finite sets

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joint work with

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- Blocks of Finite Groups and Beyond -
in honor of
Burkhard Külshammer
Correspondences, Relations

Let $X$ and $Y$ be finite sets. A correspondence from $X$ to $Y$ is a subset of $Y \times X$. Let $C(Y, X)$ denote the set of correspondences from $X$ to $Y$. A correspondence from $X$ to $X$ is called a relation on $X$. Correspondences can be composed: if $S \subseteq Z \times Y$ and $R \subseteq Y \times X$, then $S \circ R = \{ (z, x) \in Z \times X \mid \exists y \in Y, (z, y) \in S, (y, x) \in R \}$. This composition is associative. In particular $C(X, X)$ is a monoid, with identity element $\Delta_X = \{ (x, x) \mid x \in X \} \subseteq X \times X$. More generally $R \circ \Delta_X = R$ for any $Y$ and any $R \in C(Y, X)$, $\Delta_X \circ S = S$ for any $Z$ and any $S \in C(X, Z)$.
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$R \circ \Delta_X = R$ for any $Y$ and any $R \in \mathcal{C}(Y, X)$,

$\Delta_X \circ S = S$ for any $Z$ and any $S \in \mathcal{C}(X, Z)$. 
When $k$ is a commutative ring, let $kC$ be the following category: the objects of $kC$ are the finite sets, $\text{Hom}_{kC}(X, Y) = kC(Y, X)$, composition of morphisms extends composition of correspondences, the identity morphism of $X$ is $\Delta_X \in kC(X, X)$.

A correspondence functor (over $k$) is a $k$-linear functor from $kC$ to $k$-Mod. Let $\mathcal{F}_k$ denote the category of correspondence functors over $k$. It is an abelian category.
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Correspondence functors

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Examples

The functor $\mathbf{Y}^E, k : X \mapsto k C(X, E)$, e.g. $E = \emptyset$, then $\mathbf{Y}^\emptyset, k \sim = k C(\emptyset)$, $\forall X$.

$E = \{•\}$: then $\mathbf{Y}^{\{•\}}, k(X) \sim = k(2X)$, $\forall X$.

The functor $\mathbf{Y}^E, k$ is a projective object of $F_k$, for any $E$.

Direct summands of $\mathbf{Y}^E, k$: by Yoneda Lemma

$\text{End} F_k(\mathbf{Y}^E, k) \sim = k C(E, E)$.

Let $R$ be a preorder on $E$, i.e. $R \in C(E, E)$ such that $\Delta_E \subseteq R = R^2$.

Then $\mathbf{Y}^{E, k_R}: X \mapsto k C(X, E)$ is a projective object of $F_k$.

Theorem

Let $E$ be a finite set. Then $R^E := k C(E, E) \sim = \text{End} F_k(\mathbf{Y}^E, k)$ is a symmetric algebra (for an explicit symmetrizing form).
Examples

- Yoneda functors $Y_{E,k} : X \mapsto kC(X, E)$ ($E$ fixed finite set)
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  - $E = \emptyset$: then $Y_{\emptyset,k}(X) \cong k$, $\forall X$. 

The functor $Y_{E,k}$ is a projective object of $\mathcal{F}_k$, for any $E$.

Direct summands of $Y_{E,k}$:

By Yoneda Lemma $\text{End}_{\mathcal{F}_k}(Y_{E,k}) \cong kC(E,E)$.

Let $R$ be a preorder on $E$, i.e. $R \in C(E,E)$ such that $\Delta E \subseteq R = R^2$.

Then $Y_{E,k,R} : X \mapsto kC(X,E)$ is a projective object of $\mathcal{F}_k$.

Theorem

Let $E$ be a finite set.

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**Theorem**

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Theorem

Let $E$ be a finite set. Then $\mathcal{R}_E := k\mathcal{C}(E, E) \cong \text{End}_{\mathcal{F}_k}(Y_{E,k})$
Examples

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Theorem

*Let $E$ be a finite set. Then $\mathcal{R}_E := kC(E, E) \cong \text{End}_{\mathcal{F}_k}(Y_{E,k})$ is a **symmetric** algebra (for an explicit symmetrizing form).*
Bounded generation

Let $M \in \mathcal{F}_k$.

1. $M$ has bounded type if there is a finite set $E$ such that $M = \langle M(E) \rangle$.

2. $M$ is finitely generated if moreover $M(E)$ is a finitely generated $k$-module.

Theorem

Let $M \in \mathcal{F}_k$.

The following are equivalent:

1. $M$ is finitely generated.

2. $M$ is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^n Y_E^i$, $k$.

If moreover $k$ is a field, these conditions are equivalent to:

3. there exist positive real numbers $a$, $b$, $r$ such that $\dim_k M(X) \leq ab|X|^r$ for any finite set $X$ with $|X| \geq r$.

4. $M$ has finite length.
Bounded generation - Finite generation

Definition

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1. $M$ is finitely generated.

2. $M$ is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^n Y_{E_i} k$.

If moreover $k$ is a field, these conditions are equivalent to:

3. there exist positive real numbers $a, b, r$ such that $\dim_k M(X) \leq ab |X|^r$ for any finite set $X$ with $|X| \geq r$.

4. $M$ has finite length.
Definition

Let $M \in F_k$. $M$ has bounded type if there is a finite set $E$ such that $M = \langle M(E) \rangle$.

$M$ is finitely generated if moreover $M(E)$ is a finitely generated $k$-module.

Theorem

Let $M \in F_k$. The following are equivalent:

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Let $M \in \mathcal{F}_k$.

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1. $M$ is finitely generated.
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If moreover $k$ is a field, these conditions are equivalent to:

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Let $M \in \mathcal{F}_k$.

1. $M$ has **bounded type**

**Theorem**

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1. $M$ has **bounded type** if there is a finite set $E$ such that $M = \langle M(E) \rangle$.
Bounded generation - Finite generation - Finite length

Definition

Let $M \in \mathcal{F}_k$.

1. $M$ has **bounded type** if there is a finite set $E$ such that $M = \langle M(E) \rangle$.
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Bounded generation - Finite generation - Finite length

**Definition**

Let $M \in F_k$.

1. *M has bounded type* if there is a finite set $E$ such that $M = \langle M(E) \rangle$.

2. *M is finitely generated* if moreover $M(E)$ is a finitely generated $k$-module.

**Theorem**

Let $M \in F_k$.

1. The following are equivalent:
   - $M$ is finitely generated.
   - $M$ is isomorphic to a quotient of a finite direct sum $\oplus_{i=1}^n Y E_i \otimes k$.

   If moreover $k$ is a field, these conditions are equivalent to:

2. There exist positive real numbers $a$, $b$, $r$ such that $\dim_k M(X) \leq ab |X|$ for any finite set $X$ with $|X| \geq r$.

3. $M$ has finite length.
Definition

Let \( M \in \mathcal{F}_k \).

1. \( M \) has **bounded type** if there is a finite set \( E \) such that \( M = \langle M(E) \rangle \).
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Theorem

Let \( M \in \mathcal{F}_k \).

The following are equivalent:

1. \( M \) is finitely generated.
2. \( M \) is isomorphic to a quotient of a finite direct sum \( \bigoplus_{i=1}^{n} Y_E \), \( k \).

If moreover \( k \) is a field, these conditions are equivalent to:

3. There exist positive real numbers \( a, b, r \) such that \( \dim_k M(X) \leq ab |X| \) for any finite set \( X \) with \( |X| \geq r \).
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**Definition**

Let $M \in \mathcal{F}_k$.

1. *$M$ has bounded type* if there is a finite set $E$ such that $M = \langle M(E) \rangle$.
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**Theorem**

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1. $M$ is finitely generated.
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**Definition**

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**Definition**

Let $M \in \mathcal{F}_k$.

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Let $M \in \mathcal{F}_k$. The following are equivalent:

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3. there exist positive real numbers $a, b, r$ such that $\dim_k M(X) \leq ab^{|X|}$ for any finite set $X$ with $|X| \geq r$. 

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Definition

Let $M \in \mathcal{F}_k$.

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4. $M$ has finite length.
The noetherian case

Let $M \in \mathcal{F}_k$ and $E$ be a finite set. Define $M(E) = M(E) / \sum_{|F| < |E|} k \mathcal{C}(E,F) M(F)$.

Theorem
Let $k$ be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let $E$ and $F$ be finite sets.

1. If $L = \langle L(F) \rangle$ and $M(E) \neq 0$, then $|E| \leq 2|F|$.
2. If $L = \langle L(F) \rangle$ and $|E| \geq 2|F|$, then $M = \langle M(E) \rangle$.
3. If $L$ has bounded type, then $M$ has bounded type.
4. If $L$ is finitely generated, then $M$ is finitely generated.

Corollary
Functors of bounded type form an abelian subcategory $\mathcal{F}_b$ of $\mathcal{F}_k$.

Finitely generated functors form an abelian subcategory $\mathcal{F}_f$ of $\mathcal{F}_b$. 

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Let $M \in \mathcal{F}_k$ and $E$ be a finite set.
The noetherian case

Let $M \in \mathcal{F}_k$ and $E$ be a finite set. Define
\[ \overline{M}(E) = M(E) / \]
Let $M \in F_k$ and $E$ be a finite set. Define

$$\overline{M}(E) = M(E)/ \sum_{|F| < |E|} kC(E, F)M(F).$$
The noetherian case

Let \( M \in \mathcal{F}_k \) and \( E \) be a finite set. Define

\[
\overline{M}(E) = M(E) / \sum_{|F|<|E|} kC(E,F)M(F).
\]

**Theorem**

*Let \( k \) be a noetherian (commutative) ring*
The noetherian case

Let $M \in \mathcal{F}_k$ and $E$ be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F| < |E|} kC(E, F)M(F).$$

**Theorem**

*Let $k$ be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$*

1. If $L = \langle L(F) \rangle$ and $M(E) \neq 0$, then $|E| \leq 2|F|$.
2. If $L = \langle L(F) \rangle$ and $|E| \geq 2|F|$, then $M = \langle M(E) \rangle$.
3. If $L$ has bounded type, then $M$ has bounded type.
4. If $L$ is finitely generated, then $M$ is finitely generated.

**Corollary**

Functors of bounded type form an abelian subcategory $\mathcal{F}_b k$ of $\mathcal{F}_k$.

Finitely generated functors form an abelian subcategory $\mathcal{F}_f k$ of $\mathcal{F}_b k$. 
Let $M \in \mathcal{F}_k$ and $E$ be a finite set. Define

$$\overline{M}(E) = M(E)/ \sum_{|F|<|E|} kC(E, F)M(F).$$

**Theorem**

Let $k$ be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let $E$ and $F$ be finite sets.

1. If $L = \langle L(F) \rangle$ and $M(E) \neq 0$, then $|E| \leq 2|F|$.
2. If $L = \langle L(F) \rangle$ and $|E| \geq 2|F|$, then $M = \langle M(E) \rangle$.
3. If $L$ has bounded type, then $M$ has bounded type.
4. If $L$ is finitely generated, then $M$ is finitely generated.

**Corollary**

Functors of bounded type form an abelian subcategory $\mathcal{F}_b k$ of $\mathcal{F}_k$.

Finitely generated functors form an abelian subcategory $\mathcal{F}_f k$ of $\mathcal{F}_b k$. 
The noetherian case

Let $M \in \mathcal{F}_k$ and $E$ be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F| < |E|} kC(E, F)M(F).$$

**Theorem**

Let $k$ be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let $E$ and $F$ be finite sets.

1. If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$
The noetherian case

Let $M \in \mathcal{F}_k$ and $E$ be a finite set. Define

$$
\overline{M}(E) = M(E)/ \sum_{|F| < |E|} kC(E,F)M(F).
$$

Theorem

Let $k$ be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let $E$ and $F$ be finite sets.

1. If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.

Corollary

Functors of bounded type form an abelian subcategory $\mathcal{F}_{b_k}$ of $\mathcal{F}_k$.

Finitely generated functors form an abelian subcategory $\mathcal{F}_{f_k}$ of $\mathcal{F}_{b_k}$.
Let \( M \in \mathcal{F}_k \) and \( E \) be a finite set. Define
\[
\overline{M}(E) = M(E) \cap \sum_{|F| < |E|} kC(E, F)M(F).
\]

**Theorem**

Let \( k \) be a noetherian ring, let \( M \subseteq L \) in \( \mathcal{F}_k \), and let \( E \) and \( F \) be finite sets.

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2. If \( L = \langle L(F) \rangle \) and \( |E| \geq 2^{|F|} \)
The noetherian case

Let \( M \in \mathcal{F}_k \) and \( E \) be a finite set. Define

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Let $M \in \mathcal{F}_k$ and $E$ be a finite set. Define

$$\overline{M}(E) = M(E)/\sum_{|F|<|E|} kC(E,F)M(F).$$

**Theorem**

Let $k$ be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let $E$ and $F$ be finite sets.

1. If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.
2. If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$, then $M = \langle M(E) \rangle$.
3. If $L$ has bounded type

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Let $M \in \mathcal{F}_k$ and $E$ be a finite set. Define

$$\overline{M}(E) = M(E)/ \sum_{|F|<|E|} kC(E,F)M(F).$$

**Theorem**

Let $k$ be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let $E$ and $F$ be finite sets.

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3. If $L$ has bounded type, then $M$ has bounded type.
Let $M \in \mathcal{F}_k$ and $E$ be a finite set. Define

$$\overline{M}(E) = M(E)/\sum_{|F|<|E|} kC(E, F)M(F).$$

**Theorem**

Let $k$ be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let $E$ and $F$ be finite sets.

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2. If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$, then $M = \langle M(E) \rangle$.
3. If $L$ has bounded type, then $M$ has bounded type.
4. If $L$ is finitely generated.

**Corollary**

Functors of bounded type form an abelian subcategory $\mathcal{F}_b_k$ of $\mathcal{F}_k$.

Finitely generated functors form an abelian subcategory $\mathcal{F}_f_k$ of $\mathcal{F}_b_k$. 
Let $M \in \mathcal{F}_k$ and $E$ be a finite set. Define

$$\overline{M}(E) = M(E)/ \sum_{|F| < |E|} kC(E, F)M(F).$$

**Theorem**

Let $k$ be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let $E$ and $F$ be finite sets.

1. If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.
2. If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$, then $M = \langle M(E) \rangle$.
3. If $L$ has bounded type, then $M$ has bounded type.
4. If $L$ is finitely generated, then $M$ is finitely generated.
The noetherian case

Let $M \in \mathcal{F}_k$ and $E$ be a finite set. Define

$$M(E) = M(E)/ \sum_{|F| < |E|} kC(E, F)M(F).$$

Theorem

Let $k$ be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let $E$ and $F$ be finite sets.

1. If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.
2. If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$, then $M = \langle M(E) \rangle$.
3. If $L$ has bounded type, then $M$ has bounded type.
4. If $L$ is finitely generated, then $M$ is finitely generated.

Corollary

Functors of bounded type form an abelian subcategory $\mathcal{F}_{b,k}$ of $\mathcal{F}_k$. Finitely generated functors form an abelian subcategory $\mathcal{F}_{f,k}$ of $\mathcal{F}_{b,k}$.
Let $M \in \mathcal{F}_k$ and $E$ be a finite set. Define

$$\overline{M}(E) = M(E)/\sum_{|F|<|E|} kC(E,F)M(F).$$

**Theorem**

Let $k$ be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let $E$ and $F$ be finite sets.

1. If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.
2. If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$, then $M = \langle M(E) \rangle$.
3. If $L$ has bounded type, then $M$ has bounded type.
4. If $L$ is finitely generated, then $M$ is finitely generated.

**Corollary**

*Functors of bounded type*
The noetherian case

Let $M \in \mathcal{F}_k$ and $E$ be a finite set. Define

$$\overline{M}(E) = \frac{M(E)}{\sum_{|F| < |E|} kC(E,F)M(F)}.$$

**Theorem**

Let $k$ be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let $E$ and $F$ be finite sets.

1. If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.
2. If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$, then $M = \langle M(E) \rangle$.
3. If $L$ has bounded type, then $M$ has bounded type.
4. If $L$ is finitely generated, then $M$ is finitely generated.

**Corollary**

Functors of bounded type form an abelian subcategory $\mathcal{F}_k^b$ of $\mathcal{F}_k$. 
The noetherian case

Let $M \in \mathcal{F}_k$ and $E$ be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F| < |E|} kC(E, F)M(F).$$

**Theorem**

Let $k$ be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let $E$ and $F$ be finite sets.

1. If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.
2. If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$, then $M = \langle M(E) \rangle$.
3. If $L$ has bounded type, then $M$ has bounded type.
4. If $L$ is finitely generated, then $M$ is finitely generated.

**Corollary**

Functors of bounded type form an abelian subcategory $\mathcal{F}_k^b$ of $\mathcal{F}_k$. Finitely generated functors
The noetherian case

Let $M \in \mathcal{F}_k$ and $E$ be a finite set. Define

$$\overline{M}(E) = M(E)/ \sum_{|F| < |E|} kC(E, F)M(F).$$

**Theorem**

Let $k$ be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let $E$ and $F$ be finite sets.

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2. If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$, then $M = \langle M(E) \rangle$.
3. If $L$ has bounded type, then $M$ has bounded type.
4. If $L$ is finitely generated, then $M$ is finitely generated.

**Corollary**

Functors of bounded type form an abelian subcategory $\mathcal{F}_k^b$ of $\mathcal{F}_k$. Finitely generated functors form an abelian subcategory $\mathcal{F}_k^f$ of $\mathcal{F}_k^b$. 
Let $E$ be a finite set. The evaluation functor $M \mapsto M(E) \in R^E$-Mod has a left adjoint $V \mapsto L_E(V)$, defined by $X \mapsto L_E(V)(X) := kC(X,E) \otimes_R E V$.

If $V$ is projective (resp. indecomposable), so is $L_E(V)$.

If $M$ is projective in $F_k$, and $M = \langle M(E) \rangle$, then $M \sim L_{F,F}(M(F))$ for any finite set $F$ with $|F| \geq |E|$, and $M(F)$ is a projective $R_F$-module.

If $k$ is a field, any finitely generated projective in $F_k$ is also injective.

$F_{f,k}$ has infinite global dimension.
Let $E$ be a finite set.
Let $E$ be a finite set. The evaluation functor
\[ M \in \mathcal{F}_k \mapsto M(E) \in \mathcal{R}_E\text{-Mod} \]
If $V$ is projective (resp. indecomposable), so is $L_E(V)$.
If $M$ is projective in $\mathcal{F}_k$, and $M = \langle M(E) \rangle$, then $M \cong L_{\mathcal{F}_k}(M(F))$ for any finite set $F$ with $|F| \geq |E|$, and $M(F)$ is a projective $\mathcal{R}_F$-module.
If $k$ is a field, any finitely generated projective in $\mathcal{F}_k$ is also injective.
$\mathcal{F}_k$ has infinite global dimension.
Let $E$ be a finite set. The evaluation functor

$$M \in \mathcal{F}_k \mapsto M(E) \in \mathcal{R}_E\text{-Mod}$$

has a left adjoint $V \mapsto \mathcal{L}_{E,V}$.

If $V$ is projective (resp. indecomposable), so is $\mathcal{L}_{E,V}$.

If $M$ is projective in $\mathcal{F}_k$, and $M = \langle M(E) \rangle$, then $M \simeq L_{\mathcal{F}_k}(M(F))$ for any finite set $F$ with $|F| \geq |E|$, and $M(F)$ is a projective $\mathcal{R}_F$-module.

If $k$ is a field, any finitely generated projective in $\mathcal{F}_k$ is also injective.

$\mathcal{F}_k$ has infinite global dimension.
Let $E$ be a finite set. The evaluation functor
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\[ X \mapsto \mathcal{L}_{E,V}(X) := kC(X, E) \otimes_{\mathcal{R}_E} V. \]
Let $E$ be a finite set. The evaluation functor 

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If $V$ is projective

If $k$ is a field, any finitely generated projective in $\mathcal{F}_k$ is also injective. $\mathcal{F}_f k$ has infinite global dimension.
Let $E$ be a finite set. The evaluation functor

$$M \in \mathcal{F}_k \mapsto M(E) \in \mathcal{R}_E\text{-Mod}$$

has a left adjoint $V \mapsto \mathcal{L}_{E,V}$, defined by

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If $V$ is projective (resp. indecomposable)

If $k$ is a field, any finitely generated projective in $\mathcal{F}_k$ is also injective.

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If $V$ is projective (resp. indecomposable), so is $\mathcal{L}_{E,V}$. 
Let $E$ be a finite set. The evaluation functor

$$M \in \mathcal{F}_k \mapsto M(E) \in \mathcal{R}_E\text{-Mod}$$

has a left adjoint $V \mapsto \mathcal{L}_{E,V}$, defined by

$$X \mapsto \mathcal{L}_{E,V}(X) := kC(X, E) \otimes_{\mathcal{R}_E} V.$$

If $V$ is projective (resp. indecomposable), so is $\mathcal{L}_{E,V}$.

If $M$ is projective in $\mathcal{F}_k$
Let $E$ be a finite set. The evaluation functor

$$M \in \mathcal{F}_k \mapsto M(E) \in \mathcal{R}_E\text{-}\text{Mod}$$

has a left adjoint $V \mapsto \mathcal{L}_{E,V}$, defined by

$$X \mapsto \mathcal{L}_{E,V}(X) := kC(X, E) \otimes_{\mathcal{R}_E} V.$$

- If $V$ is projective (resp. indecomposable), so is $\mathcal{L}_{E,V}$.
- If $M$ is projective in $\mathcal{F}_k$, and $M = \langle M(E) \rangle$.

If $k$ is a field, any finitely generated projective in $\mathcal{F}_k$ is also injective.

$\mathcal{F}_k$ has infinite global dimension.
Let $E$ be a finite set. The evaluation functor

$$M \in \mathcal{F}_k \mapsto M(E) \in \mathcal{R}_E\text{-Mod}$$

has a left adjoint $V \mapsto \mathcal{L}_{E,V}$, defined by

$$X \mapsto \mathcal{L}_{E,V}(X) := k\mathcal{C}(X, E) \otimes_{\mathcal{R}_E} V.$$

If $V$ is projective (resp. indecomposable), so is $\mathcal{L}_{E,V}$.

If $M$ is projective in $\mathcal{F}_k$, and $M = \langle M(E) \rangle$, then $M \cong \mathcal{L}_{F,M(F)}$ for any finite set $F$ with $|F| \geq |E|$.
Let $E$ be a finite set. The evaluation functor

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If $M$ is projective in $\mathcal{F}_k$, and $M = \langle M(E) \rangle$, then $M \cong \mathcal{L}_{F,M(F)}$ for any finite set $F$ with $|F| \geq |E|$, and $M(F)$ is a projective $\mathcal{R}_F$-module.
Let $E$ be a finite set. The evaluation functor
\[ M \in \mathcal{F}_k \mapsto M(E) \in \mathcal{R}_E\text{-Mod} \]
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\[ X \mapsto \mathcal{L}_{E,V}(X) := k \mathcal{C}(X,E) \otimes_{\mathcal{R}_E} V. \]
If $V$ is projective (resp. indecomposable), so is $\mathcal{L}_{E,V}$.
If $M$ is projective in $\mathcal{F}_k$, and $M = \langle M(E) \rangle$, then $M \cong \mathcal{L}_{F,M(F)}$ for any finite set $F$ with $|F| \geq |E|$, and $M(F)$ is a projective $\mathcal{R}_F$-module.
If $k$ is a field, any finitely generated projective in $\mathcal{F}_k$ is also injective.
Let $E$ be a finite set. The evaluation functor

$$M \in \mathcal{F}_k \mapsto M(E) \in \mathcal{R}_E\text{-Mod}$$

has a left adjoint $V \mapsto \mathcal{L}_{E,V}$, defined by

$$X \mapsto \mathcal{L}_{E,V}(X) := kC(X, E) \otimes_{\mathcal{R}_E} V.$$ 

If $V$ is projective (resp. indecomposable), so is $\mathcal{L}_{E,V}$.

If $M$ is projective in $\mathcal{F}_k$, and $M = \langle M(E) \rangle$, then $M \cong \mathcal{L}_{F,M(F)}$ for any finite set $F$ with $|F| \geq |E|$, and $M(F)$ is a projective $\mathcal{R}_F$-module.

If $k$ is a field, any finitely generated projective in $\mathcal{F}_k$ is also injective. $\mathcal{F}_k^f$ has infinite global dimension.
If $V$ is simple, then $L_E, V$ has a unique maximal subfunctor $J_E, V$, so $S_E, V = L_E, V / J_E, V$ is a simple functor. Conversely, if $S \in F_k$ is simple, and if $V = S(E) \neq 0$, then $V$ is a simple $R_E$-module, and $S \sim S_E, V$.

If moreover $E$ is minimal such that $S(E) \neq 0$, then $V = S(E)$ is a module for the algebra of essential relations on $E_E = kC(E, E) / \sum |F| < |E| kC(F, E)$. 
If $V$ is simple

Conversely, if $S \in F_k$ is simple, and if $V = S(E) \neq 0$, then $V$ is a simple $R$-module, and $S \sim S(E)$. If moreover $E$ is minimal such that $S(E) \neq 0$, then $V = S(E)$ is a module for the algebra of essential relations on $E = kC(E,F) / \sum |F| < |E| kC(F,E)$. 
If $V$ is simple, then $\mathcal{L}_{E,V}$ has a unique maximal subfunctor $J_{E,V}$.
If $V$ is simple, then $\mathcal{L}_{E,V}$ has a unique maximal subfunctor $J_{E,V}$, so $S_{E,V} = \mathcal{L}_{E,V}/J_{E,V}$ is a simple functor.
If $V$ is simple, then $\mathcal{L}_{E,V}$ has a unique maximal subfunctor $J_{E,V}$, so $S_{E,V} = \mathcal{L}_{E,V}/J_{E,V}$ is a simple functor.

Conversely, if $S \in \mathcal{F}_k$ is simple...
If $V$ is simple, then $\mathcal{L}_{E,V}$ has a unique maximal subfunctor $J_{E,V}$, so $S_{E,V} = \mathcal{L}_{E,V}/J_{E,V}$ is a simple functor.

Conversely, if $S \in \mathcal{F}_k$ is simple, and if $V = S(E) \neq 0$
If $V$ is simple, then $\mathcal{L}_{E,V}$ has a unique maximal subfunctor $J_{E,V}$, so $S_{E,V} = \mathcal{L}_{E,V}/J_{E,V}$ is a simple functor.

Conversely, if $S \in \mathcal{F}_k$ is simple, and if $V = S(E) \neq 0$, then $V$ is a simple $R_E$-module.
If $V$ is simple, then $\mathcal{L}_{E,V}$ has a unique maximal subfunctor $J_{E,V}$, so $S_{E,V} = \mathcal{L}_{E,V} / J_{E,V}$ is a simple functor.

Conversely, if $S \in \mathcal{F}_k$ is simple, and if $V = S(E) \neq 0$, then $V$ is a simple $\mathcal{R}_E$-module, and $S \cong S_{E,V}$.
If $V$ is simple, then $\mathcal{L}_{E,V}$ has a unique maximal subfunctor $J_{E,V}$, so $S_{E,V} = \mathcal{L}_{E,V}/J_{E,V}$ is a simple functor.

Conversely, if $S \in \mathcal{F}_k$ is simple, and if $V = S(E) \neq 0$, then $V$ is a simple $\mathcal{R}_E$-module, and $S \cong S_{E,V}$.

If moreover $E$ is minimal such that $S(E) \neq 0$
If $V$ is simple, then $\mathcal{L}_{E,V}$ has a unique maximal subfunctor $J_{E,V}$, so $S_{E,V} = \mathcal{L}_{E,V}/J_{E,V}$ is a simple functor.

Conversely, if $S \in \mathcal{F}_k$ is simple, and if $V = S(E) \neq 0$, then $V$ is a simple $\mathcal{R}_E$-module, and $S \cong S_{E,V}$.

If moreover $E$ is minimal such that $S(E) \neq 0$, then $V = S(E)$ is a module for the algebra of essential relations on $E$. 
If $V$ is simple, then $\mathcal{L}_{E,V}$ has a unique maximal subfunctor $J_{E,V}$, so $S_{E,V} = \mathcal{L}_{E,V}/J_{E,V}$ is a simple functor.

Conversely, if $S \in \mathcal{F}_k$ is simple, and if $V = S(E) \neq 0$, then $V$ is a simple $\mathcal{R}_E$-module, and $S \cong S_{E,V}$.

If moreover $E$ is minimal such that $S(E) \neq 0$, then $V = S(E)$ is a module for the algebra of essential relations on $E$

$$\mathcal{E}_E = k\mathcal{C}(E,E)/ \sum_{|F|<|E|} k\mathcal{C}(E,F)\mathcal{C}(F,E).$$
Evaluation - Simple functors

- If $V$ is simple, then $\mathcal{L}_{E,V}$ has a unique maximal subfunctor $J_{E,V}$, so $S_{E,V} = \mathcal{L}_{E,V} / J_{E,V}$ is a simple functor.

- Conversely, if $S \in \mathcal{F}_k$ is simple, and if $V = S(E) \neq 0$, then $V$ is a simple $\mathcal{R}_E$-module, and $S \cong S_{E,V}$.

- If moreover $E$ is minimal such that $S(E) \neq 0$, then $V = S(E)$ is a module for the algebra of essential relations on $E$

$$\mathcal{E}_E = k\mathcal{C}(E, E)/\sum_{|F|<|E|} k\mathcal{C}(E, F)\mathcal{C}(F, E).$$
Simple functors

Theorem

There is a bijection

Simple correspondence

functors over $k$

up to isomorphism

$\leftrightarrow$

Triples $(E, R, W)$

$\begin{cases}
E \text{ finite set} \\
R \text{ partial order on } E \\
W \text{ simple } k\text{Aut}(E, R)\text{-module}
\end{cases}$

$\uparrow$ up to isomorphism.

Examples:

Let $k$ be a field.

The functor $Y_{\emptyset}, k$ is simple (and projective, and injective), isomorphic to $S_{\emptyset}, \text{tot}, k$.

The functor $Y_{\bullet}, k$ is semisimple (and projective, and injective), isomorphic to $S_{\emptyset}, \text{tot}, k \oplus S_{\bullet}, \text{tot}, k$. 


Theorem

There is a bijection between simple functors over $k$ up to isomorphism and triples $(E, R, W)$:

- $E$ is a finite set,
- $R$ is a partial order on $E$,
- $W$ is a simple $k$-module over $\text{Aut}(E, R)$.

**Examples:**

Let $k$ be a field. The functor $Y_{\emptyset, k}$ is simple (and projective, and injective), isomorphic to $S_{\emptyset, \text{tot}, k}$. The functor $Y_{\bullet, k}$ is semisimple (and projective, and injective), isomorphic to $S_{\emptyset, \text{tot}, k} \oplus S_{\bullet, \text{tot}, k}$. 

Serge Bouc (CNRS-LAMFA)
Simple functors

Theorem

There is a bijection

Simple correspondence

functors over $k$ up to isomorphism

$\leftrightarrow$

Triples $(E, R, W)$

$\begin{cases}
E \text{ finite set} \\
R \text{ partial order on } E \\
W \text{ simple } k\text{Aut}(E, R) - \text{module}
\end{cases}$

Examples:

Let $k$ be a field.

The functor $Y^\emptyset, k$ is simple (and projective, and injective), isomorphic to $S^\emptyset, \text{tot}, k$.

The functor $Y^\bullet, k$ is semisimple (and projective, and injective), isomorphic to $S^\emptyset, \text{tot}, k \oplus S^\bullet, \text{tot}, k$.
Theorem

There is a bijection
Simple functors

Theorem

There is a bijection

Simple correspondence functors over \( k \)

up to isomorphism

Examples:

Let \( k \) be a field. The functor \( Y^{\emptyset}, k \) is simple (and projective, and injective), isomorphic to \( S^{\emptyset}, \text{tot}, k \).

The functor \( Y^{\bullet}, k \) is semisimple (and projective, and injective), isomorphic to \( S^{\emptyset}, \text{tot}, k \oplus S^{\bullet}, \text{tot}, k \).
Simple functors

Theorem

There is a bijection

Simple correspondence functors over $k$

up to isomorphism

Triples $(E, R, W)$

Examples:

Let $k$ be a field. The functor $Y_{\emptyset}, k$ is simple (and projective, and injective), isomorphic to $S_{\emptyset}, \text{tot}, k$. The functor $Y_{\emptyset}, k$ is semisimple (and projective, and injective), isomorphic to $S_{\emptyset}, \text{tot}, k \oplus S_{\emptyset}, \text{tot}, k$. 
Simple functors

Theorem

There is a bijection

Simple correspondence
functors over \( k \)

up to isomorphism

\[
\begin{align*}
\text{Triples } & (E, R, W) \\
\{ & \\
E & \text{ finite set}
\end{align*}
\]

Examples:
Let \( k \) be a field.
The functor \( Y_{\emptyset}, k \)
is simple (and projective, and injective),
isomorphic to \( S_{\emptyset}, \text{tot}, k \).
The functor \( Y_{\cdot}, k \)
is semisimple (and projective, and injective),
isomorphic to \( S_{\emptyset}, \text{tot}, k \) \( \oplus \) \( S_{\cdot}, \text{tot}, k \).
Simple functors

Theorem

There is a bijection

Simple correspondence functors over $k$

up to isomorphism

$\leftrightarrow$

Triples $(E, R, W)$

$\begin{cases} 
E \text{ finite set} \\
R \text{ partial order on } E
\end{cases}$
Theorem

There is a bijection

Simple correspondence functors over $k$

up to isomorphism

$\leftrightarrow$

Triples $(E, R, W)$

\[
\begin{cases}
E \text{ finite set} \\
R \text{ partial order on } E \\
W \text{ simple } k\text{Aut}(E, R)-\text{module}
\end{cases}
\]
Simple functors

**Theorem**

*There is a bijection*

Simple correspondence functors over $k$

up to isomorphism

**Triples** $(E, R, W)$

\[
\begin{align*}
E & \text{ finite set} \\
R & \text{ partial order on } E \\
W & \text{ simple } k\text{Aut}(E, R)-\text{module} \\
& \text{up to isomorphism.}
\end{align*}
\]
Simple functors

**Theorem**

There is a bijection

Simple correspondence
functors over $k$

*up to isomorphism*

$S_{E,R,W} \leftrightarrow (E, R, W)$

**Triples $(E, R, W)$**

\[
\begin{align*}
E & \text{ finite set} \\
R & \text{ partial order on } E \\
W & \text{ simple } k\text{Aut}(E, R)\text{-module} \\
\text{up to isomorphism.}
\end{align*}
\]

Examples:

Let $k$ be a field.

The functor $Y \emptyset, k$ is simple (and projective, and injective), isomorphic to $S \emptyset, \text{tot}, k$.

The functor $Y \bullet, k$ is semisimple (and projective, and injective), isomorphic to $S \emptyset, \text{tot}, k \oplus S \bullet, \text{tot}, k$. 

Serge Bouc (CNRS-LAMFA)

Representations of finite sets

Jena, 2015/07/25
Simple functors

Theorem

There is a bijection

Simple correspondence functors over $k$ up to isomorphism

$S_{E,R,W} \leftrightarrow$ Triangles $(E, R, W)$

$E$ finite set
$R$ partial order on $E$
$W$ simple $k\text{Aut}(E, R)$-module up to isomorphism.

Examples:

Let $k$ be a field.
The functor $Y_{\emptyset}, k$ is simple (and projective, and injective), isomorphic to $S_{\emptyset}, \text{tot}, k$.
The functor $Y_{\bullet}, k$ is semisimple (and projective, and injective), isomorphic to $S_{\emptyset}, \text{tot}, k \oplus S_{\bullet}, \text{tot}, k$. 
Simple functors

**Theorem**

*There is a bijection*

Simple correspondence functors over $k$ *up to isomorphism*

$S_{E,R,W}$ ↔ Triples $(E, R, W)$

- $E$ finite set
- $R$ partial order on $E$
- $W$ simple $k\operatorname{Aut}(E, R)$-module *up to isomorphism."

**Examples:** Let $k$ be a field.
Simple functors

**Theorem**

There is a bijection

Simple correspondence

functors over \( k \)

up to isomorphism

\[ S_{E,R,W} \leftrightarrow \text{Triples } (E, R, W) \]

\[ \begin{cases} 
E \text{ finite set} \\
R \text{ partial order on } E \\
W \text{ simple } k\text{Aut}(E, R)\text{-module up to isomorphism.}
\end{cases} \]

**Examples:** Let \( k \) be a field.

- The functor \( Y_{\emptyset, k} \)
Simple functors

**Theorem**

There is a bijection between simple correspondence functors over $k$ up to isomorphism and triples $(E, R, W)$ where:

- $E$ is a finite set
- $R$ is a partial order on $E$
- $W$ is a simple $k\text{Aut}(E, R)$-module up to isomorphism.

Let $k$ be a field. Then:

- The functor $Y_{\emptyset, k}$ is simple.

Examples: Let $k$ be a field. Then:

- The functor $Y_{\emptyset, k}$ is simple.
Theorem

There is a bijection

Simple correspondence
functors over $k$
up to isomorphism

$S_{E,R,W} \iff (E, R, W)$

Triples $(E, R, W)$

- $E$ finite set
- $R$ partial order on $E$
- $W$ simple $k\text{Aut}(E, R)$-module
up to isomorphism.

Examples: Let $k$ be a field.

- The functor $Y_{\emptyset,k}$ is simple (and projective
Simple functors

**Theorem**

*There is a bijection*

Simple correspondence functors over $k$ up to isomorphism

\[ S_{E,R,W} \leftrightarrow \begin{cases} 
E \text{ finite set} \\
R \text{ partial order on } E \\
W \text{ simple } k\text{Aut}(E,R)-\text{module} 
\end{cases} \text{ up to isomorphism.} \]

**Examples:** Let $k$ be a field.

- The functor $Y_{\emptyset,k}$ is simple (and projective, and injective)
Simple functors

**Theorem**

There is a bijection

Simple correspondence
functors over \( k \)

up to isomorphism

\[
S_{E,R,W} \leftrightarrow \text{Triples } (E, R, W)
\]

\[
\begin{align*}
E & \text{ finite set} \\
R & \text{ partial order on } E \\
W & \text{ simple } k\text{Aut}(E, R)-\text{module} \\
& \text{up to isomorphism.}
\end{align*}
\]

**Examples:** Let \( k \) be a field.

- The functor \( Y_{\emptyset, k} \) is simple (and projective, and injective), isomorphic to \( S_{\emptyset, \text{tot}, k} \).
**Theorem**

There is a bijection

Simple correspondence

functors over \( k \)

denote a bijection

up to isomorphism

\[ S_{E,R,W} \leftrightarrow (E, R, W) \]

**Examples:** Let \( k \) be a field.

- The functor \( Y_{\emptyset,k} \) is simple (and projective, and injective),
  isomorphic to \( S_{\emptyset,tot,k} \).
- The functor \( Y_{\bullet,k} \)
Simple functors

**Theorem**

There is a bijection

Simple correspondence functors over $k$

up to isomorphism

$$S_{E,R,W} \leftrightarrow (E, R, W)$$

**Triples** $(E, R, W)$

- $E$ finite set
- $R$ partial order on $E$
- $W$ simple $k\operatorname{Aut}(E, R)$-module

up to isomorphism.

**Examples:** Let $k$ be a field.

- The functor $Y_{\emptyset,k}$ is simple (and projective, and injective), isomorphic to $S_{\emptyset,\text{tot},k}$.
- The functor $Y_{\bullet,k}$ is semisimple
Simple functors

**Theorem**

There is a bijection

\[
S_{E,R,W} \leftrightarrow \left\{ \begin{array}{l}
E \text{ finite set} \\
R \text{ partial order on } E \\
W \text{ simple } kAut(E, R)-\text{module up to isomorphism.}
\end{array} \right.
\]

**Examples:** Let \( k \) be a field.

- The functor \( Y_{\emptyset,k} \) is simple (and projective, and injective), isomorphic to \( S_{\emptyset,tot,k} \).
- The functor \( Y_{\bullet,k} \) is semisimple (and projective...
**Theorem**

There is a bijection

Simple correspondence functors over $k$

up to isomorphism

$S_{E,R,W} \leftrightarrow \text{Triples } (E, R, W)$

- $E$ finite set
- $R$ partial order on $E$
- $W$ simple $k\text{Aut}(E, R)$-module

up to isomorphism.

**Examples:** Let $k$ be a field.

- The functor $Y_{\emptyset, k}$ is simple (and projective, and injective), isomorphic to $S_{\emptyset, \text{tot}, k}$.
- The functor $Y_{\bullet, k}$ is semisimple (and projective, and injective)
Simple functors

**Theorem**

There is a bijection

Simple correspondence functors over $k$

up to isomorphism

$$S_{E,R,W} \leftrightarrow (E, R, W)$$

**Triples** $(E, R, W)$

- $E$ finite set
- $R$ partial order on $E$
- $W$ simple $k\text{Aut}(E, R)$-module

up to isomorphism.

**Examples:** Let $k$ be a field.

- The functor $Y_{\emptyset,k}$ is simple (and projective, and injective), isomorphic to $S_{\emptyset,\text{tot},k}$.
- The functor $Y_{\bullet,k}$ is semisimple (and projective, and injective), isomorphic to $S_{\emptyset,\text{tot},k} \oplus S_{\bullet,\text{tot},k}$. 
Theorem

Let \( k \) be a noetherian ring, let \( M, N \in \mathcal{F}_k \), and let \( E, F \) be finite sets.

1. If \( M = \langle M(E) \rangle \), then for \( |F| \geq 2|E| \), the evaluation map
   \[ \text{Hom}_{\mathcal{F}_k}(M, N) \rightarrow \text{Hom}_{\mathcal{R}_F}(M(F), N(F)) \]
   is an isomorphism.

2. If \( M \) has bounded type, then for any \( i \in \mathbb{N} \), there exists \( n_i \in \mathbb{N} \) such that if \( |F| \geq n_i \), the map
   \[ \text{Ext}^i_{\mathcal{F}_k}(M, N) \rightarrow \text{Ext}^i_{\mathcal{R}_F}(M(F), N(F)) \]
   is an isomorphism.
Theorem

Let $k$ be a noetherian ring, let $M, N \in \mathcal{F}_k$, and let $E, F$ be finite sets.

1. If $M = \langle M(E) \rangle$, then for $|F| \geq 2|E|$, the evaluation map $\text{Hom}_{\mathcal{F}_k}(M, N) \to \text{Hom}_{\mathcal{R}_F}(M(F), N(F))$ is an isomorphism.

2. If $M$ has bounded type, then for any $i \in \mathbb{N}$, there exists $n_i \in \mathbb{N}$ such that if $|F| \geq n_i$, the map $\text{Ext}^i_{\mathcal{F}_k}(M, N) \to \text{Ext}^i_{\mathcal{R}_F}(M(F), N(F))$ is an isomorphism.
Theorem

Let $k$ be a noetherian ring, let $M, N \in F_k$, and let $E, F$ be finite sets.

1. If $M = \langle M(E) \rangle$
Evaluation - Stability

Theorem

Let $k$ be a noetherian ring, let $M, N \in \mathcal{F}_k$, and let $E, F$ be finite sets.

1. If $M = \langle M(E) \rangle$, then for $|F| \geq 2^{|E|}$, the evaluation map $\text{Hom}_{\mathcal{F}_k}(M, N) \to \text{Hom}_{\mathcal{R}_F}(M(F), N(F))$ is an isomorphism.

2. If $M$ has bounded type, then for any $i \in \mathbb{N}$, there exists $n_i \in \mathbb{N}$ such that if $|F| \geq n_i$, the map $\text{Ext}^i_{\mathcal{F}_k}(M, N) \to \text{Ext}^i_{\mathcal{R}_F}(M(F), N(F))$ is an isomorphism.
Theorem

Let $k$ be a noetherian ring, let $M, N \in \mathcal{F}_k$, and let $E, F$ be finite sets.

1. If $M = \langle M(E) \rangle$, then for $|F| \geq 2^{|E|}$, the evaluation map
   \[ \text{Hom}_{\mathcal{F}_k}(M, N) \to \text{Hom}_{R_F}(M(F), N(F)) \]
Theorem

Let $k$ be a noetherian ring, let $M, N \in \mathcal{F}_k$, and let $E, F$ be finite sets.

1. If $M = \langle M(E) \rangle$, then for $|F| \geq 2^{|E|}$, the evaluation map
   \[
   \text{Hom}_{\mathcal{F}_k}(M, N) \to \text{Hom}_{\mathcal{R}_F}(M(F), N(F))
   \]
   is an isomorphism.
Theorem

Let $k$ be a noetherian ring, let $M, N \in \mathcal{F}_k$, and let $E, F$ be finite sets.

1. If $M = \langle M(E) \rangle$, then for $|F| \geq 2^{|E|}$, the evaluation map
   \[ \text{Hom}_{\mathcal{F}_k}(M, N) \to \text{Hom}_{\mathcal{R}_F}(M(F), N(F)) \]
   is an isomorphism.

2. If $M$ has bounded type
Theorem

Let $k$ be a noetherian ring, let $M, N \in \mathcal{F}_k$, and let $E, F$ be finite sets.

1. If $M = \langle M(E) \rangle$, then for $|F| \geq 2^{|E|}$, the evaluation map
   \[
   \text{Hom}_{\mathcal{F}_k}(M, N) \rightarrow \text{Hom}_{\mathcal{R}_F}(M(F), N(F))
   \]
   is an isomorphism.

2. If $M$ has bounded type, then for any $i \in \mathbb{N}$, there exists $n_i \in \mathbb{N}$
**Theorem**

Let $k$ be a noetherian ring, let $M, N \in \mathcal{F}_k$, and let $E, F$ be finite sets.

1. If $M = \langle M(E) \rangle$, then for $|F| \geq 2^{|E|}$, the evaluation map

   $$\text{Hom}_{\mathcal{F}_k}(M, N) \rightarrow \text{Hom}_{\mathcal{R}_F}(M(F), N(F))$$

   is an isomorphism.

2. If $M$ has bounded type, then for any $i \in \mathbb{N}$, there exists $n_i \in \mathbb{N}$ such that if $|F| \geq n_i$
Theorem

Let $k$ be a noetherian ring, let $M, N \in \mathcal{F}_k$, and let $E, F$ be finite sets.

1. If $M = \langle M(E) \rangle$, then for $|F| \geq 2^{|E|}$, the evaluation map
   \[ \text{Hom}_{\mathcal{F}_k}(M, N) \to \text{Hom}_{\mathcal{R}_F}(M(F), N(F)) \]
   is an isomorphism.

2. If $M$ has bounded type, then for any $i \in \mathbb{N}$, there exists $n_i \in \mathbb{N}$ such that if $|F| \geq n_i$, the map
   \[ \text{Ext}^i_{\mathcal{F}_k}(M, N) \to \text{Ext}^i_{\mathcal{R}_F}(M(F), N(F)) \]
Theorem

Let $k$ be a noetherian ring, let $M, N \in \mathcal{F}_k$, and let $E, F$ be finite sets.

1. If $M = \langle M(E) \rangle$, then for $|F| \geq 2^{|E|}$, the evaluation map
   
   $$\text{Hom}_{\mathcal{F}_k}(M, N) \to \text{Hom}_{\mathcal{R}_F}(M(F), N(F))$$

   is an isomorphism.

2. If $M$ has bounded type, then for any $i \in \mathbb{N}$, there exists $n_i \in \mathbb{N}$ such that if $|F| \geq n_i$, the map

   $$\text{Ext}^i_{\mathcal{F}_k}(M, N) \to \text{Ext}^i_{\mathcal{R}_F}(M(F), N(F))$$

   is an isomorphism.
Evaluation - Stability

**Theorem**

Let $k$ be a noetherian ring, let $M, N \in \mathcal{F}_k$, and let $E, F$ be finite sets.

1. If $M = \langle M(E) \rangle$, then for $|F| \geq 2^{|E|}$, the evaluation map
   \[
   \text{Hom}_{\mathcal{F}_k}(M, N) \to \text{Hom}_{\mathcal{R}_F}(M(F), N(F))
   \]
   is an isomorphism.

2. If $M$ has bounded type, then for any $i \in \mathbb{N}$, there exists $n_i \in \mathbb{N}$ such
   that if $|F| \geq n_i$, the map
   \[
   \text{Ext}^i_{\mathcal{F}_k}(M, N) \to \text{Ext}^i_{\mathcal{R}_F}(M(F), N(F))
   \]
   is an isomorphism.
An equivalence of categories

**Definition**

Let $G_k$ be the following category:

- The objects are pairs $(E, U)$, where $E$ is a finite set, and $U$ is an $R_E$-module.
- A morphism $(E, U) \to (F, V)$ is a morphism of $R_E$-modules $U \to kC(E, F) \otimes R_F V$.

The composition of $U \to kC(E, F) \otimes R_F V$ and $V \to kC(F, G) \otimes R_G W$ is $U \to kC(E, F) \otimes R_F V \to kC(F, G) \otimes R_G W \to kC(E, G) \otimes R_G W$.

The identity morphism of $(E, U)$ is $U \sim \to kC(E, E) \otimes R_E U$.

**Theorem 1**

The assignment $(E, U) \mapsto L_E, U$ is a fully faithful $k$-linear functor $G_k \to F_k$.

**Theorem 2**

When $k$ is noetherian, it is an equivalence of categories. In particular $G_k$ is abelian.
Definition

Let $\mathcal{G}_k$ be the following category:

- The objects are pairs $(E, U)$, where $E$ is a finite set and $U$ is an $R_E$-module.
- A morphism $(E, U) \to (F, V)$ is a morphism of $R_E$-modules $U \to kC(E, F) \otimes R_F V$.
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- The identity morphism of $(E, U)$ is $U \sim \to kC(E, E) \otimes R_E U$.

Theorem

1. The assignment $(E, U) \mapsto L_E, U$ is a fully faithful $k$-linear functor $\mathcal{G}_k \to \mathcal{F}_k$.
2. When $k$ is noetherian, it is an equivalence of categories. In particular $\mathcal{G}_k$ is abelian.
An equivalence of categories

**Definition**

Let $G_k$ be the following category:
- the **objects** are pairs $(E, U)$

The composition of $U \to k \text{C}(E, F) \otimes R F V$ and $V \to k \text{C}(F, G) \otimes R G W$ is $U \to k \text{C}(E, F) \otimes R F V \to k \text{C}(E, F) \otimes R F k \text{C}(F, G) \otimes R G W \to k \text{C}(E, G) \otimes R G W$.

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**Theorem**

1. The assignment $(E, U) \to \text{L}_E, U$ is a fully faithful $k$-linear functor $G_k \to F_b k$.
2. When $k$ is noetherian, it is an equivalence of categories. In particular $G_k$ is abelian.
An equivalence of categories

**Definition**

Let $G_k$ be the following category:
- the objects are pairs $(E, U)$, where $E$ is a finite set

**Theorem**

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An equivalence of categories

Definition

Let $G_k$ be the following category:
- the objects are pairs $(E, U)$, where $E$ is a finite set, and $U$ is an $R_E$-module.

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1. The assignment $(E, U) \mapsto L^E, U$ is a fully faithful $k$-linear functor $G_k \rightarrow F^k$.
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Serge Bouc (CNRS-LAMFA)
Representations of finite sets
Jena, 2015/07/25
Definition

Let $G_k$ be the following category:
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An equivalence of categories

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Let $\mathcal{G}_k$ be the following category:
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1. The assignment $(E, U) \mapsto L(E, U)$ is a fully faithful $k$-linear functor $\mathcal{G}_k \to \mathcal{F}_k$.
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Serge Bouc (CNRS-LAMFA)  Representations of finite sets  Jena, 2015/07/25
An equivalence of categories

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An equivalence of categories

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An equivalence of categories

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Theorem 1

The assignment $(E, U) \mapsto L^E, U$ is a fully faithful $k$-linear functor $\mathcal{G}_k \to \mathcal{F}_k$.

When $k$ is noetherian, it is an equivalence of categories. In particular $\mathcal{G}_k$ is abelian.
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An equivalence of categories

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An equivalence of categories

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- the identity morphism of $(E, U)$ is $U \to kC(E, E) \otimes_{\mathcal{R}_E} U$. 

Theorem 1

The assignment $(E, U) \to \mathcal{L}_E$, $U$ is a fully faithful $k$-linear functor $G_k \to F_k$. When $k$ is noetherian, it is an equivalence of categories. In particular $G_k$ is abelian.
An equivalence of categories

Definition

Let $G_k$ be the following category:
- the objects are pairs $(E, U)$, where $E$ is a finite set, and $U$ is an $\mathcal{R}_E$-module.
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- the identity morphism of $(E, U)$ is $U \to kC(E, E) \otimes_{\mathcal{R}_E} U$.

Theorem

1. The assignment $(E, U) \mapsto \mathcal{L}_{E, U}$ is a fully faithful $k$-linear functor $G_k \to \mathcal{F}^b_k$. 
An equivalence of categories

Definition

Let $G_k$ be the following category:

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- the identity morphism of $(E, U)$ is $U \overset{\text{id}}{\to} kC(E, E) \otimes_{R_E} U$.

Theorem

1. The assignment $(E, U) \mapsto \mathcal{L}_{E, U}$ is a fully faithful $k$-linear functor $G_k \to \mathcal{F}_k^b$.
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Theorem

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2. When $k$ is noetherian, it is an equivalence of categories. In particular $G_k$ is abelian.
Functors and lattices

Let $T = (T, \lor, \land)$ be a finite lattice.

For a finite set $X$, set $F_T(X) = k(T \times X)$.

For $R \in C(Y, X)$ and $\phi: X \to T$, define $R \phi: Y \to T$ by $\forall y \in Y, (R \phi)(y) = \bigvee \{ y, x \in R \phi(x) \}$.

Theorem 1 $F_T$ is a correspondence functor.

Theorem 2 $F_T$ is projective in $F_k \iff T$ is distributive.

Let $kL$ be the following category:

The objects of $kL$ are the finite lattices.

$Hom_{kL}(T, T') = \{ f: T \to T' | f(\bigvee t \in A t) = \bigvee t \in A f(t), \forall A \subseteq T \}$.

Theorem The assignment $T \mapsto \rightarrow F_T$ is a fully faithful $k$-linear functor $kL \to F_k$. 

Serge Bouc (CNRS-LAMFA)

Representations of finite sets

Jena, 2015/07/25
Let $T = (T, \lor, \land)$ be a finite lattice.
Functors and lattices

- Let $T = (T, \lor, \land)$ be a finite lattice.
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Theorem 1 $F_T$ is a correspondence functor.

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- The objects of $kL$ are the finite lattices.
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Let $T = (T, \lor, \land)$ be a finite lattice.

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$$\forall y \in Y, (R\varphi)(y) = \bigvee_{(y,x) \in R} (T^X)(x).$$
Functors and lattices

- Let $T = (T, \vee, \wedge)$ be a finite lattice.
  - For a finite set $X$, set $F_T(X) = k(T^X)$.
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Theorem 1

$F_T$ is a correspondence functor.

Theorem 2

$F_T$ is projective in $F^k$ $\iff$ $T$ is distributive.

Let $kL$ be the following category:

The objects of $kL$ are the finite lattices.

Hom $kL(T, T') = k\{f : T \to T' | f(\bigvee t \in A t) = \bigvee f(t), \forall A \subseteq T\}$.

Theorem

The assignment $T \mapsto F_T$ is a fully faithful $k$-linear functor $kL \to F^k$.
Functors and lattices

- Let \( T = (T, \lor, \land) \) be a finite lattice.
  - For a finite set \( X \), set \( F_T(X) = k(T^X) \).
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**Theorem**

1. $F_T$ is a correspondence functor.
Functors and lattices

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Functors and lattices

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Serge Bouc (CNRS-LAMFA) Representation of finite sets Jena, 2015/07/25
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1. $F_T$ is a correspondence functor.
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Let $k\mathcal{L}$ be the following category:

- The objects of $k\mathcal{L}$ are the finite lattices.
- $\text{Hom}_{k\mathcal{L}}(T, T') = k\{f : T \to T' | f(\bigvee t) = \bigvee_{t \in A} f(t), \forall A \subseteq T\}.$
Let \( T = (T, \lor, \land) \) be a finite lattice.

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Functors and lattices

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Theorem
Functors and lattices

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**Theorem**

The assignment $T \mapsto F_T$
Functors and lattices

- Let $T = (T, \lor, \land)$ be a finite lattice.
  - For a finite set $X$, set $F_T(X) = k(T^X)$.
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    \[\forall y \in Y, (R\varphi)(y) = \bigvee_{(y,x) \in R} \varphi(x) .\]

Theorem

1. $F_T$ is a correspondence functor.
2. $F_T$ is projective in $\mathcal{F}_k \iff T$ is distributive.

- Let $k\mathcal{L}$ be the following category:
  - The objects of $k\mathcal{L}$ are the finite lattices.
  - $\text{Hom}_{k\mathcal{L}}(T, T') = k\{f : T \to T' \mid f(\bigvee_t t) = \bigvee_{t \in A} f(t), \forall A \subseteq T\}.$

Theorem

The assignment $T \mapsto F_T$ is a functor $k\mathcal{L} \to \mathcal{F}_k$. 
Functors and lattices

- Let $T = (T, \lor, \land)$ be a finite lattice.
  - For a finite set $X$, set $F_T(X) = k(T^X)$.
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**Theorem**

The assignment $T \mapsto F_T$ is a $k$-linear functor $k\mathcal{L} \to \mathcal{F}_k$. 
Let $T = (T, \lor, \land)$ be a finite lattice.

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Theorem

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Theorem

The assignment $T \mapsto F_T$ is a fully faithful $k$-linear functor $k\mathcal{L} \to \mathcal{F}_k$. 
A subfunctor of $F_T$

Let $T$ be a finite lattice. Recall that $e \in T$ is irreducible if $\forall A \subseteq T$, $e = \bigvee t \in A \Rightarrow e \in A$.

Let $\text{Irr}(T)$ be the set of irreducible elements of $T$.

For a finite set $X$, denote by $H_T(X)$ the $k$-submodule of $F_T(X) = k(T^X)$ generated by all $\phi : X \to T$ such that $\phi(X) \not\subseteq \text{Irr}(T)$.

Lemma 1

Let $Y, X$ be finite sets, let $R \in C(Y, X)$, and let $\phi : X \to T$. Then $(R\phi)(Y) \cap \text{Irr}(T) \subseteq \phi(X) \cap \text{Irr}(T)$.

2

The assignment $X \mapsto H_T(X)$ is a subfunctor of $F_T$.

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Representations of finite sets

Jena, 2015/07/25
A subfunctor of $F_T$

Let $T$ be a finite lattice.

- Recall that $e \in T$ is irreducible
A subfunctor of $F_T$

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Lemma 1

Let $Y, X$ be finite sets, let $R \in C(Y, X)$, and let $\phi : X \to T$. Then $(R \phi)(Y) \cap \text{Irr}(T) \subseteq \phi(X) \cap \text{Irr}(T)$.

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The assignment $X \mapsto H_T(X)$ is a subfunctor of $F_T$.
Let $T$ be a finite lattice.

- Recall that $e \in T$ is irreducible if $\forall A \subseteq T$, $e = \bigvee_{t \in A} t$.

Let $Irr(T)$ be the set of irreducible elements of $T$.

For a finite set $X$, denote by $H^T(X)$ the $k$-submodule of $F^T(X) = k(T)$ generated by all $\phi: X \to T$ such that $\phi(X) \not\subseteq Irr(T)$.

Lemma 1
Let $Y, X$ be finite sets, let $R \in C(Y, X)$, and let $\phi: X \to T$. Then $(R \phi)(Y) \cap Irr(T) \subseteq \phi(X) \cap Irr(T)$.

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Representations of finite sets
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Let $T$ be a finite lattice.

- Recall that $e \in T$ is irreducible if $\forall A \subseteq T$, $e = \bigvee_{t \in A} t \implies e \in A$. 

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Let $Y$, $X$ be finite sets, let $R \in C(Y, X)$, and let $\phi : X \to T$. Then $(R \phi)(Y) \cap \text{Irr}(T) \subseteq \phi(X) \cap \text{Irr}(T)$.

**The assignment** $X \mapsto H_T(X)$ **is a subfunctor of** $F_T$. 

Serge Bouc (CNRS-LAMFA) 

Representations of finite sets 

Jena, 2015/07/25
Let $T$ be a finite lattice.

- Recall that $e \in T$ is **irreducible** if $\forall A \subseteq T$, $e = \bigvee_{t \in A} t \implies e \in A$.

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**Lemma 1**

Let $Y, X$ be finite sets, let $R \in C(Y, X)$, and let $\phi : X \to T$.

Then $(R\phi)(Y) \cap Irr(T) \subseteq \phi(X) \cap Irr(T)$.

**2**

The assignment $X \mapsto \mathcal{H}_T(X)$ is a subfunctor of $F_T$. 

Let $T$ be a finite lattice.

- Recall that $e \in T$ is irreducible if $\forall A \subseteq T$, $e = \bigvee_{t \in A} t \implies e \in A$.
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A subfunctor of $F_T$

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Let $Y$, $X$ be finite sets, let $R \in \mathcal{C}(Y, X)$, and let $\phi : X \to T$.

Then $(R\phi)(Y) \cap Irr(T) \subseteq \phi(X) \cap Irr(T)$.

The assignment $X \mapsto H_T(X)$ is a subfunctor of $F_T$. 
A subfunctor of $F_T$

Let $T$ be a finite lattice.

- Recall that $e \in T$ is irreducible if $\forall A \subseteq T$, $e = \bigvee_{t \in A} t \implies e \in A$.
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A subfunctor of $F_T$

Let $T$ be a finite lattice.

- Recall that $e \in T$ is *irreducible* if $\forall A \subseteq T, e = \bigvee_{t \in A} t \implies e \in A$. Let $\text{Irr}(T)$ be the set of irreducible elements of $T$.
- For a finite set $X$, denote by $H_T(X)$ the $k$-submodule of $F_T(X) = k(T^X)$ generated by all $\varphi : X \to T$ such that $\varphi(X) \not\subseteq \text{Irr}(T)$. 

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A subfunctor of $F_T$

Let $T$ be a finite lattice.

- Recall that $e \in T$ is irreducible if $\forall A \subseteq T$, $e = \lor_{t \in A} t \implies e \in A$.
  
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**Lemma**
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A subfunctor of $F_T$

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### Lemma

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Serge Bouc (CNRS-LAMFA)  
Representations of finite sets  
Jena, 2015/07/25
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Serge Bouc (CNRS-LAMFA)  Representations of finite sets  Jena, 2015/07/25  13 / 19
The case of a total order

Let $n \in \mathbb{N}$. Set $\mathcal{N} = \{0 < 1 < \ldots < n\}$, and $\mathcal{N}^\times = \mathcal{N} - \{0\}$.

Theorem

For $n \in \mathbb{N}$, set $S(n) = \mathbb{F}_n / H_n$. Then:

1. The surjection $\mathbb{F}_n \to S(n)$ splits.
2. The functor $S(n)$ is projective.
3. If $X$ is a finite set, then $S(n)(X)$ is a free $k$-module of rank $n \sum_{i=0}^{n} (-1)^{n-i} (n \choose i) (i+1)$.
4. $\mathbb{F}_n \cong \bigoplus_{\mathcal{N}^\times} S(|A|)$.
5. If $k$ is a field, then $S(n)$ is simple (and projective, and injective), isomorphic to $S[n]$, \textup{tot}, $k$.
The case of a total order

Let $n \in \mathbb{N}$. 
The case of a total order

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Let \( n \in \mathbb{N} \). Set \( n = \{0 < 1 < \ldots < n\} \), and \([n] = n - \{0\}\).
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Let \( n \in \mathbb{N} \). Set \( n = \{0 < 1 < \ldots < n\} \), and \([n] = n - \{0\}\).

**Theorem**

1. The surjection \( F_n \to S(n) \) splits.
2. The functor \( S(n) \) is projective.
3. If \( X \) is a finite set, then \( S(n)(X) \) is a free \( k \)-module of rank \( \sum_{i=0}^{n} (-1)^i n - i (n+i) / |X| \).
4. \( F_n \cong \bigoplus_{A \subseteq [n]} S(|A|) \cong \bigoplus_{j=0}^{n} S(j)^{n-j} \).
5. If \( k \) is a field, then \( S(n) \) is simple (and projective, and injective), isomorphic to \( S(n, k)_{\text{tot}} \).
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For \( n \in \mathbb{N} \), set \( S(n) = F_n/H_n \). Then:

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3. If \( X \) is a finite set, then \( S(n)(X) \) is a free \( k \)-module of rank \( n \sum_{i=0}^{n-1} (-1)^i n_i i+1 |X| \).
4. \( F_n \cong \bigoplus_{A \subseteq [n]} S(|A|) \cong n \bigoplus_{j=0}^{n-1} S(j) \oplus (n-j) \).
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The case of a total order

Let $n \in \mathbb{N}$. Set $\underline{n} = \{0 < 1 < \ldots < n\}$, and $[n] = n - \{0\}$.

**Theorem**

For $n \in \mathbb{N}$, set $S(n) = F_{\underline{n}}/H_{\underline{n}}$. Then:

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The case of a total order

Let $n \in \mathbb{N}$. Set $\underline{n} = \{0 < 1 < \ldots < n\}$, and $[n] = n - \{0\}$.

Theorem

For $n \in \mathbb{N}$, set $S(n) = F_{\underline{n}}/H_{\underline{n}}$. Then:

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For $n \in \mathbb{N}$, set $S(n) = F_{\underline{n}}/H_{\underline{n}}$. Then:

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2. If $X$ is a finite set, then $S(\underline{n})(X)$ is a free $k$-module of rank
   $$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (i + 1)|X|.$$
The case of a total order

Let \( n \in \mathbb{N} \). Set \( \underline{n} = \{0 < 1 < \ldots < n\} \), and \( [n] = n - \{0\} \).

**Theorem**

For \( n \in \mathbb{N} \), set \( S(n) = \overline{F_n}/\overline{H_n} \). Then:

1. The surjection \( \overline{F_n} \to S(n) \) splits. The functor \( S(n) \) is projective.
2. If \( X \) is a finite set, then \( S(\underline{n})(X) \) is a free \( k \)-module of rank \( \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (i+1)^{|X|} \).
3. \( \overline{F_n} \cong \bigoplus_{A \subseteq [n]} S(|A|) \).
Let $n \in \mathbb{N}$. Set $\underline{n} = \{0 < 1 < \ldots < n\}$, and $[n] = n - \{0\}$.

**Theorem**

For $n \in \mathbb{N}$, set $S(n) = F_{\underline{n}}/H_{\underline{n}}$. Then:

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2. If $X$ is a finite set, then $S(\underline{n})(X)$ is a free $k$-module of rank
   \[
   \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (i + 1)^{|X|}.
   \]
3. $F_{\underline{n}} \cong \bigoplus_{A \subseteq [n]} S(|A|) \cong \bigoplus_{j=0}^{n} S(j) \oplus (n)$.

Serge Bouc (CNRS-LAMFA)

Representations of finite sets

Jena, 2015/07/25
The case of a total order

Let \( n \in \mathbb{N} \). Set \( \underline{n} = \{0 < 1 < \ldots < n\} \), and \( [n] = n - \{0\} \).

**Theorem**

For \( n \in \mathbb{N} \), set \( S(n) = F_{\underline{n}}/H_{\underline{n}} \). Then:

1. The surjection \( F_{\underline{n}} \to S(n) \) splits. The functor \( S(n) \) is projective.
2. If \( X \) is a finite set, then \( S(n)(X) \) is a free \( k \)-module of rank
   \[
   \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (i + 1)^{|X|}.
   \]
3. \( F_{\underline{n}} \cong \bigoplus_{A \subseteq [n]} S(|A|) \cong \bigoplus_{j=0}^{n} S(j)^{\oplus (n)} \).
4. \( \text{End}_{k\mathcal{L}}(\underline{n}) \cong \)
The case of a total order

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### Theorem

For \( n \in \mathbb{N} \), set \( S(n) = F_{\underline{n}}/H_{\underline{n}} \). Then:

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   \[
   \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (i + 1)^{|X|} .
   \]
3. \( F_{\underline{n}} \cong \bigoplus_{A \subseteq [n]} S(|A|) \cong \bigoplus_{j=0}^{n} S(j)^{\oplus (n)} \).
4. \( \text{End}_{k\mathcal{L}}(\underline{n}) \cong \text{End}_{\mathcal{F}_k}(F_{\underline{n}}) \cong \ldots \)
The case of a total order

Let $n \in \mathbb{N}$. Set $\underline{n} = \{0 < 1 < \ldots < n\}$, and $[n] = n - \{0\}$.

**Theorem**

For $n \in \mathbb{N}$, set $S(n) = F_{\underline{n}}/H_n$. Then:

1. The surjection $F_{\underline{n}} \rightarrow S(n)$ splits. The functor $S(n)$ is projective.

2. If $X$ is a finite set, then $S(n)(X)$ is a free $k$-module of rank 

\[ \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (i + 1)^{|X|}. \]

3. $F_{\underline{n}} \cong \bigoplus_{A \subseteq [n]} S(|A|) \cong \bigoplus_{j=0}^{n} S(j)^{\oplus (n)}$.

4. $\text{End}_{kL}(n) \cong \text{End}_{\mathcal{F}_k}(F_{\underline{n}}) \cong \prod_{j=0}^{n} M(n^j)(k)$.
The case of a total order

Let $n \in \mathbb{N}$. Set $\underline{n} = \{0 < 1 < \ldots < n\}$, and $[n] = n - \{0\}$.

Theorem

For $n \in \mathbb{N}$, set $S(n) = F_n/H_n$. Then:

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3. $F_n \cong \bigoplus_{A \subseteq [n]} S(|A|) \cong \bigoplus_{j=0}^{n} S(j)^{\oplus \binom{n}{j}}$.
4. $\text{End}_{k\mathcal{L}}(\underline{n}) \cong \text{End}_{\mathcal{F}_k}(F_n) \cong \prod_{j=0}^{n} M_{\binom{n}{j}}(k)$.
5. If $k$ is a field
Let $n \in \mathbb{N}$. Set $\underline{n} = \{0 < 1 < \ldots < n\}$, and $[n] = n - \{0\}$.

**Theorem**

For $n \in \mathbb{N}$, set $S(n) = F_{\underline{n}}/H_{\underline{n}}$. Then:

1. The surjection $F_{\underline{n}} \rightarrow S(n)$ splits. The functor $S(n)$ is projective.
2. If $X$ is a finite set, then $S(n)(X)$ is a free $k$-module of rank $\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (i + 1)^{|X|}$.
3. $F_{\underline{n}} \cong \bigoplus_{A \subseteq [n]} S(|A|) \cong \bigoplus_{j=0}^{n} S(j)^{\oplus (n)}$.
4. $\text{End}_{k\mathcal{L}}(\underline{n}) \cong \text{End}_{\mathcal{F}_k}(F_{\underline{n}}) \cong \prod_{j=0}^{n} M_{(n)}(k)$.
5. If $k$ is a field, then $S(n)$ is simple.
The case of a total order

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**Theorem**

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1. The surjection \( F_{\underline{n}} \to S(n) \) splits. The functor \( S(n) \) is projective.
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4. \( \text{End}_{k\mathcal{L}}(n) \cong \text{End}_{\mathcal{F}_k}(F_{\underline{n}}) \cong \prod_{j=0}^{n} M_{\binom{n}{j}}(k) \).
5. If \( k \) is a field, then \( S(n) \) is simple (and projective, and injective).
The case of a total order

Let $n \in \mathbb{N}$. Set $\underline{n} = \{0 < 1 < \ldots < n\}$, and $[n] = n - \{0\}$.

**Theorem**

For $n \in \mathbb{N}$, set $S(n) = F_\underline{n}/H_\underline{n}$. Then:

1. The surjection $F_\underline{n} \twoheadrightarrow S(n)$ splits. The functor $S(n)$ is projective.

2. If $X$ is a finite set, then $S(\underline{n})(X)$ is a free $k$-module of rank
   
   $\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (i + 1)^{|X|}$.

3. $F_\underline{n} \cong \bigoplus_{A \subseteq [n]} S(|A|) \cong \bigoplus_{j=0}^{n} S(j)^{\oplus (\binom{n}{j})}$.

4. $\text{End}_{k\mathcal{L}}(\underline{n}) \cong \text{End}_{\mathcal{F}_k}(F_\underline{n}) \cong \prod_{j=0}^{n} M(\binom{n}{j})(k)$.

5. If $k$ is a field, then $S(\underline{n})$ is simple (and projective, and injective).
The case of a total order

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Theorem

For $n \in \mathbb{N}$, set $S(n) = F_{\underline{n}}/H_{\underline{n}}$. Then:

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   \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (i + 1)^{|X|}.
   \]
3. $F_{\underline{n}} \cong \bigoplus_{A \subseteq [n]} S(|A|) \cong \bigoplus_{j=0}^{n} S(j) \oplus(n).
4. $\text{End}_{kL}(\underline{n}) \cong \text{End}_{\mathcal{F}_k}(F_{\underline{n}}) \cong \prod_{j=0}^{n} M_{\binom{n}{j}}(k).
5. If $k$ is a field, then $S(n)$ is simple (and projective, and injective), isomorphic to $S_{[n],\text{tot},k}$.
Let \((E, R)\) be a finite poset, and set \(n = |E|\).

Choose \(T\) such that \((E, R) \sim = \text{Irr}(T)\), and \(\text{Aut}(T) \sim = \text{Aut}(E, R)\).

Let \(S(E, R)\) be the subfunctor of \(F\) generated by \(\gamma\).

**Theorem 1** \(S(E, R)\) doesn't depend on the choice of \(T\), up to isomorphism.

**Theorem 2** \(\exists f = f_{E, R} \in \mathbb{N} - \{0\}\) (explicit) such that, for any finite set \(X\), the \(k\)-module \(S(E, R)(X)\) is free of rank

\[
\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (n^i + f)
\]

It is moreover a free right \(k\text{Aut}(E, R)\)-module.

**Theorem 3** Let \(W\) be a \(k\text{Aut}(E, R)\)-module. Then the assignment \(X \mapsto S(E, R)(X) \otimes k\text{Aut}(E, R)W\) is a correspondence functor, denoted by \(S(E, R, W)\).

**Theorem 4** If \(k\) is a field and \(W\) is simple, then \(S(E, R, W) \sim = S_{E, R, W}\).
Let \((E, R)\) be a finite poset.
Simple functors: the general case

- Let \((E, R)\) be a finite poset, and set \(n = |E|\).
Simple functors: the general case

Let \((E, R)\) be a finite poset, and set \(n = |E|\).
Choose a finite lattice \(T\).

Theorem 1

\(S(E, R)\) doesn't depend on the choice of \(T\), up to isomorphism.

\(\exists f = f_{E, R} \in \mathbb{N} - \{0\}\) (explicit)

such that, for any finite set \(X\), the \(k\)-module \(S(E, R)(X)\)

is free of rank \(n \sum_{i=0}^{\infty} (-1)^{n-i} \binom{n}{i} (i+f) |X|

It is moreover a free right \(k\Aut(E, R)\)-module.

Let \(W\) be a \(k\Aut(E, R)\)-module.

Then the assignment \(X \mapsto S(E, R)(X) \otimes k\Aut(E, R) W\)

is a correspondence functor, denoted by \(S(E, R, W)\).

If \(k\) is a field and \(W\) is simple, then \(S(E, R, W) \cong S_E, R, W\).
Simple functors: the general case

Let \((E, R)\) be a finite poset, and set \(n = |E|\). Choose a finite lattice \(T\) such that \((E, R) \cong \text{Irr}(T)\) as a full subposet.
Simple functors: the general case

- Let \((E, R)\) be a finite poset, and set \(n = |E|\).
  Choose \(T\) such that \((E, R) \cong \text{Irr}(T)\),

\[
\text{Theorem 1: } S(E, R) \text{ doesn't depend on the choice of } T, \text{ up to isomorphism.}
\]

\[
\exists f = f_{E, R} \in \mathbb{N} - \{0\} \text{ (explicit)}\text{ such that, for any finite set } X,
\]

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It is moreover a free right \(\text{Aut}(E, R)\)-module.

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\]

\[
\text{If } k \text{ is a field and } W \text{ is simple, then } S(E, R, W) \cong S(E, R, W).
\]
Simple functors: the general case

- Let \((E, R)\) be a finite poset, and set \(n = |E|\).
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Let \((E, R)\) be a finite poset, and set \(n = |E|\).
Choose \(T\) such that \((E, R) \cong \text{Irr}(T)\), and \(\text{Aut}(T) \cong \text{Aut}(E, R)\) (e.g. \(T = \{\text{lower ideals}(E, R)\}\)).
Simple functors: the general case

Let \((E, R)\) be a finite poset, and set \(n = |E|\).
Choose \(T\) such that \((E, R) \cong \text{Irr}(T)\), and \(\text{Aut}(T) \cong \text{Aut}(E, R)\).
Let \((E, R)\) be a finite poset, and set \(n = |E|\).
Choose \(T\) such that \((E, R) \cong \text{Irr}(T)\), and \(\text{Aut}(T) \cong \text{Aut}(E, R)\).

If \(e \in E\), let \(r(e)\) denote the unique maximal element of \([0, e]\_T\).

Let \(S(E, R)\) be the subfunctor of \(\text{F}_{\text{Top}}(E)\) generated by \(\gamma_I\).

**Theorem 1**
\(S(E, R)\) doesn't depend on the choice of \(T\), up to isomorphism.

**Theorem 2**
\(\exists f = f_{E, R} \in N - \{0\}\) (explicit) such that, for any finite set \(X\), the \(k\)-module \(S(E, R)(X)\) is free of rank \(n \sum_{i=0}^{n-1} (-1)^{n-i} n^i (i + f) |X|\).

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**Theorem 3**
Let \(W\) be a \(k \text{Aut}(E, R)\)-module. Then the assignment \(X \mapsto S(E, R)(X) \otimes k \text{Aut}(E, R) W\) is a correspondence functor, denoted by \(S(E, R, W)\).

**Theorem 4**
If \(k\) is a field and \(W\) is simple, then \(S(E, R, W) \cong S(E, R, W)\).
Simple functors: the general case

- Let \((E, R)\) be a finite poset, and set \(n = |E|\).
  Choose \(T\) such that \((E, R) \cong \text{Irr}(T)\), and \(\text{Aut}(T) \cong \text{Aut}(E, R)\).

- If \(e \in E\), let \(r(e)\) denote the unique maximal element of \([0, e[T\).

- For \(A \subseteq E\), let \(\gamma_A : E \to T\) be the function

\[
\forall e \in E, \gamma_A(e) = \]

Theorem 1: \(S(E, R)\) doesn't depend on the choice of \(T\), up to isomorphism.

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Simple functors: the general case

- Let \((E, R)\) be a finite poset, and set \(n = |E|\).
  Choose \(T\) such that \((E, R) \cong Irr(T)\), and \(Aut(T) \cong Aut(E, R)\).

- If \(e \in E\), let \(r(e)\) denote the unique maximal element of \([0, e]_T\).

- For \(A \subseteq E\), let \(\gamma_A : E \to T\) be the function
  \[
  \forall e \in E, \quad \gamma_A(e) = \begin{cases} 
  e & \text{if } e \notin A \\
  \end{cases}
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Let \((E, R)\) be a finite poset, and set \(n = |E|\).
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For $A \subseteq E$, let $\gamma_A : E \to T$ be the function

$$\forall e \in E, \gamma_A(e) = \begin{cases} e & \text{if } e \notin A \\ r(e) & \text{if } e \in A \end{cases}.$$ 

Let $\gamma = \sum_{A \subseteq E} (-1)^{|A|} \gamma_A \in k(T^E)$
Let \((E, R)\) be a finite poset, and set \(n = \mid E\mid\).

Choose \(T\) such that \((E, R) \cong \text{Irr}(T)\), and \(\text{Aut}(T) \cong \text{Aut}(E, R)\).

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- If \(e \in E\), let \(r(e)\) denote the unique maximal element of \([0, e]_T\).
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  \[
  \forall e \in E, \; \gamma_A(e) = \begin{cases} 
  e & \text{if } e \notin A \\
  r(e) & \text{if } e \in A
  \end{cases}.
  \]
- Let \(\gamma = \sum_{A \subseteq E} (-1)^{|A|} \gamma_A \in k(T^E) = F_{T^{op}}(E)\).

Let \(S(E, R)\) be the subfunctor of \(F_{T^{op}}\) generated by \(\gamma\).
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**Theorem**

- Theorem 1: \(S(E, R)\) doesn't depend on the choice of \(T\), up to isomorphism.
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- Theorem 4: If \(k\) is a field and \(W\) is simple, then \(S(E, R, W) \cong S(E, R, W)\).
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Let $(E, R)$ be a finite poset, and set $n = |E|$. Choose $T$ such that $(E, R) \cong \text{Irr}(T)$, and $\text{Aut}(T) \cong \text{Aut}(E, R)$.

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   It is moreover a free right $k\text{Aut}(E, R)$-module.
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Simple functors: the general case

- Let \((E, R)\) be a finite poset, and set \(n = |E|\).
  Choose \(T\) such that \((E, R) \cong Irr(T)\), and \(Aut(T) \cong Aut(E, R)\).
- Let \(S(E, R)\) be the subfunctor of \(F_{T^{op}}\) generated by \(\gamma\).

**Theorem**

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2. \(\exists f = f_{E,R} \in \mathbb{N} - \{0\}\) (explicit) such that, for any finite set \(X\), the \(k\)-module \(S(E, R)(X)\) is free of rank \(\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (i + f) |X|\).
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3. Let \(W\) be a \(k Aut(E, R)\)-module. Then the assignment \(X \mapsto S(E, R)(X) \otimes_{k Aut(E, R)} W\) is a correspondence functor, denoted by \(S(E, R, W)\).
Simple functors: the general case

Let \((E, R)\) be a finite poset, and set \(n = |E|\).
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4. If \(k\) is a field and \(W\) is simple \(S(E, R, W) \cong S(E, R, W)\).
Simple functors: the general case

- Let \((E, R)\) be a finite poset, and set \(n = |E|\).
  Choose \(T\) such that \((E, R) \cong \text{Irr}(T)\), and \(\text{Aut}(T) \cong \text{Aut}(E, R)\).
- Let \(S(E, R)\) be the subfunctor of \(F_{T_\text{op}}\) generated by \(\gamma\).

**Theorem**

1. \(S(E, R)\) doesn’t depend on the choice of \(T\), up to isomorphism.
2. \(\exists f = f_{E,R} \in \mathbb{N} - \{0\}\) (explicit) such that, for any finite set \(X\), the \(k\)-module \(S(E, R)(X)\) is free of rank \(\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} (i + f)|X|\).
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4. If \(k\) is a field and \(W\) is simple, then \(S(E, R, W) \cong S_{E,R,W}\).
Corollary

Let $k$ be a field.

Let $(E, R)$ be a finite poset, and $W$ be a simple $k\text{Aut}(E, R)$-module.

Let $d_W = \dim_k \text{End}_{k\text{Aut}(E, R)}(W)$.

Then for any finite set $X$,

$$\dim_k S_{E, R, W}(X) = \dim_k W \cdot d_W \cdot |\text{Aut}(E, R)| \cdot |E| \cdot \sum_{i=0}^{\mid E \mid} (-1)^{|E| - i} \cdot |E| - i \cdot (i + f_{E, R}) |X|.$$
Corollary

Let $k$ be a field. Let $(E, R)$ be a finite poset, and $W$ be a simple $k\text{Aut}(E, R)$-module. Let $d_W = \dim_k \text{End}_{k\text{Aut}(E, R)}(W)$. Then for any finite set $X$, $\dim_k S_{E, R}(X) = \dim_k W |\text{Aut}(E, R)| |E| \sum_{i=0}^{\infty} (-1)^i |E|^i f_{E, R} |X|$.
Corollary

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then for any finite set $X$,

$$
\dim_k S_{E, R, W}(X) = \dim_k W |_\text{Aut}(E, R) |_E \sum_{i=0}^{|E|} (-1)^{|E| - i} |E|^{-i} f_{E, R} |_X.
$$
Corollary

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$$\dim_k S_{E, R, W}(X) = d_W |\text{Aut}(E, R)| |E| \sum_{i=0}^{|E|} (-1)^{|E|-i} (|E|-i)^{i+f_{E, R}} |X|.$$
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$$\dim_k S_{E, R, W}(X) = \ldots$$
Corollary

Let $k$ be a field. Let $(E, R)$ be a finite poset, and $W$ be a simple $k\text{Aut}(E, R)$-module. Let $d_W = \dim_k \text{End}_{k\text{Aut}(E, R)}(W)$. Then for any finite set $X$,

$$\dim_k S_{E,R,W}(X) = \frac{\dim_k W}{d_W|\text{Aut}(E, R)|} \sum_{i=0}^{|E|} (-1)^{|E|-i} \binom{|E|}{i} (i + f_{E,R})|X|.$$
Posets of cardinality 4
Posets of cardinality 4

\[
\begin{array}{cccc}
\bullet \bullet \bullet \bullet & \bullet \bullet \circ & \circ \bullet \circ & \circ \circ \bullet \\
\circ \bullet \bullet & \circ \circ \circ & \bullet \circ \circ & \bullet \circ \circ \\
\circ \circ \circ & \circ \circ \circ & \circ \circ \circ & \circ \circ \circ \\
\bullet & \bullet & \bullet & \circ
\end{array}
\]
Splitting the diamond

The diamond is the following lattice $D$.

Over a field of characteristic different from 2, the functor $F_D$ is semisimple:

$$F_D \cong S_0 \oplus 4S_1 \oplus 4S_2 \oplus S_3 \oplus 2S_\cdot \cdot \cdot \oplus S_\cdot \cdot \cdot.$$
Splitting the diamond

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\[
\text{Over a field of characteristic different from 2, the functor } F\text{ is semisimple:}
\]
\[
F D \cong S^0 \oplus 4 S^1 \oplus 4 S^2 \oplus S^3 \oplus 2 S^{\cdots} \oplus S^{\cdots}. 
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The **diamond** is the following lattice $D$

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Splitting the diamond

The diamond is the following lattice $D$

Over a field of characteristic different from 2, the functor $F_D$ is semisimple:

$$F_D \cong S_0$$
Splitting the diamond

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Over a field of characteristic different from 2, the functor $F_D$ is semisimple:

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