Representations of finite sets and correspondences

Serge Bouc

CNRS-LAMFA
Université de Picardie

joint work with

Jacques Thévenaz

EPFL

ICRA 2018

Correspondence functors

Correspondence functors

For finite sets X and Y

Correspondence functors

For finite sets X and Y, let $\mathcal{C}(Y, X)$ denote the set of correspondences from X to Y

Correspondence functors

For finite sets X and Y, let $\mathcal{C}(Y, X)$ denote the set of correspondences from X to Y, i.e. the set of subsets of $Y \times X$.

Correspondence functors

For finite sets X and Y, let $\mathcal{C}(Y,X)$ denote the set of correspondences from X to Y, i.e. the set of subsets of $Y \times X$.
When k is a commutative ring

Correspondence functors

For finite sets X and Y, let $\mathcal{C}(Y, X)$ denote the set of correspondences from X to Y, i.e. the set of subsets of $Y \times X$.
When k is a commutative ring, let $k\mathcal{C}$ be the following category:

Correspondence functors

For finite sets X and Y, let $\mathcal{C}(Y,X)$ denote the set of correspondences from X to Y, i.e. the set of subsets of $Y \times X$.
When k is a commutative ring, let $k\mathcal{C}$ be the following category:

- the objects of $k\mathcal{C}$ are the finite sets

Correspondence functors

For finite sets X and Y, let $\mathcal{C}(Y,X)$ denote the set of correspondences from X to Y, i.e. the set of subsets of $Y \times X$.
When k is a commutative ring, let $k\mathcal{C}$ be the following category:

- the objects of $k\mathcal{C}$ are the finite sets,
- $Hom_{k\mathcal{C}}(X,Y) = k\mathcal{C}(Y,X)$

Correspondence functors

For finite sets X and Y, let $\mathcal{C}(Y, X)$ denote the set of correspondences from X to Y, i.e. the set of subsets of $Y \times X$.
When k is a commutative ring, let $k\mathcal{C}$ be the following category:

- the objects of $k\mathcal{C}$ are the finite sets,
- $Hom_{k\mathcal{C}}(X, Y) = k\mathcal{C}(Y, X)$ (free k-module with basis $\mathcal{C}(Y, X)$)

Correspondence functors

For finite sets X and Y, let $\mathcal{C}(Y, X)$ denote the set of correspondences from X to Y, i.e. the set of subsets of $Y \times X$.
When k is a commutative ring, let $k\mathcal{C}$ be the following category:

- the objects of $k\mathcal{C}$ are the finite sets,
- $Hom_{k\mathcal{C}}(X, Y) = k\mathcal{C}(Y, X)$,
- composition of morphisms extends composition of correspondences

Correspondence functors

For finite sets X and Y, let $\mathcal{C}(Y,X)$ denote the set of correspondences from X to Y, i.e. the set of subsets of $Y \times X$.
When k is a commutative ring, let $k\mathcal{C}$ be the following category:

- the objects of $k\mathcal{C}$ are the finite sets,
- $Hom_{k\mathcal{C}}(X,Y) = k\mathcal{C}(Y,X)$,
- composition of morphisms extends composition of correspondences,
- the identity morphism of X is $\Delta_X \in k\mathcal{C}(X,X)$.

Correspondence functors

For finite sets X and Y, let $\mathcal{C}(Y,X)$ denote the set of correspondences from X to Y, i.e. the set of subsets of $Y \times X$.
When k is a commutative ring, let $k\mathcal{C}$ be the following category:

- the objects of $k\mathcal{C}$ are the finite sets,
- $Hom_{k\mathcal{C}}(X,Y) = k\mathcal{C}(Y,X)$,
- composition of morphisms extends composition of correspondences,
- the identity morphism of X is $\Delta_X \in k\mathcal{C}(X,X)$.

A correspondence functor (over k) is a k-linear functor from $k\mathcal{C}$ to k-Mod.

Correspondence functors

For finite sets X and Y, let $\mathcal{C}(Y, X)$ denote the set of correspondences from X to Y, i.e. the set of subsets of $Y \times X$.
When k is a commutative ring, let $k\mathcal{C}$ be the following category:

- the objects of $k\mathcal{C}$ are the finite sets,
- $Hom_{k\mathcal{C}}(X, Y) = k\mathcal{C}(Y, X)$,
- composition of morphisms extends composition of correspondences,
- the identity morphism of X is $\Delta_X \in k\mathcal{C}(X, X)$.

A correspondence functor (over k) is a k-linear functor from $k\mathcal{C}$ to k-Mod.
Let $\mathcal{F}_k$ denote the category of correspondence functors over k.

Correspondence functors

For finite sets X and Y, let $\mathcal{C}(Y, X)$ denote the set of correspondences from X to Y, i.e. the set of subsets of $Y \times X$.

When k is a commutative ring, let $k\mathcal{C}$ be the following category:

- the objects of $k\mathcal{C}$ are the finite sets,
- $Hom_{k\mathcal{C}}(X, Y) = k\mathcal{C}(Y, X)$,
- composition of morphisms extends composition of correspondences,
- the identity morphism of X is $\Delta_X \in k\mathcal{C}(X, X)$.

A correspondence functor (over k) is a k-linear functor from $k\mathcal{C}$ to k-Mod. Let $\mathcal{F}_k$ denote the category of correspondence functors over k. It is an abelian category.

Correspondence functors

For finite sets X and Y, let $\mathcal{C}(Y, X)$ denote the set of correspondences from X to Y, i.e. the set of subsets of $Y \times X$.
When k is a commutative ring, let $k\mathcal{C}$ be the following category:

- the objects of $k\mathcal{C}$ are the finite sets,
- $Hom_{k\mathcal{C}}(X, Y) = k\mathcal{C}(Y, X)$,
- composition of morphisms extends composition of correspondences,
- the identity morphism of X is $\Delta_X \in k\mathcal{C}(X, X)$.

A correspondence functor (over k) is a k-linear functor from $k\mathcal{C}$ to k-Mod.
Let $\mathcal{F}_k$ denote the category of correspondence functors over k. It is an abelian category.

Remarks on $k\mathcal{C}$

Remarks on $k\mathcal{C}$

1. For $S \in \mathcal{C}(Y, X)$, let $S^{op} = \{(x, y) \mid (y, x) \in S\} \in \mathcal{C}(X, Y)$.

Remarks on $k\mathcal{C}$

1. For $S \in \mathcal{C}(Y, X)$, let $S^{op} = \{(x, y) \mid (y, x) \in S\} \in \mathcal{C}(X, Y)$. The functor sending each finite set to itself

Remarks on $k\mathcal{C}$

1. For $S \in \mathcal{C}(Y, X)$, let $S^{op} = \{(x, y) \mid (y, x) \in S\} \in \mathcal{C}(X, Y)$. The functor sending each finite set to itself and $S \in \mathcal{C}(Y, X)$ to $S^{op} \in \mathcal{C}(X, Y)$

Remarks on $k\mathcal{C}$

1. For $S \in \mathcal{C}(Y, X)$, let $S^{op} = \{(x, y) \mid (y, x) \in S\} \in \mathcal{C}(X, Y)$. The functor sending each finite set to itself and $S \in \mathcal{C}(Y, X)$ to $S^{op} \in \mathcal{C}(X, Y)$ induces an equivalence of k-linear categories $k\mathcal{C} \to k\mathcal{C}^{op}$.

Remarks on $k\mathcal{C}$

1. For $S \in \mathcal{C}(Y,X)$, let $S^{op} = \{(x,y) \mid (y,x) \in S\} \in \mathcal{C}(X,Y)$. The functor sending each finite set to itself and $S \in \mathcal{C}(Y,X)$ to $S^{op} \in \mathcal{C}(X,Y)$ induces an equivalence of k-linear categories $k\mathcal{C} \to k\mathcal{C}^{op}$.
2. Let X and Y be finite sets such that $|X| \leq |Y|$.

Remarks on $k\mathcal{C}$

1. For $S \in \mathcal{C}(Y, X)$, let $S^{op} = \{(x, y) \mid (y, x) \in S\} \in \mathcal{C}(X, Y)$. The functor sending each finite set to itself and $S \in \mathcal{C}(Y, X)$ to $S^{op} \in \mathcal{C}(X, Y)$ induces an equivalence of k-linear categories $k\mathcal{C} \to k\mathcal{C}^{op}$.
2. Let X and Y be finite sets such that $|X| \leq |Y|$. Let $i : X \hookrightarrow Y$ be an injective map

Remarks on $k\mathcal{C}$

1. For $S \in \mathcal{C}(Y, X)$, let $S^{op} = \{(x, y) \mid (y, x) \in S\} \in \mathcal{C}(X, Y)$. The functor sending each finite set to itself and $S \in \mathcal{C}(Y, X)$ to $S^{op} \in \mathcal{C}(X, Y)$ induces an equivalence of k-linear categories $k\mathcal{C} \to k\mathcal{C}^{op}$.

2. Let X and Y be finite sets such that $|X| \leq |Y|$. Let $i : X \hookrightarrow Y$ be an injective map, and set

$$i_* \; = \Big\{ \big(i(x), x\big) \mid x \in X \Big\}$$

Remarks on $k\mathcal{C}$

1. For $S \in \mathcal{C}(Y, X)$, let $S^{op} = \{(x, y) \mid (y, x) \in S\} \in \mathcal{C}(X, Y)$. The functor sending each finite set to itself and $S \in \mathcal{C}(Y, X)$ to $S^{op} \in \mathcal{C}(X, Y)$ induces an equivalence of k-linear categories $k\mathcal{C} \to k\mathcal{C}^{op}$.
2. Let X and Y be finite sets such that $|X| \leq |Y|$. Let $i : X \hookrightarrow Y$ be an injective map, and set

$$i_* \;\; = \big\{\big(i(x), x\big) \mid x \in X\big\} \in \mathcal{C}(Y, X)$$

Remarks on $k\mathcal{C}$

1. For $S \in \mathcal{C}(Y, X)$, let $S^{op} = \{(x, y) \mid (y, x) \in S\} \in \mathcal{C}(X, Y)$. The functor sending each finite set to itself and $S \in \mathcal{C}(Y, X)$ to $S^{op} \in \mathcal{C}(X, Y)$ induces an equivalence of k-linear categories $k\mathcal{C} \to k\mathcal{C}^{op}$.

2. Let X and Y be finite sets such that $|X| \leq |Y|$. Let $i : X \hookrightarrow Y$ be an injective map, and set

$$\begin{aligned} i_* &= \{(i(x), x) \mid x \in X\} \in \mathcal{C}(Y, X), \\ i^* &= \{(x, i(x)) \mid x \in X\} \end{aligned}$$

Remarks on $k\mathcal{C}$

1. For $S \in \mathcal{C}(Y, X)$, let $S^{op} = \{(x, y) \mid (y, x) \in S\} \in \mathcal{C}(X, Y)$. The functor sending each finite set to itself and $S \in \mathcal{C}(Y, X)$ to $S^{op} \in \mathcal{C}(X, Y)$ induces an equivalence of k-linear categories $k\mathcal{C} \to k\mathcal{C}^{op}$.

2. Let X and Y be finite sets such that $|X| \leq |Y|$. Let $i : X \hookrightarrow Y$ be an injective map, and set

$$\begin{aligned} i_* &= \big\{\big(i(x), x\big) \mid x \in X\big\} \in \mathcal{C}(Y, X), \\ i^* &= \big\{\big(x, i(x)\big) \mid x \in X\big\} \in \mathcal{C}(X, Y). \end{aligned}$$

Remarks on $k\mathcal{C}$

1. For $S \in \mathcal{C}(Y, X)$, let $S^{op} = \{(x, y) \mid (y, x) \in S\} \in \mathcal{C}(X, Y)$. The functor sending each finite set to itself and $S \in \mathcal{C}(Y, X)$ to $S^{op} \in \mathcal{C}(X, Y)$ induces an equivalence of k-linear categories $k\mathcal{C} \to k\mathcal{C}^{op}$.
2. Let X and Y be finite sets such that $|X| \leq |Y|$. Let $i : X \hookrightarrow Y$ be an injective map, and set

$$\begin{aligned} i_* &= \{(i(x), x) \mid x \in X\} \in \mathcal{C}(Y, X), \\ i^* &= \{(x, i(x)) \mid x \in X\} \in \mathcal{C}(X, Y). \end{aligned}$$

Then $i^* i_* = \Delta_X$.

Remarks on $k\mathcal{C}$

1. For $S \in \mathcal{C}(Y, X)$, let $S^{op} = \{(x, y) \mid (y, x) \in S\} \in \mathcal{C}(X, Y)$. The functor sending each finite set to itself and $S \in \mathcal{C}(Y, X)$ to $S^{op} \in \mathcal{C}(X, Y)$ induces an equivalence of k-linear categories $k\mathcal{C} \to k\mathcal{C}^{op}$.
2. Let X and Y be finite sets such that $|X| \leq |Y|$. Let $i : X \hookrightarrow Y$ be an injective map, and set

$$\begin{aligned} i_* &= \big\{\big(i(x), x\big) \mid x \in X\big\} \in \mathcal{C}(Y, X), \\ i^* &= \big\{\big(x, i(x)\big) \mid x \in X\big\} \in \mathcal{C}(X, Y). \end{aligned}$$

Then $i^* i_* = \Delta_X$. If $F \in \mathcal{F}_k$

Remarks on $k\mathcal{C}$

1. For $S \in \mathcal{C}(Y, X)$, let $S^{op} = \{(x, y) \mid (y, x) \in S\} \in \mathcal{C}(X, Y)$. The functor sending each finite set to itself and $S \in \mathcal{C}(Y, X)$ to $S^{op} \in \mathcal{C}(X, Y)$ induces an equivalence of k-linear categories $k\mathcal{C} \to k\mathcal{C}^{op}$.
2. Let X and Y be finite sets such that $|X| \leq |Y|$. Let $i : X \hookrightarrow Y$ be an injective map, and set

$$\begin{aligned} i_* &= \big\{\big(i(x), x\big) \mid x \in X\big\} \in \mathcal{C}(Y, X), \\ i^* &= \big\{\big(x, i(x)\big) \mid x \in X\big\} \in \mathcal{C}(X, Y). \end{aligned}$$

Then $i^* i_* = \Delta_X$. If $F \in \mathcal{F}_k$, then $F(X)$ is isomorphic to a direct summand of $F(Y)$.

Remarks on $k\mathcal{C}$

1. For $S \in \mathcal{C}(Y,X)$, let $S^{op} = \{(x,y) \mid (y,x) \in S\} \in \mathcal{C}(X,Y)$. The functor sending each finite set to itself and $S \in \mathcal{C}(Y,X)$ to $S^{op} \in \mathcal{C}(X,Y)$ induces an equivalence of k-linear categories $k\mathcal{C} \to k\mathcal{C}^{op}$.

2. Let X and Y be finite sets such that $|X| \leq |Y|$. Let $i : X \hookrightarrow Y$ be an injective map, and set

$$\begin{aligned} i_* &= \big\{\big(i(x),x\big) \mid x \in X\big\} \in \mathcal{C}(Y,X), \\ i^* &= \big\{\big(x,i(x)\big) \mid x \in X\big\} \in \mathcal{C}(X,Y). \end{aligned}$$

Then $i^*i_* = \Delta_X$. If $F \in \mathcal{F}_k$, then $F(X)$ is isomorphic to a direct summand of $F(Y)$. In particular $F(X) \neq 0$ implies $F(Y) \neq 0$.

Examples of correspondence functors

Examples of correspondence functors

- Yoneda functors $\mathsf{Y}_{E,k} : X \mapsto k\mathcal{C}(X, E)$ (E fixed finite set)

Examples of correspondence functors

- Yoneda functors $\mathsf{Y}_{E,k} : X \mapsto k\mathcal{C}(X, E)$ e.g.
  - $E = \emptyset$:

Examples of correspondence functors

- Yoneda functors $\mathsf{Y}_{E,k} : X \mapsto k\mathcal{C}(X, E)$ e.g.
  - $E = \emptyset$: then $\mathsf{Y}_{\emptyset,k}(X) = k\mathcal{C}(X, \emptyset)$

Examples of correspondence functors

- Yoneda functors $\mathsf{Y}_{E,k} : X \mapsto k\mathcal{C}(X,E)$ e.g.
  - $E = \emptyset$: then $\mathsf{Y}_{\emptyset,k}(X) = k\mathcal{C}(X,\emptyset) \cong k$, $\forall X$.

Examples of correspondence functors

- Yoneda functors $\mathsf{Y}_{E,k} : X \mapsto k\mathcal{C}(X, E)$ e.g.
    - $E = \emptyset$: then $\mathsf{Y}_{\emptyset,k}(X) = k\mathcal{C}(X, \emptyset) \cong k$, $\forall X$.
    - $E = \{\bullet\}$:

Examples of correspondence functors

- Yoneda functors $\mathsf{Y}_{E,k} : X \mapsto k\mathcal{C}(X, E)$ e.g.
  - $E = \emptyset$: then $\mathsf{Y}_{\emptyset,k}(X) = k\mathcal{C}(X, \emptyset) \cong k,\ \forall X$.
  - $E = \{\bullet\}$: then $\mathsf{Y}_{\bullet,k}(X) = k\mathcal{C}(X, \bullet)$

Examples of correspondence functors

- Yoneda functors $\mathsf{Y}_{E,k} : X \mapsto k\mathcal{C}(X, E)$ e.g.
  - $E = \emptyset$: then $\mathsf{Y}_{\emptyset,k}(X) = k\mathcal{C}(X, \emptyset) \cong k, \ \forall X.$
  - $E = \{\bullet\}$: then $\mathsf{Y}_{\bullet,k}(X) = k\mathcal{C}(X, \bullet) \cong k(2^X), \ \forall X.$

Examples of correspondence functors

- Yoneda functors $\mathsf{Y}_{E,k} : X \mapsto k\mathcal{C}(X,E)$ e.g.
  - $E=\emptyset$: then $\mathsf{Y}_{\emptyset,k}(X) = k\mathcal{C}(X,\emptyset) \cong k$, $\forall X$.
  - $E=\{\bullet\}$: then $\mathsf{Y}_{\bullet,k}(X) = k\mathcal{C}(X,\bullet) \cong k(2^X)$, $\forall X$.

The functor $\mathsf{Y}_{E,k}$ is a projective object of $\mathcal{F}_k$, for any E

Examples of correspondence functors

- Yoneda functors $\mathsf{Y}_{E,k} : X \mapsto k\mathcal{C}(X,E)$ e.g.
  - $E = \emptyset$: then $\mathsf{Y}_{\emptyset,k}(X) = k\mathcal{C}(X,\emptyset) \cong k$, $\forall X$.
  - $E = \{\bullet\}$: then $\mathsf{Y}_{\bullet,k}(X) = k\mathcal{C}(X,\bullet) \cong k(2^X)$, $\forall X$.

  The functor $\mathsf{Y}_{E,k}$ is a projective object of $\mathcal{F}_k$, for any E, since $Hom_{\mathcal{F}_k}(Y_{E,k}, M) \cong M(E)$, $\forall M \in \mathcal{F}_k$

Examples of correspondence functors

- Yoneda functors $\mathsf{Y}_{E,k} : X \mapsto k\mathcal{C}(X, E)$ e.g.
  - $E = \emptyset$: then $\mathsf{Y}_{\emptyset,k}(X) = k\mathcal{C}(X, \emptyset) \cong k$, $\forall X$.
  - $E = \{\bullet\}$: then $\mathsf{Y}_{\bullet,k}(X) = k\mathcal{C}(X, \bullet) \cong k(2^X)$, $\forall X$.

  The functor $\mathsf{Y}_{E,k}$ is a projective object of $\mathcal{F}_k$, for any E.

Examples of correspondence functors

- Yoneda functors $\mathsf{Y}_{E,k} : X \mapsto k\mathcal{C}(X, E)$ e.g.
  - $E = \emptyset$: then $\mathsf{Y}_{\emptyset,k}(X) = k\mathcal{C}(X, \emptyset) \cong k$, $\forall X$.
  - $E = \{\bullet\}$: then $\mathsf{Y}_{\bullet,k}(X) = k\mathcal{C}(X, \bullet) \cong k(2^X)$, $\forall X$.

  The functor $\mathsf{Y}_{E,k}$ is a projective object of $\mathcal{F}_k$, for any E.
- Direct summands of $\mathsf{Y}_{E,k}$:

Examples of correspondence functors

- Yoneda functors $\mathsf{Y}_{E,k} : X \mapsto k\mathcal{C}(X, E)$ e.g.
  - $E = \emptyset$: then $\mathsf{Y}_{\emptyset,k}(X) = k\mathcal{C}(X, \emptyset) \cong k$, $\forall X$.
  - $E = \{\bullet\}$: then $\mathsf{Y}_{\bullet,k}(X) = k\mathcal{C}(X, \bullet) \cong k(2^X)$, $\forall X$.

  The functor $\mathsf{Y}_{E,k}$ is a projective object of $\mathcal{F}_k$, for any E.
- Direct summands of $\mathsf{Y}_{E,k}$: by the Yoneda Lemma
  $End_{\mathcal{F}_k}(\mathsf{Y}_{E,k}) \cong$

Examples of correspondence functors

- Yoneda functors $\mathsf{Y}_{E,k} : X \mapsto k\mathcal{C}(X,E)$ e.g.
  - $E = \emptyset$: then $\mathsf{Y}_{\emptyset,k}(X) = k\mathcal{C}(X,\emptyset) \cong k$, $\forall X$.
  - $E = \{\bullet\}$: then $\mathsf{Y}_{\bullet,k}(X) = k\mathcal{C}(X,\bullet) \cong k(2^X)$, $\forall X$.

  The functor $\mathsf{Y}_{E,k}$ is a projective object of $\mathcal{F}_k$, for any E.
- Direct summands of $\mathsf{Y}_{E,k}$: by the Yoneda Lemma $End_{\mathcal{F}_k}(\mathsf{Y}_{E,k}) \cong k\mathcal{C}(E,E)$.

Examples of correspondence functors

- Yoneda functors $\mathsf{Y}_{E,k} : X \mapsto k\mathcal{C}(X,E)$ e.g.
  - $E=\emptyset$: then $\mathsf{Y}_{\emptyset,k}(X)=k\mathcal{C}(X,\emptyset)\cong k$, $\forall X$.
  - $E=\{\bullet\}$: then $\mathsf{Y}_{\bullet,k}(X)=k\mathcal{C}(X,\bullet)\cong k(2^X)$, $\forall X$.

  The functor $\mathsf{Y}_{E,k}$ is a projective object of $\mathcal{F}_k$, for any E.
- Direct summands of $\mathsf{Y}_{E,k}$: by the Yoneda Lemma $End_{\mathcal{F}_k}(\mathsf{Y}_{E,k})\cong k\mathcal{C}(E,E)$. Let R be a preorder on E

Examples of correspondence functors

- Yoneda functors $\mathsf{Y}_{E,k} : X \mapsto k\mathcal{C}(X,E)$ e.g.
  - $E=\emptyset$: then $\mathsf{Y}_{\emptyset,k}(X) = k\mathcal{C}(X,\emptyset) \cong k$, $\forall X$.
  - $E=\{\bullet\}$: then $\mathsf{Y}_{\bullet,k}(X) = k\mathcal{C}(X,\bullet) \cong k(2^X)$, $\forall X$.

  The functor $\mathsf{Y}_{E,k}$ is a projective object of $\mathcal{F}_k$, for any E.
- Direct summands of $\mathsf{Y}_{E,k}$: by the Yoneda Lemma $End_{\mathcal{F}_k}(\mathsf{Y}_{E,k}) \cong k\mathcal{C}(E,E)$. Let R be a preorder on E, i.e. $R \in \mathcal{C}(E,E)$ such that $\Delta_E \subseteq R = R^2$.

Examples of correspondence functors

- Yoneda functors $\mathsf{Y}_{E,k} : X \mapsto k\mathcal{C}(X,E)$ e.g.
  - $E=\emptyset$: then $\mathsf{Y}_{\emptyset,k}(X) = k\mathcal{C}(X,\emptyset) \cong k$, $\forall X$.
  - $E=\{\bullet\}$: then $\mathsf{Y}_{\bullet,k}(X) = k\mathcal{C}(X,\bullet) \cong k(2^X)$, $\forall X$.

  The functor $\mathsf{Y}_{E,k}$ is a projective object of $\mathcal{F}_k$, for any E.
- Direct summands of $\mathsf{Y}_{E,k}$: by the Yoneda Lemma $End_{\mathcal{F}_k}(\mathsf{Y}_{E,k}) \cong k\mathcal{C}(E,E)$. Let R be a preorder on E, i.e. $R \in \mathcal{C}(E,E)$ such that $\Delta_E \subseteq R = R^2$.
  Then $\mathsf{Y}_{E,k}R : X \mapsto k\mathcal{C}(X,E)R$ is a projective object of $\mathcal{F}_k$.

Examples of correspondence functors

- Yoneda functors $\mathsf{Y}_{E,k} : X \mapsto k\mathcal{C}(X,E)$ e.g.
  - $E = \emptyset$: then $\mathsf{Y}_{\emptyset,k}(X) = k\mathcal{C}(X,\emptyset) \cong k$, $\forall X$.
  - $E = \{\bullet\}$: then $\mathsf{Y}_{\bullet,k}(X) = k\mathcal{C}(X,\bullet) \cong k(2^X)$, $\forall X$.

  The functor $\mathsf{Y}_{E,k}$ is a projective object of $\mathcal{F}_k$, for any E.
- Direct summands of $\mathsf{Y}_{E,k}$: by the Yoneda Lemma $End_{\mathcal{F}_k}(\mathsf{Y}_{E,k}) \cong k\mathcal{C}(E,E)$. Let R be a preorder on E, i.e. $R \in \mathcal{C}(E,E)$ such that $\Delta_E \subseteq R = R^2$.
  Then $\mathsf{Y}_{E,k}R : X \mapsto k\mathcal{C}(X,E)R$ is a projective object of $\mathcal{F}_k$.
- For $F \in \mathcal{F}_k$, the dual $F^\natural$ of F

Examples of correspondence functors

- Yoneda functors $\mathsf{Y}_{E,k} : X \mapsto k\mathcal{C}(X,E)$ e.g.
  - $E = \emptyset$: then $\mathsf{Y}_{\emptyset,k}(X) = k\mathcal{C}(X,\emptyset) \cong k$, $\forall X$.
  - $E = \{\bullet\}$: then $\mathsf{Y}_{\bullet,k}(X) = k\mathcal{C}(X,\bullet) \cong k(2^X)$, $\forall X$.

  The functor $\mathsf{Y}_{E,k}$ is a projective object of $\mathcal{F}_k$, for any E.
- Direct summands of $\mathsf{Y}_{E,k}$: by the Yoneda Lemma $End_{\mathcal{F}_k}(\mathsf{Y}_{E,k}) \cong k\mathcal{C}(E,E)$. Let R be a preorder on E, i.e. $R \in \mathcal{C}(E,E)$ such that $\Delta_E \subseteq R = R^2$.
  Then $\mathsf{Y}_{E,k}R : X \mapsto k\mathcal{C}(X,E)R$ is a projective object of $\mathcal{F}_k$.
- For $F \in \mathcal{F}_k$, the dual $F^\natural$ of F is defined by $F^\natural(X) = Hom_k\big(F(X),k\big)$

Examples of correspondence functors

- Yoneda functors $\mathsf{Y}_{E,k} : X \mapsto k\mathcal{C}(X, E)$ e.g.
  - $E = \emptyset$: then $\mathsf{Y}_{\emptyset,k}(X) = k\mathcal{C}(X, \emptyset) \cong k$, $\forall X$.
  - $E = \{\bullet\}$: then $\mathsf{Y}_{\bullet,k}(X) = k\mathcal{C}(X, \bullet) \cong k(2^X)$, $\forall X$.

  The functor $\mathsf{Y}_{E,k}$ is a projective object of $\mathcal{F}_k$, for any E.
- Direct summands of $\mathsf{Y}_{E,k}$: by the Yoneda Lemma $End_{\mathcal{F}_k}(\mathsf{Y}_{E,k}) \cong k\mathcal{C}(E, E)$. Let R be a preorder on E, i.e. $R \in \mathcal{C}(E, E)$ such that $\Delta_E \subseteq R = R^2$.
  Then $\mathsf{Y}_{E,k}R : X \mapsto k\mathcal{C}(X, E)R$ is a projective object of $\mathcal{F}_k$.
- For $F \in \mathcal{F}_k$, the dual $F^\natural$ of F is defined by $F^\natural(X) = Hom_k\big(F(X), k\big)$ and $F^\natural(S) = {}^tF(S^{op})$ for $S \in \mathcal{C}(Y, X)$.

Simple functors

- Let X be a finite set

Simple functors

- Let X be a finite set, and V be a simple $k\mathcal{C}(X, X)$-module.

Simple functors

- Let X be a finite set, and V be a simple $k\mathcal{C}(X, X)$-module. Then the functor $L_{X,V}$

Simple functors

- Let X be a finite set, and V be a simple $k\mathcal{C}(X,X)$-module. Then the functor $L_{X,V}$ (recall that $L_{X,V}(Y) = k\mathcal{C}(Y,X) \otimes_{k\mathcal{C}(X,X)} V$)

Simple functors

- Let X be a finite set, and V be a simple $k\mathcal{C}(X,X)$-module. Then the functor $L_{X,V}$ admits a unique maximal proper subfunctor $J_{X,V}$

Simple functors

- Let X be a finite set, and V be a simple $k\mathcal{C}(X,X)$-module. Then the functor $L_{X,V}$ admits a unique maximal proper subfunctor $J_{X,V}$, and $S_{X,V} = L_{X,V}/J_{X,V}$ is a simple functor

Simple functors

- Let X be a finite set, and V be a simple $k\mathcal{C}(X,X)$-module. Then the functor $L_{X,V}$ admits a unique maximal proper subfunctor $J_{X,V}$, and $S_{X,V} = L_{X,V}/J_{X,V}$ is a simple functor, such that $S_{X,V}(X) \cong V$.

Simple functors

- Let X be a finite set, and V be a simple $k\mathcal{C}(X,X)$-module. Then the functor $L_{X,V}$ admits a unique maximal proper subfunctor $J_{X,V}$, and $S_{X,V} = L_{X,V}/J_{X,V}$ is a simple functor, such that $S_{X,V}(X) \cong V$.
- Conversely, if S is a simple functor such that $S(X) \neq 0$

Simple functors

- Let X be a finite set, and V be a simple $k\mathcal{C}(X, X)$-module. Then the functor $L_{X,V}$ admits a unique maximal proper subfunctor $J_{X,V}$, and $S_{X,V} = L_{X,V}/J_{X,V}$ is a simple functor, such that $S_{X,V}(X) \cong V$.
- Conversely, if S is a simple functor such that $S(X) \neq 0$, then $V = S(X)$ is a simple $k\mathcal{C}(X, X)$-module

Simple functors

- Let X be a finite set, and V be a simple $k\mathcal{C}(X,X)$-module. Then the functor $L_{X,V}$ admits a unique maximal proper subfunctor $J_{X,V}$, and $S_{X,V} = L_{X,V}/J_{X,V}$ is a simple functor, such that $S_{X,V}(X) \cong V$.
- Conversely, if S is a simple functor such that $S(X) \neq 0$, then $V = S(X)$ is a simple $k\mathcal{C}(X,X)$-module, and $S \cong S_{X,V}$.

Simple functors

- Let X be a finite set, and V be a simple $k\mathcal{C}(X,X)$-module. Then the functor $L_{X,V}$ admits a unique maximal proper subfunctor $J_{X,V}$, and $S_{X,V} = L_{X,V}/J_{X,V}$ is a simple functor, such that $S_{X,V}(X) \cong V$.
- Conversely, if S is a simple functor such that $S(X) \neq 0$, then $V = S(X)$ is a simple $k\mathcal{C}(X,X)$-module, and $S \cong S_{X,V}$.
- If S is a simple functor, and if E is a set of minimal cardinality such that $S(E) \neq 0$

Simple functors

- Let X be a finite set, and V be a simple $k\mathcal{C}(X,X)$-module. Then the functor $L_{X,V}$ admits a unique maximal proper subfunctor $J_{X,V}$, and $S_{X,V} = L_{X,V}/J_{X,V}$ is a simple functor, such that $S_{X,V}(X) \cong V$.
- Conversely, if S is a simple functor such that $S(X) \neq 0$, then $V = S(X)$ is a simple $k\mathcal{C}(X,X)$-module, and $S \cong S_{X,V}$.
- If S is a simple functor, and if E is a set of minimal cardinality such that $S(E) \neq 0$, then $S(E)$ is a simple module for the algebra $\mathcal{E}_E$ of

  essential relations on E

Simple functors

- Let X be a finite set, and V be a simple $k\mathcal{C}(X,X)$-module. Then the functor $L_{X,V}$ admits a unique maximal proper subfunctor $J_{X,V}$, and $S_{X,V} = L_{X,V}/J_{X,V}$ is a simple functor, such that $S_{X,V}(X) \cong V$.
- Conversely, if S is a simple functor such that $S(X) \neq 0$, then $V = S(X)$ is a simple $k\mathcal{C}(X,X)$-module, and $S \cong S_{X,V}$.
- If S is a simple functor, and if E is a set of minimal cardinality such that $S(E) \neq 0$, then $S(E)$ is a simple module for the algebra $\mathcal{E}_E$ of essential relations on E ($S(E) \xrightarrow{S(R)} S(E)$ factoring as $S(E) \searrow S(Y) = 0 \nearrow S(E)$ if $E \xrightarrow{R} E$ factoring as $E \searrow Y \nearrow E$ with $|Y| < |E|$).

Simple functors

- Let X be a finite set, and V be a simple $k\mathcal{C}(X,X)$-module. Then the functor $L_{X,V}$ admits a unique maximal proper subfunctor $J_{X,V}$, and $S_{X,V} = L_{X,V}/J_{X,V}$ is a simple functor, such that $S_{X,V}(X) \cong V$.
- Conversely, if S is a simple functor such that $S(X) \neq 0$, then $V = S(X)$ is a simple $k\mathcal{C}(X,X)$-module, and $S \cong S_{X,V}$.
- If S is a simple functor, and if E is a set of minimal cardinality such that $S(E) \neq 0$, then $S(E)$ is a simple module for the algebra $\mathcal{E}_E$ of essential relations on E.

Simple functors

- Let X be a finite set, and V be a simple $k\mathcal{C}(X,X)$-module. Then the functor $L_{X,V}$ admits a unique maximal proper subfunctor $J_{X,V}$, and $S_{X,V} = L_{X,V}/J_{X,V}$ is a simple functor, such that $S_{X,V}(X) \cong V$.
- Conversely, if S is a simple functor such that $S(X) \neq 0$, then $V = S(X)$ is a simple $k\mathcal{C}(X,X)$-module, and $S \cong S_{X,V}$.
- If S is a simple functor, and if E is a set of minimal cardinality such that $S(E) \neq 0$, then $S(E)$ is a simple module for the algebra $\mathcal{E}_E$ of

  essential relations on E.
- The simple $\mathcal{E}_E$-modules are parametrized by pairs (R, W)

Simple functors

- Let X be a finite set, and V be a simple $k\mathcal{C}(X,X)$-module. Then the functor $L_{X,V}$ admits a unique maximal proper subfunctor $J_{X,V}$, and $S_{X,V} = L_{X,V}/J_{X,V}$ is a simple functor, such that $S_{X,V}(X) \cong V$.
- Conversely, if S is a simple functor such that $S(X) \neq 0$, then $V = S(X)$ is a simple $k\mathcal{C}(X,X)$-module, and $S \cong S_{X,V}$.
- If S is a simple functor, and if E is a set of minimal cardinality such that $S(E) \neq 0$, then $S(E)$ is a simple module for the algebra $\mathcal{E}_E$ of

  essential relations on E.
- The simple $\mathcal{E}_E$-modules are parametrized by pairs (R, W), where R is an order on E

Simple functors

- Let X be a finite set, and V be a simple $k\mathcal{C}(X,X)$-module. Then the functor $L_{X,V}$ admits a unique maximal proper subfunctor $J_{X,V}$, and $S_{X,V} = L_{X,V}/J_{X,V}$ is a simple functor, such that $S_{X,V}(X) \cong V$.
- Conversely, if S is a simple functor such that $S(X) \neq 0$, then $V = S(X)$ is a simple $k\mathcal{C}(X,X)$-module, and $S \cong S_{X,V}$.
- If S is a simple functor, and if E is a set of minimal cardinality such that $S(E) \neq 0$, then $S(E)$ is a simple module for the algebra $\mathcal{E}_E$ of essential relations on E.
- The simple $\mathcal{E}_E$-modules are parametrized by pairs (R,W), where R is an order on E, and W is a simple $kAut(E,R)$-module.

Simple functors

- Let X be a finite set, and V be a simple $k\mathcal{C}(X,X)$-module. Then the functor $L_{X,V}$ admits a unique maximal proper subfunctor $J_{X,V}$, and $S_{X,V} = L_{X,V}/J_{X,V}$ is a simple functor, such that $S_{X,V}(X) \cong V$.
- Conversely, if S is a simple functor such that $S(X) \neq 0$, then $V = S(X)$ is a simple $k\mathcal{C}(X,X)$-module, and $S \cong S_{X,V}$.
- If S is a simple functor, and if E is a set of minimal cardinality such that $S(E) \neq 0$, then $S(E)$ is a simple module for the algebra $\mathcal{E}_E$ of essential relations on E.
- The simple $\mathcal{E}_E$-modules are parametrized by pairs (R, W), where R is an order on E, and W is a simple $kAut(E,R)$-module.
- The simple correspondence functors over k are parametrized by triples (E, R, W)

Simple functors

- Let X be a finite set, and V be a simple $k\mathcal{C}(X,X)$-module. Then the functor $L_{X,V}$ admits a unique maximal proper subfunctor $J_{X,V}$, and $S_{X,V} = L_{X,V}/J_{X,V}$ is a simple functor, such that $S_{X,V}(X) \cong V$.
- Conversely, if S is a simple functor such that $S(X) \neq 0$, then $V = S(X)$ is a simple $k\mathcal{C}(X,X)$-module, and $S \cong S_{X,V}$.
- If S is a simple functor, and if E is a set of minimal cardinality such that $S(E) \neq 0$, then $S(E)$ is a simple module for the algebra $\mathcal{E}_E$ of

  essential relations on E.
- The simple $\mathcal{E}_E$-modules are parametrized by pairs (R,W), where R is an order on E, and W is a simple $kAut(E,R)$-module.
- The simple correspondence functors over k are parametrized by triples (E,R,W), where E is a finite set

Simple functors

- Let X be a finite set, and V be a simple $k\mathcal{C}(X,X)$-module. Then the functor $L_{X,V}$ admits a unique maximal proper subfunctor $J_{X,V}$, and $S_{X,V} = L_{X,V}/J_{X,V}$ is a simple functor, such that $S_{X,V}(X) \cong V$.
- Conversely, if S is a simple functor such that $S(X) \neq 0$, then $V = S(X)$ is a simple $k\mathcal{C}(X,X)$-module, and $S \cong S_{X,V}$.
- If S is a simple functor, and if E is a set of minimal cardinality such that $S(E) \neq 0$, then $S(E)$ is a simple module for the algebra $\mathcal{E}_E$ of

  essential relations on E.
- The simple $\mathcal{E}_E$-modules are parametrized by pairs (R, W), where R is an order on E, and W is a simple $kAut(E,R)$-module.
- The simple correspondence functors over k are parametrized by triples (E, R, W), where E is a finite set, R is an order on E

Simple functors

- Let X be a finite set, and V be a simple $k\mathcal{C}(X,X)$-module. Then the functor $L_{X,V}$ admits a unique maximal proper subfunctor $J_{X,V}$, and $S_{X,V} = L_{X,V}/J_{X,V}$ is a simple functor, such that $S_{X,V}(X) \cong V$.
- Conversely, if S is a simple functor such that $S(X) \neq 0$, then $V = S(X)$ is a simple $k\mathcal{C}(X,X)$-module, and $S \cong S_{X,V}$.
- If S is a simple functor, and if E is a set of minimal cardinality such that $S(E) \neq 0$, then $S(E)$ is a simple module for the algebra $\mathcal{E}_E$ of

  essential relations on E.
- The simple $\mathcal{E}_E$-modules are parametrized by pairs (R, W), where R is an order on E, and W is a simple $kAut(E,R)$-module.
- The simple correspondence functors over k are parametrized by triples (E, R, W), where E is a finite set, R is an order on E, and W is a simple $kAut(E,R)$-module.

Simple functors

- Let X be a finite set, and V be a simple $k\mathcal{C}(X,X)$-module. Then the functor $L_{X,V}$ admits a unique maximal proper subfunctor $J_{X,V}$, and $S_{X,V} = L_{X,V}/J_{X,V}$ is a simple functor, such that $S_{X,V}(X) \cong V$.
- Conversely, if S is a simple functor such that $S(X) \neq 0$, then $V = S(X)$ is a simple $k\mathcal{C}(X,X)$-module, and $S \cong S_{X,V}$.
- If S is a simple functor, and if E is a set of minimal cardinality such that $S(E) \neq 0$, then $S(E)$ is a simple module for the algebra $\mathcal{E}_E$ of

  essential relations on E.
- The simple $\mathcal{E}_E$-modules are parametrized by pairs (R, W), where R is an order on E, and W is a simple $kAut(E,R)$-module.
- The simple correspondence functors over k are parametrized by triples (E, R, W), where E is a finite set, R is an order on E, and W is a simple $kAut(E,R)$-module. Notation: $(E,R,W) \mapsto S_{E,R,W}$.

Bounded generation

Bounded generation - Finite generation

Bounded generation - Finite generation

Definition

Let $M \in \mathcal{F}_k$.

Bounded generation - Finite generation

Definition

Let $M \in \mathcal{F}_k$.

1. Let $(E_i)_{i \in I}$ be a sequence of finite sets

Bounded generation - Finite generation

Definition

Let $M \in \mathcal{F}_k$.

1. Let $(E_i)_{i \in I}$ be a sequence of finite sets, and for each $i \in I$, let $m_i \in M(E_i)$.

Bounded generation - Finite generation

Definition

Let $M \in \mathcal{F}_k$.

1. Let $(E_i)_{i \in I}$ be a sequence of finite sets, and for each $i \in I$, let $m_i \in M(E_i)$. We say that M is generated by $(m_i)_{i \in I}$

Definition

Let $M \in \mathcal{F}_k$.

1. Let $(E_i)_{i \in I}$ be a sequence of finite sets, and for each $i \in I$, let $m_i \in M(E_i)$. We say that M is generated by $(m_i)_{i \in I}$, and write $M = \langle m_i \rangle_{i \in I}$

Bounded generation - Finite generation

Definition

Let $M \in \mathcal{F}_k$.

1. Let $(E_i)_{i \in I}$ be a sequence of finite sets, and for each $i \in I$, let $m_i \in M(E_i)$. We say that M is generated by $(m_i)_{i \in I}$, and write $M = \langle m_i \rangle_{i \in I}$, if for any finite set X and any $m \in M(X)$

Definition

Let $M \in \mathcal{F}_k$.

1. Let $(E_i)_{i\in I}$ be a sequence of finite sets, and for each $i \in I$, let $m_i \in M(E_i)$. We say that M is generated by $(m_i)_{i\in I}$, and write $M = \langle m_i \rangle_{i\in I}$, if for any finite set X and any $m \in M(X)$, there exists a finite subsequence $J \subseteq I$

Bounded generation - Finite generation

Definition

Let $M \in \mathcal{F}_k$.

1. Let $(E_i)_{i \in I}$ be a sequence of finite sets, and for each $i \in I$, let $m_i \in M(E_i)$. We say that M is generated by $(m_i)_{i \in I}$, and write $M = \langle m_i \rangle_{i \in I}$, if for any finite set X and any $m \in M(X)$, there exists a finite subsequence $J \subseteq I$ and elements $\alpha_j \in k\mathcal{C}(X, E_j)$, for $j \in J$

Bounded generation - Finite generation

Definition

Let $M \in \mathcal{F}_k$.

1. Let $(E_i)_{i\in I}$ be a sequence of finite sets, and for each $i \in I$, let $m_i \in M(E_i)$. We say that M is generated by $(m_i)_{i\in I}$, and write $M = \langle m_i \rangle_{i\in I}$, if for any finite set X and any $m \in M(X)$, there exists a finite subsequence $J \subseteq I$ and elements $\alpha_j \in k\mathcal{C}(X, E_j)$, for $j \in J$, such that $m = \sum_{j\in J} M(\alpha_j)(m_j)$.

Definition

Let $M \in \mathcal{F}_k$.

1. Let $(E_i)_{i\in I}$ be a sequence of finite sets, and for each $i \in I$, let $m_i \in M(E_i)$. We say that M is generated by $(m_i)_{i\in I}$, and write $M = \langle m_i \rangle_{i\in I}$, if for any finite set X and any $m \in M(X)$, there exists a finite subsequence $J \subseteq I$ and elements $\alpha_j \in k\mathcal{C}(X, E_j)$, for $j \in J$, such that $m = \sum_{j\in J} M(\alpha_j)(m_j)$.
2. M is finitely generated

Definition

Let $M \in \mathcal{F}_k$.

1. Let $(E_i)_{i\in I}$ be a sequence of finite sets, and for each $i \in I$, let $m_i \in M(E_i)$. We say that M is generated by $(m_i)_{i\in I}$, and write $M = \langle m_i\rangle_{i\in I}$, if for any finite set X and any $m \in M(X)$, there exists a finite subsequence $J \subseteq I$ and elements $\alpha_j \in k\mathcal{C}(X, E_j)$, for $j \in J$, such that $m = \sum_{j\in J} M(\alpha_j)(m_j)$.
2. M is finitely generated if $M = \langle m_i\rangle_{i\in I}$, where I is finite.

Bounded generation - Finite generation

Definition

Let $M \in \mathcal{F}_k$.

1. Let $(E_i)_{i\in I}$ be a sequence of finite sets, and for each $i \in I$, let $m_i \in M(E_i)$. We say that M is generated by $(m_i)_{i\in I}$, and write $M = \langle m_i \rangle_{i\in I}$, if for any finite set X and any $m \in M(X)$, there exists a finite subsequence $J \subseteq I$ and elements $\alpha_j \in k\mathcal{C}(X, E_j)$, for $j \in J$, such that $m = \sum_{j\in J} M(\alpha_j)(m_j)$.
2. M is finitely generated if $M = \langle m_i \rangle_{i\in I}$, where I is finite.
3. M has bounded type

Bounded generation - Finite generation

Definition

Let $M \in \mathcal{F}_k$.

1. Let $(E_i)_{i\in I}$ be a sequence of finite sets, and for each $i \in I$, let $m_i \in M(E_i)$. We say that M is generated by $(m_i)_{i\in I}$, and write $M = \langle m_i \rangle_{i\in I}$, if for any finite set X and any $m \in M(X)$, there exists a finite subsequence $J \subseteq I$ and elements $\alpha_j \in k\mathcal{C}(X, E_j)$, for $j \in J$, such that $m = \sum_{j\in J} M(\alpha_j)(m_j)$.
2. M is finitely generated if $M = \langle m_i \rangle_{i\in I}$, where I is finite.
3. M has bounded type if there is a finite set E such that $M = \langle M(E) \rangle$.

Bounded generation - Finite generation

Definition

Let $M \in \mathcal{F}_k$.

1. Let $(E_i)_{i\in I}$ be a sequence of finite sets, and for each $i \in I$, let $m_i \in M(E_i)$. We say that M is generated by $(m_i)_{i\in I}$, and write $M = \langle m_i \rangle_{i\in I}$, if for any finite set X and any $m \in M(X)$, there exists a finite subsequence $J \subseteq I$ and elements $\alpha_j \in k\mathcal{C}(X, E_j)$, for $j \in J$, such that $m = \sum\limits_{j\in J} M(\alpha_j)(m_j)$.
2. M is finitely generated if $M = \langle m_i \rangle_{i\in I}$, where I is finite.
3. M has bounded type if there is a finite set E such that $M = \langle M(E) \rangle$.

Example: The Yoneda functor $Y_{E,k}$ is finitely generated

Bounded generation - Finite generation

Definition

Let $M \in \mathcal{F}_k$.

1. Let $(E_i)_{i \in I}$ be a sequence of finite sets, and for each $i \in I$, let $m_i \in M(E_i)$. We say that M is generated by $(m_i)_{i \in I}$, and write $M = \langle m_i \rangle_{i \in I}$, if for any finite set X and any $m \in M(X)$, there exists a finite subsequence $J \subseteq I$ and elements $\alpha_j \in k\mathcal{C}(X, E_j)$, for $j \in J$, such that $m = \sum_{j \in J} M(\alpha_j)(m_j)$.
2. M is finitely generated if $M = \langle m_i \rangle_{i \in I}$, where I is finite.
3. M has bounded type if there is a finite set E such that $M = \langle M(E) \rangle$.

Example: The Yoneda functor $Y_{E,k}$ is finitely generated (by the single element $\Delta_E \in Y_{E,k}(E) = k\mathcal{C}(E, E)$).

Bounded generation - Finite generation

Definition

Let $M \in \mathcal{F}_k$.

1. Let $(E_i)_{i\in I}$ be a sequence of finite sets, and for each $i \in I$, let $m_i \in M(E_i)$. We say that M is generated by $(m_i)_{i\in I}$, and write $M = \langle m_i \rangle_{i\in I}$, if for any finite set X and any $m \in M(X)$, there exists a finite subsequence $J \subseteq I$ and elements $\alpha_j \in k\mathcal{C}(X, E_j)$, for $j \in J$, such that $m = \sum_{j\in J} M(\alpha_j)(m_j)$.
2. M is finitely generated if $M = \langle m_i \rangle_{i\in I}$, where I is finite.
3. M has bounded type if there is a finite set E such that $M = \langle M(E) \rangle$.

Example: The Yoneda functor $Y_{E,k}$ is finitely generated (by the single element $\Delta_E \in Y_{E,k}(E) = k\mathcal{C}(E, E)$).
The functor $L_{E,V}$ has bounded type

Bounded generation - Finite generation

Definition

Let $M \in \mathcal{F}_k$.

1. Let $(E_i)_{i\in I}$ be a sequence of finite sets, and for each $i \in I$, let $m_i \in M(E_i)$. We say that M is generated by $(m_i)_{i\in I}$, and write $M = \langle m_i \rangle_{i\in I}$, if for any finite set X and any $m \in M(X)$, there exists a finite subsequence $J \subseteq I$ and elements $\alpha_j \in k\mathcal{C}(X, E_j)$, for $j \in J$, such that $m = \sum_{j\in J} M(\alpha_j)(m_j)$.
2. M is finitely generated if $M = \langle m_i \rangle_{i\in I}$, where I is finite.
3. M has bounded type if there is a finite set E such that $M = \langle M(E) \rangle$.

Example: The Yoneda functor $Y_{E,k}$ is finitely generated (by the single element $\Delta_E \in Y_{E,k}(E) = k\mathcal{C}(E, E)$).
The functor $L_{E,V}$ has bounded type, generated by $L_{E,V}(E) \cong V$.

Bounded generation - Finite generation

Definition

Let $M \in \mathcal{F}_k$.

1. Let $(E_i)_{i \in I}$ be a sequence of finite sets, and for each $i \in I$, let $m_i \in M(E_i)$. We say that M is generated by $(m_i)_{i \in I}$, and write $M = \langle m_i \rangle_{i \in I}$, if for any finite set X and any $m \in M(X)$, there exists a finite subsequence $J \subseteq I$ and elements $\alpha_j \in k\mathcal{C}(X, E_j)$, for $j \in J$, such that $m = \sum\limits_{j \in J} M(\alpha_j)(m_j)$.
2. M is finitely generated if $M = \langle m_i \rangle_{i \in I}$, where I is finite.
3. M has bounded type if there is a finite set E such that $M = \langle M(E) \rangle$.

Example: The Yoneda functor $Y_{E,k}$ is finitely generated (by the single element $\Delta_E \in Y_{E,k}(E) = k\mathcal{C}(E, E)$).
The functor $L_{E,V}$ has bounded type, generated by $L_{E,V}(E) \cong V$. It is finitely generated

Bounded generation - Finite generation

Definition

Let $M \in \mathcal{F}_k$.

1. Let $(E_i)_{i\in I}$ be a sequence of finite sets, and for each $i \in I$, let $m_i \in M(E_i)$. We say that M is generated by $(m_i)_{i\in I}$, and write $M = \langle m_i \rangle_{i\in I}$, if for any finite set X and any $m \in M(X)$, there exists a finite subsequence $J \subseteq I$ and elements $\alpha_j \in k\mathcal{C}(X, E_j)$, for $j \in J$, such that $m = \sum_{j\in J} M(\alpha_j)(m_j)$.
2. M is finitely generated if $M = \langle m_i \rangle_{i\in I}$, where I is finite.
3. M has bounded type if there is a finite set E such that $M = \langle M(E) \rangle$.

Example: The Yoneda functor $Y_{E,k}$ is finitely generated (by the single element $\Delta_E \in Y_{E,k}(E) = k\mathcal{C}(E, E)$).
The functor $L_{E,V}$ has bounded type, generated by $L_{E,V}(E) \cong V$. It is finitely generated if and only if V is finitely generated.

Finite generation

Finite generation

Theorem

Let $M \in \mathcal{F}_k$.

Finite generation

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

Finite generation

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*

Finite generation

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*
2. *M is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^{n} Y_{E_i,k}$.*

Finite generation

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*
2. *M is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^{n} Y_{E_i,k}$.*
3. *M is isomorphic to a quotient of a finite direct sum $(Y_{E,k})^{\oplus n}$.*

Finite generation

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*
2. *M is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^{n} Y_{E_i,k}$.*
3. *M is isomorphic to a quotient of a finite direct sum $(Y_{E,k})^{\oplus n}$.*

If moreover k is a field, these conditions are equivalent to:

Finite generation

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*
2. *M is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^{n} Y_{E_i,k}$.*
3. *M is isomorphic to a quotient of a finite direct sum $(Y_{E,k})^{\oplus n}$.*

If moreover k is a field, these conditions are equivalent to:

4. *there exist positive real numbers a, b, r such that $\dim_k M(X) \leq ab^{|X|}$ for any finite set X with $|X| \geq r$.*

Finite generation

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*
2. *M is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^{n} Y_{E_i,k}$.*
3. *M is isomorphic to a quotient of a finite direct sum $(Y_{E,k})^{\oplus n}$.*

If moreover k is a field, these conditions are equivalent to:

4. *there exist positive real numbers a, b, r such that $\dim_k M(X) \leq ab^{|X|}$ for any finite set X with $|X| \geq r$.*
5. *M has finite length.*

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*
2. *M is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^{n} Y_{E_i,k}$.*
3. *M is isomorphic to a quotient of a finite direct sum $(Y_{E,k})^{\oplus n}$.*

If moreover k is a field, these conditions are equivalent to:

4. *there exist positive real numbers a, b, r such that $\dim_k M(X) \leq ab^{|X|}$ for any finite set X with $|X| \geq r$.*
5. *M has finite length.*

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*
2. *M is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^{n} Y_{E_i,k}$.*
3. *M is isomorphic to a quotient of a finite direct sum $(Y_{E,k})^{\oplus n}$.*

If moreover k is a field, these conditions are equivalent to:

4. *there exist positive real numbers a, b, r such that $\dim_k M(X) \leq ab^{|X|}$ for any finite set X with $|X| \geq r$.*
5. *M has finite length.*

Proof (sketch):

Theorem

Let $M \in \mathcal{F}_k$*. The following are equivalent:*

1. M *is finitely generated.*
2. M *is isomorphic to a quotient of a finite direct sum* $\bigoplus_{i=1}^{n} Y_{E_i,k}$.
3. M *is isomorphic to a quotient of a finite direct sum* $(Y_{E,k})^{\oplus n}$.

If moreover k *is a field, these conditions are equivalent to:*

4. *there exist positive real numbers* a, b, r *such that* $\dim_k M(X) \leq ab^{|X|}$ *for any finite set* X *with* $|X| \geq r$.
5. M *has finite length.*

Proof (sketch): $\bullet$ $1 \Leftrightarrow 2$

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*
2. *M is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^{n} Y_{E_i,k}$.*
3. *M is isomorphic to a quotient of a finite direct sum $(Y_{E,k})^{\oplus n}$.*

If moreover k is a field, these conditions are equivalent to:

4. *there exist positive real numbers a, b, r such that $\dim_k M(X) \leq ab^{|X|}$ for any finite set X with $|X| \geq r$.*
5. *M has finite length.*

Proof (sketch): • $1 \Leftrightarrow 2 \Leftrightarrow 3$

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*
2. *M is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^{n} Y_{E_i,k}$.*
3. *M is isomorphic to a quotient of a finite direct sum $(Y_{E,k})^{\oplus n}$.*

If moreover k is a field, these conditions are equivalent to:

4. *there exist positive real numbers a, b, r such that $\dim_k M(X) \leq ab^{|X|}$ for any finite set X with $|X| \geq r$.*
5. *M has finite length.*

Proof (sketch): • $1 \Leftrightarrow 2 \Leftrightarrow 3 \Rightarrow 4$ are easy

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*
2. *M is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^{n} Y_{E_i,k}$.*
3. *M is isomorphic to a quotient of a finite direct sum $(Y_{E,k})^{\oplus n}$.*

If moreover k is a field, these conditions are equivalent to:

4. *there exist positive real numbers a, b, r such that $\dim_k M(X) \leq ab^{|X|}$ for any finite set X with $|X| \geq r$.*
5. *M has finite length.*

Proof (sketch): • $1 \Leftrightarrow 2 \Leftrightarrow 3 \Rightarrow 4$ are easy, as $dim_k Y_{E,k}(X) = (2^{|E|})^{|X|}$.

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*
2. *M is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^{n} Y_{E_i,k}$.*
3. *M is isomorphic to a quotient of a finite direct sum $(Y_{E,k})^{\oplus n}$.*

If moreover k is a field, these conditions are equivalent to:

4. *there exist positive real numbers a, b, r such that $\dim_k M(X) \leq ab^{|X|}$ for any finite set X with $|X| \geq r$.*
5. *M has finite length.*

Proof (sketch): • $1 \Leftrightarrow 2 \Leftrightarrow 3 \Rightarrow 4$ are easy, as $dim_k Y_{E,k}(X) = (2^{|E|})^{|X|}$.
• Let $S_{E,R,W}$ be a simple correspondence functor,

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*
2. *M is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^{n} Y_{E_i,k}$.*
3. *M is isomorphic to a quotient of a finite direct sum $(Y_{E,k})^{\oplus n}$.*

If moreover k is a field, these conditions are equivalent to:

4. *there exist positive real numbers a, b, r such that $\dim_k M(X) \leq ab^{|X|}$ for any finite set X with $|X| \geq r$.*
5. *M has finite length.*

Proof (sketch): • $1 \Leftrightarrow 2 \Leftrightarrow 3 \Rightarrow 4$ are easy, as $dim_k Y_{E,k}(X) = (2^{|E|})^{|X|}$.
• Let $S_{E,R,W}$ be a simple correspondence functor, and $V = S_{E,R,W}(E)$.

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*
2. *M is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^{n} Y_{E_i,k}$.*
3. *M is isomorphic to a quotient of a finite direct sum $(Y_{E,k})^{\oplus n}$.*

If moreover k is a field, these conditions are equivalent to:

4. *there exist positive real numbers a, b, r such that $\dim_k M(X) \leq ab^{|X|}$ for any finite set X with $|X| \geq r$.*
5. *M has finite length.*

Proof (sketch): • $1 \Leftrightarrow 2 \Leftrightarrow 3 \Rightarrow 4$ are easy, as $dim_k Y_{E,k}(X) = (2^{|E|})^{|X|}$.

• Let $S_{E,R,W}$ be a simple correspondence functor, and $V = S_{E,R,W}(E)$.

- There exists $c > 0$ and $N \in \mathbb{N}$ such that

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*
2. *M is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^{n} Y_{E_i,k}$.*
3. *M is isomorphic to a quotient of a finite direct sum $(Y_{E,k})^{\oplus n}$.*

If moreover k is a field, these conditions are equivalent to:

4. *there exist positive real numbers a, b, r such that $\dim_k M(X) \leq ab^{|X|}$ for any finite set X with $|X| \geq r$.*
5. *M has finite length.*

Proof (sketch): • $1 \Leftrightarrow 2 \Leftrightarrow 3 \Rightarrow 4$ are easy, as $dim_k Y_{E,k}(X) = (2^{|E|})^{|X|}$.

• Let $S_{E,R,W}$ be a simple correspondence functor, and $V = S_{E,R,W}(E)$.

- There exists $c > 0$ and $N \in \mathbb{N}$ such that $c|E|^{|X|} \leq dim_k S_{E,R,W}(X) \leq (2^{|E|})^{|X|}$ if $|X| \geq N$.

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*
2. *M is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^{n} Y_{E_i,k}$.*
3. *M is isomorphic to a quotient of a finite direct sum $(Y_{E,k})^{\oplus n}$.*

If moreover k is a field, these conditions are equivalent to:

4. *there exist positive real numbers a, b, r such that $\dim_k M(X) \leq ab^{|X|}$ for any finite set X with $|X| \geq r$.*
5. *M has finite length.*

Proof (sketch): • $1 \Leftrightarrow 2 \Leftrightarrow 3 \Rightarrow 4$ are easy, as $dim_k Y_{E,k}(X) = (2^{|E|})^{|X|}$.

• Let $S_{E,R,W}$ be a simple correspondence functor, and $V = S_{E,R,W}(E)$.

- There exists $c > 0$ and $N \in \mathbb{N}$ such that $c|E|^{|X|} \leq dim_k S_{E,R,W}(X) \leq (2^{|E|})^{|X|}$ if $|X| \geq N$. Hence $5 \Rightarrow 4$.

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*
2. *M is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^{n} Y_{E_i,k}$.*
3. *M is isomorphic to a quotient of a finite direct sum $(Y_{E,k})^{\oplus n}$.*

If moreover k is a field, these conditions are equivalent to:

4. *there exist positive real numbers a, b, r such that $\dim_k M(X) \leq ab^{|X|}$ for any finite set X with $|X| \geq r$.*
5. *M has finite length.*

Proof (sketch): • $1 \Leftrightarrow 2 \Leftrightarrow 3 \Rightarrow 4$ are easy, as $dim_k Y_{E,k}(X) = (2^{|E|})^{|X|}$.

• Let $S_{E,R,W}$ be a simple correspondence functor, and $V = S_{E,R,W}(E)$.

- There exists $c > 0$ and $N \in \mathbb{N}$ such that $c|E|^{|X|} \leq dim_k S_{E,R,W}(X) \leq (2^{|E|})^{|X|}$ if $|X| \geq N$. Hence $5 \Rightarrow 4$.
- If $F \in \mathcal{F}_k$

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*
2. *M is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^{n} Y_{E_i,k}$.*
3. *M is isomorphic to a quotient of a finite direct sum $(Y_{E,k})^{\oplus n}$.*

If moreover k is a field, these conditions are equivalent to:

4. *there exist positive real numbers a, b, r such that $\dim_k M(X) \leq ab^{|X|}$ for any finite set X with $|X| \geq r$.*
5. *M has finite length.*

Proof (sketch): • $1 \Leftrightarrow 2 \Leftrightarrow 3 \Rightarrow 4$ are easy, as $dim_k Y_{E,k}(X) = (2^{|E|})^{|X|}$.

- Let $S_{E,R,W}$ be a simple correspondence functor, and $V = S_{E,R,W}(E)$.
  - There exists $c > 0$ and $N \in \mathbb{N}$ such that $c|E|^{|X|} \leq dim_k S_{E,R,W}(X) \leq (2^{|E|})^{|X|}$ if $|X| \geq N$. Hence $5 \Rightarrow 4$.
  - If $F \in \mathcal{F}_k$, then V is a subquotient of $F(E)$

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*
2. *M is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^{n} Y_{E_i,k}$.*
3. *M is isomorphic to a quotient of a finite direct sum $(Y_{E,k})^{\oplus n}$.*

If moreover k is a field, these conditions are equivalent to:

4. *there exist positive real numbers a, b, r such that $\dim_k M(X) \leq ab^{|X|}$ for any finite set X with $|X| \geq r$.*
5. *M has finite length.*

Proof (sketch): • $1 \Leftrightarrow 2 \Leftrightarrow 3 \Rightarrow 4$ are easy, as $dim_k Y_{E,k}(X) = (2^{|E|})^{|X|}$.

- Let $S_{E,R,W}$ be a simple correspondence functor, and $V = S_{E,R,W}(E)$.
  - There exists $c > 0$ and $N \in \mathbb{N}$ such that $c|E|^{|X|} \leq dim_k S_{E,R,W}(X) \leq (2^{|E|})^{|X|}$ if $|X| \geq N$. Hence $5 \Rightarrow 4$.
  - If $F \in \mathcal{F}_k$, then V is a subquotient of $F(E)$ if and only if $S_{E,R,W}$ is a subquotient of F.

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*
2. *M is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^{n} Y_{E_i,k}$.*
3. *M is isomorphic to a quotient of a finite direct sum $(Y_{E,k})^{\oplus n}$.*

If moreover k is a field, these conditions are equivalent to:

4. *there exist positive real numbers a, b, r such that $\dim_k M(X) \leq ab^{|X|}$ for any finite set X with $|X| \geq r$.*
5. *M has finite length.*

Proof (sketch): • $1 \Leftrightarrow 2 \Leftrightarrow 3 \Rightarrow 4$ are easy, as $dim_k Y_{E,k}(X) = (2^{|E|})^{|X|}$.

• Let $S_{E,R,W}$ be a simple correspondence functor, and $V = S_{E,R,W}(E)$.

- There exists $c > 0$ and $N \in \mathbb{N}$ such that $c|E|^{|X|} \leq dim_k S_{E,R,W}(X) \leq (2^{|E|})^{|X|}$ if $|X| \geq N$. Hence $5 \Rightarrow 4$.
- If $F \in \mathcal{F}_k$, then V is a subquotient of $F(E)$ if and only if $S_{E,R,W}$ is a subquotient of F. Hence (.../...)

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*
2. *M is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^{n} Y_{E_i,k}$.*
3. *M is isomorphic to a quotient of a finite direct sum $(Y_{E,k})^{\oplus n}$.*

If moreover k is a field, these conditions are equivalent to:

4. *there exist positive real numbers a, b, r such that $\dim_k M(X) \leq ab^{|X|}$ for any finite set X with $|X| \geq r$.*
5. *M has finite length.*

Proof (sketch): • $1 \Leftrightarrow 2 \Leftrightarrow 3 \Rightarrow 4$ are easy, as $dim_k Y_{E,k}(X) = (2^{|E|})^{|X|}$.

• Let $S_{E,R,W}$ be a simple correspondence functor, and $V = S_{E,R,W}(E)$.

- There exists $c > 0$ and $N \in \mathbb{N}$ such that $c|E|^{|X|} \leq dim_k S_{E,R,W}(X) \leq (2^{|E|})^{|X|}$ if $|X| \geq N$. Hence $5 \Rightarrow 4$.
- If $F \in \mathcal{F}_k$, then V is a subquotient of $F(E)$ if and only if $S_{E,R,W}$ is a subquotient of F. Hence $(\ldots/\ldots)$ $4 \Rightarrow 1$

Theorem

Let $M \in \mathcal{F}_k$. The following are equivalent:

1. *M is finitely generated.*
2. *M is isomorphic to a quotient of a finite direct sum $\bigoplus_{i=1}^{n} Y_{E_i,k}$.*
3. *M is isomorphic to a quotient of a finite direct sum $(Y_{E,k})^{\oplus n}$.*

If moreover k is a field, these conditions are equivalent to:

4. *there exist positive real numbers a, b, r such that $\dim_k M(X) \leq ab^{|X|}$ for any finite set X with $|X| \geq r$.*
5. *M has finite length.*

Proof (sketch): • $1 \Leftrightarrow 2 \Leftrightarrow 3 \Rightarrow 4$ are easy, as $dim_k Y_{E,k}(X) = (2^{|E|})^{|X|}$.

• Let $S_{E,R,W}$ be a simple correspondence functor, and $V = S_{E,R,W}(E)$.

- There exists $c > 0$ and $N \in \mathbb{N}$ such that $c|E|^{|X|} \leq dim_k S_{E,R,W}(X) \leq (2^{|E|})^{|X|}$ if $|X| \geq N$. Hence $5 \Rightarrow 4$.
- If $F \in \mathcal{F}_k$, then V is a subquotient of $F(E)$ if and only if $S_{E,R,W}$ is a subquotient of F. Hence $(\ldots/\ldots)$ $4 \Rightarrow 1$ and $4 \Rightarrow 5$.

The noetherian case

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set.

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set. Define

$$\overline{M}(E) = M(E)/$$

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F|<|E|} k\mathcal{C}(E,F)M(F).$$

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F|<|E|} k\mathcal{C}(E,F)M(F).$$

Theorem

Let k be a noetherian (commutative) ring

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F|<|E|} k\mathcal{C}(E,F)M(F).$$

Theorem

Let k be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F|<|E|} k\mathcal{C}(E,F)M(F).$$

Theorem

Let k be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let E and F be finite sets.

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F|<|E|} k\mathcal{C}(E,F)M(F).$$

Theorem

Let k be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let E and F be finite sets.

1. *If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$*

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F|<|E|} k\mathcal{C}(E,F)M(F).$$

Theorem

Let k be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let E and F be finite sets.

1. *If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.*

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F|<|E|} k\mathcal{C}(E,F)M(F).$$

Theorem

Let k be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let E and F be finite sets.

1. *If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.*
2. *If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$*

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F|<|E|} k\mathcal{C}(E,F)M(F).$$

Theorem

Let k be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let E and F be finite sets.

1. *If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.*
2. *If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$, then $M = \langle M(E) \rangle$.*

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F|<|E|} k\mathcal{C}(E,F)M(F).$$

Theorem

Let k be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let E and F be finite sets.

1. *If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.*
2. *If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$, then $M = \langle M(E) \rangle$.*
3. *If L has bounded type*

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F|<|E|} k\mathcal{C}(E,F)M(F).$$

Theorem

Let k be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let E and F be finite sets.

1. *If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.*
2. *If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$, then $M = \langle M(E) \rangle$.*
3. *If L has bounded type, then M has bounded type.*

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F|<|E|} k\mathcal{C}(E,F)M(F).$$

Theorem

Let k be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let E and F be finite sets.

1. *If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.*
2. *If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$, then $M = \langle M(E) \rangle$.*
3. *If L has bounded type, then M has bounded type.*
4. *If L is finitely generated*

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F|<|E|} k\mathcal{C}(E,F)M(F).$$

Theorem

Let k be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let E and F be finite sets.

1. *If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.*
2. *If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$, then $M = \langle M(E) \rangle$.*
3. *If L has bounded type, then M has bounded type.*
4. *If L is finitely generated, then M is finitely generated.*

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F|<|E|} k\mathcal{C}(E,F)M(F).$$

Theorem

Let k be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let E and F be finite sets.

1. *If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.*
2. *If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$, then $M = \langle M(E) \rangle$.*
3. *If L has bounded type, then M has bounded type.*
4. *If L is finitely generated, then M is finitely generated.*

Proof (sketch):

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F|<|E|} k\mathcal{C}(E,F)M(F).$$

Theorem

Let k be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let E and F be finite sets.

1. *If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.*
2. *If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$, then $M = \langle M(E) \rangle$.*
3. *If L has bounded type, then M has bounded type.*
4. *If L is finitely generated, then M is finitely generated.*

Proof (sketch): Assertion 1 by localization

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F|<|E|} k\mathcal{C}(E,F)M(F).$$

Theorem

Let k be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let E and F be finite sets.

1. *If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.*
2. *If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$, then $M = \langle M(E) \rangle$.*
3. *If L has bounded type, then M has bounded type.*
4. *If L is finitely generated, then M is finitely generated.*

Proof (sketch): Assertion 1 by localization + Artin-Rees lemma.

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F|<|E|} k\mathcal{C}(E,F)M(F).$$

Theorem

Let k be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let E and F be finite sets.

1. *If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.*
2. *If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$, then $M = \langle M(E) \rangle$.*
3. *If L has bounded type, then M has bounded type.*
4. *If L is finitely generated, then M is finitely generated.*

Proof (sketch): Assertion 1 by localization + Artin-Rees lemma. Then 1 $\Rightarrow$ 2 $\Rightarrow$ 3, 4 easy.

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F|<|E|} k\mathcal{C}(E,F)M(F).$$

Theorem

Let k be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let E and F be finite sets.

1. *If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.*
2. *If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$, then $M = \langle M(E) \rangle$.*
3. *If L has bounded type, then M has bounded type.*
4. *If L is finitely generated, then M is finitely generated.*

Corollary

Functors of bounded type

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F|<|E|} k\mathcal{C}(E,F)M(F).$$

Theorem

Let k be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let E and F be finite sets.

1. *If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.*
2. *If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$, then $M = \langle M(E) \rangle$.*
3. *If L has bounded type, then M has bounded type.*
4. *If L is finitely generated, then M is finitely generated.*

Corollary

Functors of bounded type form an abelian subcategory $\mathcal{F}_k^b$ of $\mathcal{F}_k$.

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F|<|E|} k\mathcal{C}(E,F)M(F).$$

Theorem

Let k be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let E and F be finite sets.

1. *If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.*
2. *If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$, then $M = \langle M(E) \rangle$.*
3. *If L has bounded type, then M has bounded type.*
4. *If L is finitely generated, then M is finitely generated.*

Corollary

Functors of bounded type form an abelian subcategory $\mathcal{F}_k^b$ of $\mathcal{F}_k$. Finitely generated functors

The noetherian case

Let $M \in \mathcal{F}_k$ and E be a finite set. Define

$$\overline{M}(E) = M(E) / \sum_{|F|<|E|} k\mathcal{C}(E,F)M(F).$$

Theorem

Let k be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let E and F be finite sets.

1. *If $L = \langle L(F) \rangle$ and $\overline{M}(E) \neq 0$, then $|E| \leq 2^{|F|}$.*
2. *If $L = \langle L(F) \rangle$ and $|E| \geq 2^{|F|}$, then $M = \langle M(E) \rangle$.*
3. *If L has bounded type, then M has bounded type.*
4. *If L is finitely generated, then M is finitely generated.*

Corollary

Functors of bounded type form an abelian subcategory $\mathcal{F}_k^b$ of $\mathcal{F}_k$. Finitely generated functors form an abelian subcategory $\mathcal{F}_k^f$ of $\mathcal{F}_k^b$.

The noetherian case

The noetherian case

Proposition

Let k be a noetherian ring.

The noetherian case

Proposition

Let k be a noetherian ring.

1. *If $M \in \mathcal{F}_k^f$*

The noetherian case

Proposition

Let k be a noetherian ring.

1. *If $M \in \mathcal{F}_k^f$, then $End_{\mathcal{F}_k}(M)$ is a finitely generated k-module.*

The noetherian case

Proposition

Let k be a noetherian ring.

1. *If $M \in \mathcal{F}_k^f$, then $End_{\mathcal{F}_k}(M)$ is a finitely generated k-module.*
2. *For any $M, N \in \mathcal{F}_k^f$*

The noetherian case

Proposition

Let k be a noetherian ring.

1. *If $M \in \mathcal{F}_k^f$, then $End_{\mathcal{F}_k}(M)$ is a finitely generated k-module.*
2. *For any $M, N \in \mathcal{F}_k^f$, the k-module $Hom_{\mathcal{F}_k}(M, N)$ is finitely generated.*

The noetherian case

Proposition

Let k be a noetherian ring.

1. *If $M \in \mathcal{F}_k^f$, then $End_{\mathcal{F}_k}(M)$ is a finitely generated k-module.*
2. *For any $M, N \in \mathcal{F}_k^f$, the k-module $Hom_{\mathcal{F}_k}(M, N)$ is finitely generated.*
3. *If k is a field, then the Krull-Remak-Schmidt theorem holds for finitely generated correspondence functors over k.*

The noetherian case

Proposition

Let k be a noetherian ring.

1. *If* $M \in \mathcal{F}_k^f$, *then* $End_{\mathcal{F}_k}(M)$ *is a finitely generated* k*-module.*
2. *For any* $M, N \in \mathcal{F}_k^f$, *the* k*-module* $Hom_{\mathcal{F}_k}(M, N)$ *is finitely generated.*
3. *If* k *is a field, then the Krull-Remak-Schmidt theorem holds for finitely generated correspondence functors over* k*.*

Proof: 1) M is a quotient of a projective functor $\bigoplus_{i=1}^{n} k\mathcal{C}(-, E)$

The noetherian case

Proposition

Let k be a noetherian ring.

1. *If $M \in \mathcal{F}_k^f$, then $End_{\mathcal{F}_k}(M)$ is a finitely generated k-module.*
2. *For any $M, N \in \mathcal{F}_k^f$, the k-module $Hom_{\mathcal{F}_k}(M, N)$ is finitely generated.*
3. *If k is a field, then the Krull-Remak-Schmidt theorem holds for finitely generated correspondence functors over k.*

Proof: 1) M is a quotient of a projective functor $\bigoplus_{i=1}^{n} k\mathcal{C}(-, E)$, so $End_{\mathcal{F}_k}(M)$ is a quotient of a k-submodule of the finitely generated k-module $M_n\big(k\mathcal{C}(E, E)\big)$.

The noetherian case

Proposition

Let k be a noetherian ring.

1. *If $M \in \mathcal{F}_k^f$, then $End_{\mathcal{F}_k}(M)$ is a finitely generated k-module.*
2. *For any $M, N \in \mathcal{F}_k^f$, the k-module $Hom_{\mathcal{F}_k}(M, N)$ is finitely generated.*
3. *If k is a field, then the Krull-Remak-Schmidt theorem holds for finitely generated correspondence functors over k.*

Proof: 1) M is a quotient of a projective functor $\bigoplus_{i=1}^{n} k\mathcal{C}(-, E)$, so $End_{\mathcal{F}_k}(M)$ is a quotient of a k-submodule of the finitely generated k-module $M_n\big(k\mathcal{C}(E, E)\big)$.
2) $Hom_{\mathcal{F}_k}(M, N)$ is a direct summand of $End_{\mathcal{F}_k}(M \oplus N)$.

The noetherian case

Proposition

Let k be a noetherian ring.

1. *If $M \in \mathcal{F}_k^f$, then $End_{\mathcal{F}_k}(M)$ is a finitely generated k-module.*
2. *For any $M, N \in \mathcal{F}_k^f$, the k-module $Hom_{\mathcal{F}_k}(M, N)$ is finitely generated.*
3. *If k is a field, then the Krull-Remak-Schmidt theorem holds for finitely generated correspondence functors over k.*

Proof: 1) M is a quotient of a projective functor $\bigoplus_{i=1}^{n} k\mathcal{C}(-, E)$, so $End_{\mathcal{F}_k}(M)$ is a quotient of a k-submodule of the finitely generated k-module $M_n\big(k\mathcal{C}(E, E)\big)$.
2) $Hom_{\mathcal{F}_k}(M, N)$ is a direct summand of $End_{\mathcal{F}_k}(M \oplus N)$.
3) Splitting $M \in \mathcal{F}_k$

The noetherian case

Proposition

Let k be a noetherian ring.

1. *If $M \in \mathcal{F}_k^f$, then $End_{\mathcal{F}_k}(M)$ is a finitely generated k-module.*
2. *For any $M, N \in \mathcal{F}_k^f$, the k-module $Hom_{\mathcal{F}_k}(M, N)$ is finitely generated.*
3. *If k is a field, then the Krull-Remak-Schmidt theorem holds for finitely generated correspondence functors over k.*

Proof: 1) M is a quotient of a projective functor $\bigoplus_{i=1}^{n} k\mathcal{C}(-, E)$, so $End_{\mathcal{F}_k}(M)$ is a quotient of a k-submodule of the finitely generated k-module $M_n\big(k\mathcal{C}(E, E)\big)$.
2) $Hom_{\mathcal{F}_k}(M, N)$ is a direct summand of $End_{\mathcal{F}_k}(M \oplus N)$.
3) Splitting $M \in \mathcal{F}_k$ amounts to splitting the identity as a sum of orthogonal idempotents

The noetherian case

Proposition

Let k be a noetherian ring.

1. *If $M \in \mathcal{F}_k^f$, then $End_{\mathcal{F}_k}(M)$ is a finitely generated k-module.*
2. *For any $M, N \in \mathcal{F}_k^f$, the k-module $Hom_{\mathcal{F}_k}(M, N)$ is finitely generated.*
3. *If k is a field, then the Krull-Remak-Schmidt theorem holds for finitely generated correspondence functors over k.*

Proof: 1) M is a quotient of a projective functor $\bigoplus_{i=1}^{n} k\mathcal{C}(-, E)$, so $End_{\mathcal{F}_k}(M)$ is a quotient of a k-submodule of the finitely generated k-module $M_n\big(k\mathcal{C}(E, E)\big)$.
2) $Hom_{\mathcal{F}_k}(M, N)$ is a direct summand of $End_{\mathcal{F}_k}(M \oplus N)$.
3) Splitting $M \in \mathcal{F}_k$ amounts to splitting the identity as a sum of orthogonal idempotents in the finite dimensional k-algebra $End_{\mathcal{F}_k}(M)$.

Evaluation - Adjunction

Evaluation - Adjunction

- Let E be a finite set

Evaluation - Adjunction

- Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E, E)$.

Evaluation - Adjunction

- Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Recall that the evaluation functor

$$M \in \mathcal{F}_k \mapsto M(E) \in \mathcal{R}_E\text{-Mod}$$

Evaluation - Adjunction

- Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Recall that the evaluation functor
$$M \in \mathcal{F}_k \mapsto M(E) \in \mathcal{R}_E\text{-Mod}$$
has a left adjoint $V \mapsto L_{E,V}$

Evaluation - Adjunction

- Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Recall that the evaluation functor
$$M \in \mathcal{F}_k \mapsto M(E) \in \mathcal{R}_E\text{-Mod}$$
has a left adjoint $V \mapsto L_{E,V}$, defined by
$$X \mapsto L_{E,V}(X) := k\mathcal{C}(X,E) \otimes_{\mathcal{R}_E} V.$$

- Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E, E)$. Recall that the evaluation functor
$$M \in \mathcal{F}_k \mapsto M(E) \in \mathcal{R}_E\text{-Mod}$$
has a left adjoint $V \mapsto L_{E,V}$, defined by
$$X \mapsto L_{E,V}(X) := k\mathcal{C}(X, E) \otimes_{\mathcal{R}_E} V.$$
In particular $L_{E,V}(E) \cong V$.

Evaluation - Adjunction

- Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Recall that the evaluation functor
$$M \in \mathcal{F}_k \mapsto M(E) \in \mathcal{R}_E\text{-Mod}$$
has a left adjoint $V \mapsto L_{E,V}$, defined by
$$X \mapsto L_{E,V}(X) := k\mathcal{C}(X,E) \otimes_{\mathcal{R}_E} V.$$
In particular $L_{E,V}(E) \cong V$.
- If M is projective in $\mathcal{F}_k$

Evaluation - Adjunction

- Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Recall that the evaluation functor
$$M \in \mathcal{F}_k \mapsto M(E) \in \mathcal{R}_E\text{-Mod}$$
has a left adjoint $V \mapsto L_{E,V}$, defined by
$$X \mapsto L_{E,V}(X) := k\mathcal{C}(X,E) \otimes_{\mathcal{R}_E} V.$$
In particular $L_{E,V}(E) \cong V$.
- If M is projective in $\mathcal{F}_k$, and $M = \langle M(E) \rangle$

Evaluation - Adjunction

- Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Recall that the evaluation functor
$$M \in \mathcal{F}_k \mapsto M(E) \in \mathcal{R}_E\text{-Mod}$$
has a left adjoint $V \mapsto L_{E,V}$, defined by
$$X \mapsto L_{E,V}(X) := k\mathcal{C}(X,E) \otimes_{\mathcal{R}_E} V.$$
In particular $L_{E,V}(E) \cong V$.
- If M is projective in $\mathcal{F}_k$, and $M = \langle M(E) \rangle$, then $M \cong L_{F,M(F)}$ for any finite set F with $|F| \geq |E|$

Evaluation - Adjunction

- Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Recall that the evaluation functor
$$M \in \mathcal{F}_k \mapsto M(E) \in \mathcal{R}_E\text{-Mod}$$
has a left adjoint $V \mapsto L_{E,V}$, defined by
$$X \mapsto L_{E,V}(X) := k\mathcal{C}(X,E) \otimes_{\mathcal{R}_E} V.$$
In particular $L_{E,V}(E) \cong V$.
- If M is projective in $\mathcal{F}_k$, and $M = \langle M(E) \rangle$, then $M \cong L_{F,M(F)}$ for any finite set F with $|F| \geq |E|$, and $M(F)$ is a projective $\mathcal{R}_F$-module.

Evaluation - Adjunction

- Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Recall that the evaluation functor
$$M \in \mathcal{F}_k \mapsto M(E) \in \mathcal{R}_E\text{-Mod}$$
has a left adjoint $V \mapsto L_{E,V}$, defined by
$$X \mapsto L_{E,V}(X) := k\mathcal{C}(X,E) \otimes_{\mathcal{R}_E} V.$$
In particular $L_{E,V}(E) \cong V$.
- If M is projective in $\mathcal{F}_k$, and $M = \langle M(E) \rangle$, then $M \cong L_{F,M(F)}$ for any finite set F with $|F| \geq |E|$, and $M(F)$ is a projective $\mathcal{R}_F$-module.
- The functor $L_{E,V}$ is projective

Evaluation - Adjunction

- Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Recall that the evaluation functor
$$M \in \mathcal{F}_k \mapsto M(E) \in \mathcal{R}_E\text{-Mod}$$
has a left adjoint $V \mapsto L_{E,V}$, defined by
$$X \mapsto L_{E,V}(X) := k\mathcal{C}(X,E) \otimes_{\mathcal{R}_E} V.$$
In particular $L_{E,V}(E) \cong V$.
- If M is projective in $\mathcal{F}_k$, and $M = \langle M(E) \rangle$, then $M \cong L_{F,M(F)}$ for any finite set F with $|F| \geq |E|$, and $M(F)$ is a projective $\mathcal{R}_F$-module.
- The functor $L_{E,V}$ is projective (resp. indecomposable)

Evaluation - Adjunction

- Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Recall that the evaluation functor
$$M \in \mathcal{F}_k \mapsto M(E) \in \mathcal{R}_E\text{-Mod}$$
has a left adjoint $V \mapsto L_{E,V}$, defined by
$$X \mapsto L_{E,V}(X) := k\mathcal{C}(X,E) \otimes_{\mathcal{R}_E} V.$$
In particular $L_{E,V}(E) \cong V$.
- If M is projective in $\mathcal{F}_k$, and $M = \langle M(E) \rangle$, then $M \cong L_{F,M(F)}$ for any finite set F with $|F| \geq |E|$, and $M(F)$ is a projective $\mathcal{R}_F$-module.
- The functor $L_{E,V}$ is projective (resp. indecomposable) if and only if V is a projective (resp. indecomposable) $\mathcal{R}_E$-module.

Symmetry

Symmetry

Theorem

1. *Let E be a finite set*

Symmetry

Theorem

1. *Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$.*

Symmetry

Theorem

1. *Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Define $t_E : \mathcal{R}_E \to k$ by*

$$\forall R \in \mathcal{C}(E,E),\ \ t_E(R) =$$

Symmetry

Theorem

1. *Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Define $t_E : \mathcal{R}_E \to k$ by*

$$\forall R \in \mathcal{C}(E,E),\ t_E(R) = \begin{cases} 1 & \text{if } R \cap \Delta_E = \emptyset, \\ & \end{cases}$$

Symmetry

Theorem

1. *Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Define $t_E : \mathcal{R}_E \to k$ by*

$$\forall R \in \mathcal{C}(E,E),\ t_E(R) = \begin{cases} 1 & \text{if } R \cap \Delta_E = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Symmetry

Theorem

1. *Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Define $t_E : \mathcal{R}_E \to k$ by*

$$\forall R \in \mathcal{C}(E,E),\ t_E(R) = \begin{cases} 1 & \text{if } R \cap \Delta_E = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Then t_E is a symmetrizing form for $\mathcal{R}_E$.

Symmetry

Theorem

1. *Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Define $t_E : \mathcal{R}_E \to k$ by*

$$\forall R \in \mathcal{C}(E,E),\ t_E(R) = \begin{cases} 1 & \text{if } R \cap \Delta_E = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

   Then t_E is a symmetrizing form for $\mathcal{R}_E$.

2. *If k is a field*

Symmetry

Theorem

1. *Let E be a finite set, and* $\mathcal{R}_E = k\mathcal{C}(E,E)$. *Define* $t_E : \mathcal{R}_E \to k$ *by*

$$\forall R \in \mathcal{C}(E,E),\ t_E(R) = \begin{cases} 1 & \textit{if } R \cap \Delta_E = \emptyset, \\ 0 & \textit{otherwise.} \end{cases}$$

   Then t_E *is a symmetrizing form for* $\mathcal{R}_E$.

2. *If k is a field (or more generally if k is self injective)*

Symmetry

Theorem

1. *Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Define $t_E : \mathcal{R}_E \to k$ by*

$$\forall R \in \mathcal{C}(E,E),\ t_E(R) = \begin{cases} 1 & \text{if } R \cap \Delta_E = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

   Then t_E is a symmetrizing form for $\mathcal{R}_E$.

2. *If k is a field (or more generally if k is self injective), any finitely generated projective in $\mathcal{F}_k$ is also injective.*

Symmetry

Theorem

1. *Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Define $t_E : \mathcal{R}_E \to k$ by*

$$\forall R \in \mathcal{C}(E,E),\ t_E(R) = \begin{cases} 1 & \text{if } R \cap \Delta_E = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

   Then t_E is a symmetrizing form for $\mathcal{R}_E$.

2. *If k is a field (or more generally if k is self injective), any finitely generated projective in $\mathcal{F}_k$ is also injective. In particular $\mathcal{F}_k^f$ has infinite global dimension.*

Symmetry

Theorem

1. *Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Define $t_E : \mathcal{R}_E \to k$ by*

$$\forall R \in \mathcal{C}(E,E),\ t_E(R) = \begin{cases} 1 & \textit{if } R \cap \Delta_E = \emptyset, \\ 0 & \textit{otherwise.} \end{cases}$$

   Then t_E is a symmetrizing form for $\mathcal{R}_E$.

2. *If k is a field (or more generally if k is self injective), any finitely generated projective in $\mathcal{F}_k$ is also injective. In particular $\mathcal{F}_k^f$ has infinite global dimension.*

Proof:

Symmetry

Theorem

1. *Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Define $t_E : \mathcal{R}_E \to k$ by*

$$\forall R \in \mathcal{C}(E,E),\ t_E(R) = \begin{cases} 1 & \text{if } R \cap \Delta_E = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

   Then t_E is a symmetrizing form for $\mathcal{R}_E$.

2. *If k is a field (or more generally if k is self injective), any finitely generated projective in $\mathcal{F}_k$ is also injective. In particular $\mathcal{F}_k^f$ has infinite global dimension.*

Proof: 1) Let $R, S \in \mathcal{C}(E,E)$.

Symmetry

Theorem

1. *Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Define $t_E : \mathcal{R}_E \to k$ by*

$$\forall R \in \mathcal{C}(E,E),\ t_E(R) = \begin{cases} 1 & \text{if } R \cap \Delta_E = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

   Then t_E is a symmetrizing form for $\mathcal{R}_E$.

2. *If k is a field (or more generally if k is self injective), any finitely generated projective in $\mathcal{F}_k$ is also injective. In particular $\mathcal{F}_k^f$ has infinite global dimension.*

Proof: 1) Let $R, S \in \mathcal{C}(E,E)$. Then $t_E(RS) = 1 \Leftrightarrow R \subseteq (E \times E) - S^{op}$.

Symmetry

Theorem

1. *Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Define $t_E : \mathcal{R}_E \to k$ by*

$$\forall R \in \mathcal{C}(E,E),\ t_E(R) = \begin{cases} 1 & \text{if } R \cap \Delta_E = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

   Then t_E is a symmetrizing form for $\mathcal{R}_E$.

2. *If k is a field (or more generally if k is self injective), any finitely generated projective in $\mathcal{F}_k$ is also injective. In particular $\mathcal{F}_k^f$ has infinite global dimension.*

Proof: 1) Let $R, S \in \mathcal{C}(E,E)$. Then $t_E(RS) = 1 \Leftrightarrow R \subseteq (E \times E) - S^{op}$. The matrix $\big(t_E(RS)\big)_{R,S \in \mathcal{C}(E,E)}$

Symmetry

Theorem

1. *Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Define $t_E : \mathcal{R}_E \to k$ by*

$$\forall R \in \mathcal{C}(E,E),\ t_E(R) = \begin{cases} 1 & \text{if } R \cap \Delta_E = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

   Then t_E is a symmetrizing form for $\mathcal{R}_E$.

2. *If k is a field (or more generally if k is self injective), any finitely generated projective in $\mathcal{F}_k$ is also injective. In particular $\mathcal{F}_k^f$ has infinite global dimension.*

Proof: 1) Let $R, S \in \mathcal{C}(E,E)$. Then $t_E(RS) = 1 \Leftrightarrow R \subseteq (E \times E) - S^{op}$. The matrix $\big(t_E(RS)\big)_{R,S \in \mathcal{C}(E,E)}$ is the product of a permutation matrix $(S \mapsto (E \times E) - S^{op})$

Symmetry

Theorem

1. *Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Define $t_E : \mathcal{R}_E \to k$ by*

$$\forall R \in \mathcal{C}(E,E),\ t_E(R) = \begin{cases} 1 & \text{if } R \cap \Delta_E = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

   Then t_E is a symmetrizing form for $\mathcal{R}_E$.

2. *If k is a field (or more generally if k is self injective), any finitely generated projective in $\mathcal{F}_k$ is also injective. In particular $\mathcal{F}_k^f$ has infinite global dimension.*

Proof: 1) Let $R, S \in \mathcal{C}(E,E)$. Then $t_E(RS) = 1 \Leftrightarrow R \subseteq (E \times E) - S^{op}$. The matrix $\big(t_E(RS)\big)_{R,S \in \mathcal{C}(E,E)}$ is the product of a permutation matrix $(S \mapsto (E \times E) - S^{op})$ with the matrix of an order $(\subseteq)$

Symmetry

Theorem

1. *Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Define $t_E : \mathcal{R}_E \to k$ by*
$$\forall R \in \mathcal{C}(E,E),\ t_E(R) = \begin{cases} 1 & \text{if } R \cap \Delta_E = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$
Then t_E is a symmetrizing form for $\mathcal{R}_E$.
2. *If k is a field (or more generally if k is self injective), any finitely generated projective in $\mathcal{F}_k$ is also injective. In particular $\mathcal{F}_k^f$ has infinite global dimension.*

Proof: 1) Let $R, S \in \mathcal{C}(E,E)$. Then $t_E(RS) = 1 \Leftrightarrow R \subseteq (E \times E) - S^{op}$. The matrix $\big(t_E(RS)\big)_{R,S \in \mathcal{C}(E,E)}$ is the product of a permutation matrix $(S \mapsto (E \times E) - S^{op})$ with the matrix of an order $(\subseteq)$, hence it is invertible (over $\mathbb{Z}$).

Symmetry

Theorem

1. *Let E be a finite set, and $\mathcal{R}_E = k\mathcal{C}(E,E)$. Define $t_E : \mathcal{R}_E \to k$ by*

$$\forall R \in \mathcal{C}(E,E),\ t_E(R) = \begin{cases} 1 & \textit{if } R \cap \Delta_E = \emptyset, \\ 0 & \textit{otherwise.} \end{cases}$$

   Then t_E is a symmetrizing form for $\mathcal{R}_E$.

2. *If k is a field (or more generally if k is self injective), any finitely generated projective in $\mathcal{F}_k$ is also injective. In particular $\mathcal{F}_k^f$ has infinite global dimension.*

Proof: 1) Let $R, S \in \mathcal{C}(E,E)$. Then $t_E(RS) = 1 \Leftrightarrow R \subseteq (E \times E) - S^{op}$. The matrix $\big(t_E(RS)\big)_{R,S \in \mathcal{C}(E,E)}$ is the product of a permutation matrix $(S \mapsto (E \times E) - S^{op})$ with the matrix of an order $(\subseteq)$, hence it is invertible (over $\mathbb{Z}$).
2) A similar argument shows that the Yoneda functor $k\mathcal{C}(-,E)$ is selfdual.

More on projective functors

More on projective functors

Theorem

Let k be a field

More on projective functors

Theorem

Let k be a field, and $M \in \mathcal{F}_k^f$.

More on projective functors

Theorem

Let k be a field, and $M \in \mathcal{F}_k^f$. The following are equivalent:

More on projective functors

Theorem

Let k be a field, and $M \in \mathcal{F}_k^f$. The following are equivalent:

1. *M is projective and indecomposable.*

More on projective functors

Theorem

Let k be a field, and $M \in \mathcal{F}_k^f$. The following are equivalent:

1. *M is projective and indecomposable.*
2. *M is projective and admits a unique maximal proper subfunctor.*

More on projective functors

Theorem

Let k be a field, and $M \in \mathcal{F}_k^f$. The following are equivalent:

1. *M is projective and indecomposable.*
2. *M is projective and admits a unique maximal proper subfunctor.*
3. *M is projective and admits a unique minimal non-zero subfunctor.*

More on projective functors

Theorem

Let k be a field, and $M \in \mathcal{F}_k^f$. The following are equivalent:

1. *M is projective and indecomposable.*
2. *M is projective and admits a unique maximal proper subfunctor.*
3. *M is projective and admits a unique minimal non-zero subfunctor.*
4. *M is injective and indecomposable.*

More on projective functors

Theorem

Let k be a field, and $M \in \mathcal{F}_k^f$. The following are equivalent:

1. *M is projective and indecomposable.*
2. *M is projective and admits a unique maximal proper subfunctor.*
3. *M is projective and admits a unique minimal non-zero subfunctor.*
4. *M is injective and indecomposable.*
5. *M is injective and admits a unique maximal proper subfunctor.*

More on projective functors

Theorem

Let k be a field, and $M \in \mathcal{F}_k^f$. The following are equivalent:

1. *M is projective and indecomposable.*
2. *M is projective and admits a unique maximal proper subfunctor.*
3. *M is projective and admits a unique minimal non-zero subfunctor.*
4. *M is injective and indecomposable.*
5. *M is injective and admits a unique maximal proper subfunctor.*
6. *M is injective and admits a unique minimal non-zero subfunctor.*

More on projective functors

Theorem

Let k be a field, and $M \in \mathcal{F}_k^f$. The following are equivalent:

1. *M is projective and indecomposable.*
2. *M is projective and admits a unique maximal proper subfunctor.*
3. *M is projective and admits a unique minimal non-zero subfunctor.*
4. *M is injective and indecomposable.*
5. *M is injective and admits a unique maximal proper subfunctor.*
6. *M is injective and admits a unique minimal non-zero subfunctor.*

More on projective functors

Theorem

Let k be a field, and $M \in \mathcal{F}_k^f$. The following are equivalent:

1. *M is projective and indecomposable.*
2. *M is projective and admits a unique maximal proper subfunctor.*
3. *M is projective and admits a unique minimal non-zero subfunctor.*
4. *M is injective and indecomposable.*
5. *M is injective and admits a unique maximal proper subfunctor.*
6. *M is injective and admits a unique minimal non-zero subfunctor.*

Theorem

Let k be a field.

1. *Let $M \in \mathcal{F}_k^f$ be a projective functor.*

More on projective functors

Theorem

Let k be a field, and $M \in \mathcal{F}_k^f$. The following are equivalent:

1. *M is projective and indecomposable.*
2. *M is projective and admits a unique maximal proper subfunctor.*
3. *M is projective and admits a unique minimal non-zero subfunctor.*
4. *M is injective and indecomposable.*
5. *M is injective and admits a unique maximal proper subfunctor.*
6. *M is injective and admits a unique minimal non-zero subfunctor.*

Theorem

Let k be a field.

1. *Let $M \in \mathcal{F}_k^f$ be a projective functor. Then $M/Rad(M) \cong Soc(M)$.*

More on projective functors

Theorem

Let k be a field, and $M \in \mathcal{F}_k^f$. The following are equivalent:

1. *M is projective and indecomposable.*
2. *M is projective and admits a unique maximal proper subfunctor.*
3. *M is projective and admits a unique minimal non-zero subfunctor.*
4. *M is injective and indecomposable.*
5. *M is injective and admits a unique maximal proper subfunctor.*
6. *M is injective and admits a unique minimal non-zero subfunctor.*

Theorem

Let k be a field.

1. *Let $M \in \mathcal{F}_k^f$ be a projective functor. Then $M/Rad(M) \cong Soc(M)$.*
2. *Let $M, N \in \mathcal{F}_k^f$ be projective functors.*

More on projective functors

Theorem

Let k be a field, and $M \in \mathcal{F}_k^f$. The following are equivalent:

1. *M is projective and indecomposable.*
2. *M is projective and admits a unique maximal proper subfunctor.*
3. *M is projective and admits a unique minimal non-zero subfunctor.*
4. *M is injective and indecomposable.*
5. *M is injective and admits a unique maximal proper subfunctor.*
6. *M is injective and admits a unique minimal non-zero subfunctor.*

Theorem

Let k be a field.

1. *Let $M \in \mathcal{F}_k^f$ be a projective functor. Then $M/Rad(M) \cong Soc(M)$.*
2. *Let $M, N \in \mathcal{F}_k^f$ be projective functors. Then*
$$\dim_k Hom_{\mathcal{F}_k}(M, N) = \dim_k Hom_{\mathcal{F}_k}(N, M)$$

Evaluation - Stability

Evaluation - Stability

Theorem

Let k be a noetherian ring, let $M, N \in \mathcal{F}_k$, and let E, F be finite sets.

Theorem

Let k be a noetherian ring, let $M, N \in \mathcal{F}_k$, and let E, F be finite sets.

1. *If $M = \langle M(E) \rangle$*

Evaluation - Stability

Theorem

Let k be a noetherian ring, let $M, N \in \mathcal{F}_k$, and let E, F be finite sets.

1. *If $M = \langle M(E) \rangle$, then for $|F| \geq 2^{|E|}$*

Evaluation - Stability

Theorem

Let k be a noetherian ring, let $M, N \in \mathcal{F}_k$, and let E, F be finite sets.

1. *If $M = \langle M(E) \rangle$, then for $|F| \geq 2^{|E|}$, the evaluation map*
$$Hom_{\mathcal{F}_k}(M, N) \to Hom_{\mathcal{R}_F}\big(M(F), N(F)\big)$$

Evaluation - Stability

Theorem

Let k be a noetherian ring, let $M, N \in \mathcal{F}_k$, and let E, F be finite sets.

1. *If $M = \langle M(E) \rangle$, then for $|F| \geq 2^{|E|}$, the evaluation map*
$$Hom_{\mathcal{F}_k}(M, N) \to Hom_{\mathcal{R}_F}\big(M(F), N(F)\big)$$
is an isomorphism.

Evaluation - Stability

Theorem

Let k be a noetherian ring, let $M, N \in \mathcal{F}_k$, and let E, F be finite sets.

1. *If $M = \langle M(E) \rangle$, then for $|F| \geq 2^{|E|}$, the evaluation map*
$$Hom_{\mathcal{F}_k}(M, N) \to Hom_{\mathcal{R}_F}\big(M(F), N(F)\big)$$
is an isomorphism.
2. *If M has bounded type*

Evaluation - Stability

Theorem

Let k be a noetherian ring, let $M, N \in \mathcal{F}_k$, and let E, F be finite sets.

1. *If $M = \langle M(E) \rangle$, then for $|F| \geq 2^{|E|}$, the evaluation map*
$$Hom_{\mathcal{F}_k}(M, N) \to Hom_{\mathcal{R}_F}\big(M(F), N(F)\big)$$
is an isomorphism.
2. *If M has bounded type, then for any $i \in \mathbb{N}$, there exists $n_i \in \mathbb{N}$*

Evaluation - Stability

Theorem

Let k be a noetherian ring, let $M, N \in \mathcal{F}_k$, and let E, F be finite sets.

1. *If $M = \langle M(E) \rangle$, then for $|F| \geq 2^{|E|}$, the evaluation map*
$$Hom_{\mathcal{F}_k}(M, N) \to Hom_{\mathcal{R}_F}\big(M(F), N(F)\big)$$
is an isomorphism.
2. *If M has bounded type, then for any $i \in \mathbb{N}$, there exists $n_i \in \mathbb{N}$ such that if $|F| \geq n_i$*

Evaluation - Stability

Theorem

Let k be a noetherian ring, let $M, N \in \mathcal{F}_k$, and let E, F be finite sets.

1. *If $M = \langle M(E) \rangle$, then for $|F| \geq 2^{|E|}$, the evaluation map*
$$Hom_{\mathcal{F}_k}(M, N) \to Hom_{\mathcal{R}_F}\big(M(F), N(F)\big)$$
is an isomorphism.
2. *If M has bounded type, then for any $i \in \mathbb{N}$, there exists $n_i \in \mathbb{N}$ such that if $|F| \geq n_i$, the map*
$$Ext^i_{\mathcal{F}_k}(M, N) \to Ext^i_{\mathcal{R}_F}\big(M(F), N(F)\big)$$

Evaluation - Stability

Theorem

Let k be a noetherian ring, let $M, N \in \mathcal{F}_k$, and let E, F be finite sets.

1. *If $M = \langle M(E) \rangle$, then for $|F| \geq 2^{|E|}$, the evaluation map*
$$Hom_{\mathcal{F}_k}(M, N) \to Hom_{\mathcal{R}_F}\big(M(F), N(F)\big)$$
is an isomorphism.
2. *If M has bounded type, then for any $i \in \mathbb{N}$, there exists $n_i \in \mathbb{N}$ such that if $|F| \geq n_i$, the map*
$$Ext^i_{\mathcal{F}_k}(M, N) \to Ext^i_{\mathcal{R}_F}\big(M(F), N(F)\big)$$
is an isomorphism.

An equivalence of categories

An equivalence of categories

Definition

Let $\mathcal{G}_k$ be the following category:

An equivalence of categories

Definition

Let $\mathcal{G}_k$ be the following category:

- the objects are pairs (E, U)

An equivalence of categories

Definition

Let $\mathcal{G}_k$ be the following category:

- the objects are pairs (E, U), where E is a finite set

An equivalence of categories

Definition

Let $\mathcal{G}_k$ be the following category:

- the objects are pairs (E, U), where E is a finite set, and U is an $\mathcal{R}_E$-module.

An equivalence of categories

Definition

Let $\mathcal{G}_k$ be the following category:

- the objects are pairs (E, U), where E is a finite set, and U is an $\mathcal{R}_E$-module.
- a morphism $(E, U) \to (F, V)$

An equivalence of categories

Definition

Let $\mathcal{G}_k$ be the following category:

- the objects are pairs (E, U), where E is a finite set, and U is an $\mathcal{R}_E$-module.
- a morphism $(E, U) \to (F, V)$ is a morphism of $\mathcal{R}_E$-modules $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V$.

An equivalence of categories

Definition

Let $\mathcal{G}_k$ be the following category:

- the objects are pairs (E, U), where E is a finite set, and U is an $\mathcal{R}_E$-module.
- a morphism $(E, U) \to (F, V)$ is a morphism of $\mathcal{R}_E$-modules $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V$.
- the composition of

$$(E, U) \to (F, V)$$

An equivalence of categories

Definition

Let $\mathcal{G}_k$ be the following category:

- the objects are pairs (E, U), where E is a finite set, and U is an $\mathcal{R}_E$-module.
- a morphism $(E, U) \to (F, V)$ is a morphism of $\mathcal{R}_E$-modules $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V$.
- the composition of

$$U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V$$

An equivalence of categories

Definition

Let $\mathcal{G}_k$ be the following category:

- the objects are pairs (E, U), where E is a finite set, and U is an $\mathcal{R}_E$-module.
- a morphism $(E, U) \to (F, V)$ is a morphism of $\mathcal{R}_E$-modules $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V$.
- the composition of

$$U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V \text{ and } (F, V) \to (G, W)$$

An equivalence of categories

Definition

Let $\mathcal{G}_k$ be the following category:

- the objects are pairs (E, U), where E is a finite set, and U is an $\mathcal{R}_E$-module.
- a morphism $(E, U) \to (F, V)$ is a morphism of $\mathcal{R}_E$-modules $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V$.
- the composition of

$$U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V \text{ and } V \to k\mathcal{C}(F, G) \otimes_{\mathcal{R}_G} W$$

An equivalence of categories

Definition

Let $\mathcal{G}_k$ be the following category:

- the objects are pairs (E, U), where E is a finite set, and U is an $\mathcal{R}_E$-module.
- a morphism $(E, U) \to (F, V)$ is a morphism of $\mathcal{R}_E$-modules $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V$.
- the composition of

$$U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V \text{ and } V \to k\mathcal{C}(F, G) \otimes_{\mathcal{R}_G} W$$

  is $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V$

An equivalence of categories

Definition

Let $\mathcal{G}_k$ be the following category:

- the objects are pairs (E, U), where E is a finite set, and U is an $\mathcal{R}_E$-module.
- a morphism $(E, U) \to (F, V)$ is a morphism of $\mathcal{R}_E$-modules $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V$.
- the composition of

$$U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V \text{ and } V \to k\mathcal{C}(F, G) \otimes_{\mathcal{R}_G} W$$

is $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} k\mathcal{C}(F, G) \otimes_{\mathcal{R}_G} W$

An equivalence of categories

Definition

Let $\mathcal{G}_k$ be the following category:

- the objects are pairs (E, U), where E is a finite set, and U is an $\mathcal{R}_E$-module.
- a morphism $(E, U) \to (F, V)$ is a morphism of $\mathcal{R}_E$-modules $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V$.
- the composition of

$$U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V \text{ and } V \to k\mathcal{C}(F, G) \otimes_{\mathcal{R}_G} W$$

is $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} k\mathcal{C}(F, G) \otimes_{\mathcal{R}_G} W$

An equivalence of categories

Definition

Let $\mathcal{G}_k$ be the following category:

- the objects are pairs (E, U), where E is a finite set, and U is an $\mathcal{R}_E$-module.
- a morphism $(E, U) \to (F, V)$ is a morphism of $\mathcal{R}_E$-modules $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V$.
- the composition of

$$U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V \text{ and } V \to k\mathcal{C}(F, G) \otimes_{\mathcal{R}_G} W$$

is $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} k\mathcal{C}(F, G) \otimes_{\mathcal{R}_G} W$

$$\to k\mathcal{C}(E, G)$$

An equivalence of categories

Definition

Let $\mathcal{G}_k$ be the following category:

- the objects are pairs (E, U), where E is a finite set, and U is an $\mathcal{R}_E$-module.
- a morphism $(E, U) \to (F, V)$ is a morphism of $\mathcal{R}_E$-modules $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V$.
- the composition of

$$U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V \text{ and } V \to k\mathcal{C}(F, G) \otimes_{\mathcal{R}_G} W$$

$$\text{is } U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} k\mathcal{C}(F, G) \otimes_{\mathcal{R}_G} W \to k\mathcal{C}(E, G) \otimes_{\mathcal{R}_G} W$$

An equivalence of categories

Definition

Let $\mathcal{G}_k$ be the following category:

- the objects are pairs (E, U), where E is a finite set, and U is an $\mathcal{R}_E$-module.
- a morphism $(E, U) \to (F, V)$ is a morphism of $\mathcal{R}_E$-modules $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V$.
- the composition of

$$U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V \text{ and } V \to k\mathcal{C}(F, G) \otimes_{\mathcal{R}_G} W$$

is $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} k\mathcal{C}(F, G) \otimes_{\mathcal{R}_G} W$

$$\to k\mathcal{C}(E, G) \otimes_{\mathcal{R}_G} W$$

- the identity morphism of (E, U)

An equivalence of categories

Definition

Let $\mathcal{G}_k$ be the following category:

- the objects are pairs (E, U), where E is a finite set, and U is an $\mathcal{R}_E$-module.
- a morphism $(E, U) \to (F, V)$ is a morphism of $\mathcal{R}_E$-modules $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V$.
- the composition of
$$U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V \text{ and } V \to k\mathcal{C}(F, G) \otimes_{\mathcal{R}_G} W$$
is $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} k\mathcal{C}(F, G) \otimes_{\mathcal{R}_G} W$
$$\to k\mathcal{C}(E, G) \otimes_{\mathcal{R}_G} W$$
- the identity morphism of (E, U) is $U \stackrel{\cong}{\to} k\mathcal{C}(E, E) \otimes_{\mathcal{R}_E} U$.

An equivalence of categories

Definition

Let $\mathcal{G}_k$ be the following category:

- the objects are pairs (E, U), where E is a finite set, and U is an $\mathcal{R}_E$-module.
- a morphism $(E, U) \to (F, V)$ is a morphism of $\mathcal{R}_E$-modules $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V$.
- the composition of

$$U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V \text{ and } V \to k\mathcal{C}(F, G) \otimes_{\mathcal{R}_G} W$$

is $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} k\mathcal{C}(F, G) \otimes_{\mathcal{R}_G} W$
$\to k\mathcal{C}(E, G) \otimes_{\mathcal{R}_G} W$

- the identity morphism of (E, U) is $U \stackrel{\cong}{\to} k\mathcal{C}(E, E) \otimes_{\mathcal{R}_E} U$.

Theorem

1. *The assignment* $(E, U) \mapsto L_{E,U}$ *is a fully faithful k-linear functor* $\mathcal{G}_k \to \mathcal{F}_k^b$.

An equivalence of categories

Definition

Let $\mathcal{G}_k$ be the following category:

- the objects are pairs (E, U), where E is a finite set, and U is an $\mathcal{R}_E$-module.
- a morphism $(E, U) \to (F, V)$ is a morphism of $\mathcal{R}_E$-modules $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V$.
- the composition of
$$U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V \text{ and } V \to k\mathcal{C}(F, G) \otimes_{\mathcal{R}_G} W$$
is $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} k\mathcal{C}(F, G) \otimes_{\mathcal{R}_G} W$
$$\to k\mathcal{C}(E, G) \otimes_{\mathcal{R}_G} W$$
- the identity morphism of (E, U) is $U \overset{\cong}{\to} k\mathcal{C}(E, E) \otimes_{\mathcal{R}_E} U$.

Theorem

1. *The assignment* $(E, U) \mapsto L_{E,U}$ *is a fully faithful* k*-linear functor* $\mathcal{G}_k \to \mathcal{F}_k^b$.
2. *When* k *is noetherian, it is an equivalence of categories*

An equivalence of categories

Definition

Let $\mathcal{G}_k$ be the following category:

- the objects are pairs (E, U), where E is a finite set, and U is an $\mathcal{R}_E$-module.
- a morphism $(E, U) \to (F, V)$ is a morphism of $\mathcal{R}_E$-modules $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V$.
- the composition of

$$U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V \text{ and } V \to k\mathcal{C}(F, G) \otimes_{\mathcal{R}_G} W$$

is $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} k\mathcal{C}(F, G) \otimes_{\mathcal{R}_G} W$

$$\to k\mathcal{C}(E, G) \otimes_{\mathcal{R}_G} W$$

- the identity morphism of (E, U) is $U \xrightarrow{\cong} k\mathcal{C}(E, E) \otimes_{\mathcal{R}_E} U$.

Theorem

1. *The assignment $(E, U) \mapsto L_{E,U}$ is a fully faithful k-linear functor $\mathcal{G}_k \to \mathcal{F}_k^b$.*
2. *When k is noetherian, it is an equivalence of categories. In particular $\mathcal{G}_k$ is abelian.*