Extensions of simple biset functors

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Biset functors over R form an abelian category $\mathcal{F}_R$.

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Simple functors and B-groups

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B_2-groups

Let G be a finite group of odd order.

For $N \trianglelefteq G$, set $m_{G,N} = \sum_{\substack{X \leq G \\ XN = G}} \mu(X, G) \in \mathbb{F}_2$

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There exists a group $\beta_2(G)$ such that

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THANK YOU!