Diagonal p-permutation functors

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Abstract

Let k be an algebraically closed field of positive characteristic p, and $\mathbb{F}$ be an algebraically closed field of characteristic 0. We consider the $\mathbb{F}$-linear category $\mathbb{F}pp_k^\Delta$ of finite groups, in which the set of morphisms from G to H is the $\mathbb{F}$-linear extension $\mathbb{F}T^\Delta(H,G)$ of the Grothendieck group $T^\Delta(H,G)$ of p-permutation (kH,kG)-bimodules with (twisted) diagonal vertices. The $\mathbb{F}$-linear functors from $\mathbb{F}pp_k^\Delta$ to $\mathbb{F}$-Mod are called *diagonal p-permutation functors*. They form an abelian category $\mathcal{F}_{pp_k}^\Delta$.

We study in particular the functor $\mathbb{F}T^\Delta$ sending a finite group G to the Grothendieck group $\mathbb{F}T(G)$ of p-permutation kG-modules, and show that $\mathbb{F}T^\Delta$ is a semisimple object of $\mathcal{F}_{pp_k}^\Delta$, equal to the direct sum of specific simple functors parametrized by isomorphism classes of pairs (P,s) of a finite p-group P and a generator s of a p'-subgroup acting faithfully on P. This leads to a precise description of the evaluations of these simple functors. In particular, we show that the simple functor indexed by the trivial pair $(1,1)$ is isomorphic to the functor sending a finite group G to $\mathbb{F}K_0(kG)$, where $K_0(kG)$ is the group of projective kG-modules.

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1. Introduction

Let p be a prime number. Throughout we denote by $\mathbb{F}$ an algebraically closed field of characteristic zero, and by k an algebraically closed field of characteristic p. The p-permutation modules play a crucial role in the study of modular representation theory of finite groups. A splendid Rickard equivalence, introduced by Rickard [8], between blocks of finite group algebras is given by a chain complex consisting of p-permutation bimodules. Also a p-permutation equivalence, introduced by Boltje

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and Xu [1], and studied extensively later by Boltje and Perepelitsky [7], is an element in the Grothendieck group of p-permutation bimodules.

In [5], Ducellier studied p-permutation functors: Consider the category $\mathbb{F}pp_k$ where the objects are finite groups and the morphisms between groups G and H are given by the Grothendieck group $\mathbb{F}\otimes_{\mathbb{Z}}T(H,G)$ of p-permutation (kH,kG)-bimodules. A p-permutation functor is an $\mathbb{F}$-linear functor from $\mathbb{F}pp_k$ to $\mathbb{F}$-Mod. The indecomposable direct summands of the bimodules that appears in a p-permutation equivalence between blocks of finite group algebras have twisted diagonal vertices. Therefore, inspired by the work of Ducellier, we consider a category with less morphisms: Let $\mathbb{F}pp_k^\Delta$ be a category where the objects are finite groups and the morphisms between groups G and H are given by the Grothendieck group $\mathbb{F}\otimes_{\mathbb{Z}}T^\Delta(H,G)$ of p-permutation (kH,kG)-bimodules whose indecomposable direct summands have twisted diagonal vertices. An $\mathbb{F}$-linear functor from $\mathbb{F}pp_k^\Delta$ to $\mathbb{F}$-Mod is called a *diagonal p-permutation functor*.

By [2], the simple diagonal p-permutation functors are parametrized by the pairs (G,V) of a finite group G and a simple module V of the essential algebra $\mathcal{E}^\Delta(G)=\mathrm{End}_{\mathbb{F}pp_k^\Delta}(G)/I$ at G, where I is the ideal generated by the morphisms that factor through groups of smaller order. We show that the essential algebra $\mathcal{E}^\Delta(G)$ is isomorphic to the essential algebra studied in [5]. As a result this implies that the essential algebra $\mathcal{E}^\Delta(G)$ is non-zero if and only if the group G is of the form $P\rtimes\langle s\rangle$ where P is a p-group and s is a generator of a p'-cyclic group acting faithfully on P. Moreover in that case there is an algebra isomorphism $\mathcal{E}^\Delta(G)\cong\big(\mathbb{F}[X]/\Phi_n[X]\big)\rtimes\mathrm{Out}(G)$ where n is the order of s. See Theorem 3.3.

We also study the functor $\mathbb{F}T^\Delta$ that sends a finite group G to the Grothendieck group $\mathbb{F}T(G)$ of p-permutation kG-modules. We describe the subfunctor lattice (Theorem 5.11) and simple quotients (Proposition 5.15) of $\mathbb{F}T^\Delta$. We also give a description for the $\mathbb{F}$-dimension of the evaluations of simple quotients of $\mathbb{F}T^\Delta$ at a finite group G (Theorem 5.18). Moreover we prove that the simple functor $S_{1,1}$ that corresponds to the pair $(1,1)$ is isomorphic to the functor that sends a finite group G to the $\mathbb{F}$-linear extension $\mathbb{F}K_0(kG)$ of the Grothendieck group of projective kG-modules (Theorem 5.20).

2. Preliminaries

Let G and H be finite groups. We denote by $p_1:G\times H\to G$ and $p_2:G\times H\to H$ the canonical projections. Let $X\leqslant G\times H$ be a subgroup. We define the subgroups $k_1(X):=p_1(X\cap\ker(p_2))$ and $k_2(X):=p_2(X\cap\ker(p_1))$ of $p_1(X)$ and $p_2(X)$, respectively. Note that $k_1(X)\times k_2(X)$ is a normal subgroup of X. Moreover, $k_i(X)$ is a normal subgroup of $p_i(X)$ and one has a canonical isomorphism $X/(k_1(X) \times k_2(X)) \to p_i(X)/k_i(X)$ induced by the projection map p_i for $i = 1, 2$.

Let $\phi : P \to Q$ be an isomorphism between subgroups $P \leqslant G$ and $Q \leqslant H$. Then $\{(\phi(x), x) : x \in P\}$ is a subgroup of $H \times G$ and a subgroup of that form is called a *twisted diagonal* subgroup of $H \times G$. Note that a subgroup $X \leqslant H \times G$ is a twisted diagonal subgroup if and only if $k_1(X) = 1$ and $k_2(X) = 1$.

Let P be a subgroup of G and M be a kG-module. We denote by M^P the k-vector space of P-fixed points of M. If $Q \leqslant P$ is a subgroup, then the map $\mathrm{Tr}_Q^P : M^Q \to M^P$ defined by $\mathrm{Tr}(m) = \sum_{x \in [P/Q]} x \cdot m$ is called the *relative trace map*. The quotient

$$M[P] := M^P / \sum_{Q<P} \mathrm{Tr}_Q^P(M^Q)$$

is called the *Brauer quotient* of M at P. Note that $M[P]$ is a $k\overline{N}_G(P)$-module, where $\overline{N}_G(P) := N_G(P)/P$. We have $M[P] = 0$ if P is not a p-group.

A (kG, kH)-bimodule M can be viewed as a $k(G \times H)$-module via $(g, h) \cdot m := gmh^{-1}$, for $(g, h) \in G \times H$ and $m \in M$. Similarly a $k(G \times H)$-module can be viewed as a (kG, kH)-bimodule. We will usually switch between these two points of views.

Definition 2.1. *Let G be a finite group. A kG-module M is called a permutation module, if M has a G-stable k-basis. A p-permutation kG-module is a kG-module M such that $\mathrm{Res}_S^G M$ is a permutation kS-module for a Sylow p-subgroup S of G.*

For a finite group G we denote by $T(G)$ the Grothendieck group of p-permutation kG-modules with respect to direct sum decompositions. If M is a p-permutation kG-module, then the class of M in $T(G)$ will be abusively denoted by M. The group $T(G)$ has a ring structure induced by the tensor product of modules over k, and $T(G)$ will be called the *ring of p-permutation modules* of G, for short. If H is another finite group, we set $T(G, H) := T(G \times H)$. We denote by $T^\Delta(G, H)$ the subgroup of $T(G, H)$ spanned by p-permutation $k(G \times H)$-modules whose indecomposable direct summands have twisted diagonal vertices.

Let $\mathcal{P}_{G,p}$ denote the set of pairs (P, E) where P is a p-subgroup of G and E is a projective indecomposable $k\overline{N}_G(P)$-module. The group G acts on the set $\mathcal{P}_{G,p}$ via conjugation and we denote by $[\mathcal{P}_{G,p}]$ a set of representatives of G-orbits of $\mathcal{P}_{G,p}$. For $(P, E) \in \mathcal{P}_{G,p}$, let $M_{P,E}$ denote the unique (up to isomorphism) indecomposable p-permutation kG-module with the property that $M_{P,E}[P] \cong E$. Note that $M_{P,E}$ has the group P as a vertex [4, Theorem 3.2]. We denote by $\mathcal{P}^\Delta_{G\times H,p}$ the set of pairs $(P, E) \in \mathcal{P}_{G\times H,p}$ where P is a twisted diagonal p-subgroup of $G \times H$.

Remark 2.2. *The isomorphism classes of the modules $M_{P,E}$ where $(P, E) \in \mathcal{P}^\Delta_{G\times H,p}$ form a $\mathbb{Z}$-basis for $T^\Delta(G, H)$.*

Definition 2.3. *[5, Definition 2.3.1] Let (P, s) be a pair where P is a p-group and s is a generator of a p'-cyclic group acting on P. We denote the semidirect product $P \rtimes \langle s\rangle$ by $\langle Ps\rangle$. Let (Q, t) be another such pair. We say that the pairs (P, s) and (Q, t) are isomorphic if there are group isomorphisms $\phi : P \to Q$ and $\psi : \langle s\rangle \to \langle t\rangle$ such that $\psi(s) = q \cdot t$ for some $q \in Q$ and $\phi(s \cdot u) = \psi(s) \cdot \phi(u)$ for all $u \in P$. In that case we write $(P, s) \simeq (Q, t)$.*

Lemma 2.4. *[5, Proposition 2.3.3] Let (P, s) and (Q, t) be two pairs. Then $(P, s) \simeq (Q, t)$ if and only if there is a group isomorphism $f : \langle Ps\rangle \to \langle Qt\rangle$ such that $f(s)$ is conjugate to t.*

Let $\mathcal{Q}_{G,p}$ denote the set of pairs (P, s) where P is a p-subgroup of G and $s \in N_G(P)$ is a p'-element. In that case $\langle Ps\rangle$ denotes the semidirect product $P \rtimes \langle s\rangle$ where the action of $\langle s\rangle$ on P is induced by conjugation. The group G acts on the set $\mathcal{Q}_{G,p}$ and we denote by $[\mathcal{Q}_{G,p}]$ a set of representatives of G-orbits. We denote by $\mathcal{Q}^{\Delta}_{G\times H,p}$ the set of pairs $(P, s) \in \mathcal{Q}_{G\times H,p}$ where P is a twisted diagonal p-subgroup of $G \times H$.

For any pair $(P, s) \in \mathcal{Q}_{G,p}$ let $\tau^G_{P,s}$ denote the additive map $T(G) \to \mathbb{F}$ that sends a p-permutation kG-module M to the value of the Brauer character of $M[P]$ at s. The map $\tau^G_{P,s}$ is a ring homomorphism and it extends to an $\mathbb{F}$-algebra homomorphism $\tau^G_{P,s} : \mathbb{F} \otimes_{\mathbb{Z}} T(G) \to \mathbb{F}$. The set $\{\tau^G_{P,s} : (P, s) \in [\mathcal{Q}_{G,p}]\}$ is the set of all species from $\mathbb{F}T(G) := \mathbb{F} \otimes_{\mathbb{Z}} T(G)$ to $\mathbb{F}$ [3, Proposition 2.18].

The algebra $\mathbb{F}T(G)$ is split semisimple and its primitive idempotents $F^G_{P,s}$ are indexed by pairs $(P, s) \in [\mathcal{Q}_{G,p}]$ [3, Corollary 2.19]. If $\phi : \langle s\rangle \to k^\times$ is a group homomorphism, we denote by k_ϕ the $k\langle s\rangle$-module k on which the element s acts as multiplication by $\phi(s)$. Let $\widehat{\langle s\rangle} = \mathrm{Hom}(\langle s\rangle, k^\times)$ denote the set of group homomorphisms. By [3, Theorem 4.12] we have the idempotent formula

$$F^G_{P,s} = \frac{1}{|P||s||C_{\overline{N}_G(P)}(s)|} \sum_{\substack{\varphi\in\widehat{\langle s\rangle}\\ L\leqslant\langle Ps\rangle\\ PL=\langle Ps\rangle}} \tilde{\varphi}(s^{-1})|L|\mu(L, \langle Ps\rangle)\mathrm{Ind}^G_L k^{\langle Ps\rangle}_{L,\varphi},$$

where $k^{\langle Ps\rangle}_{L,\varphi} = \mathrm{Res}^{\langle Ps\rangle}_L \mathrm{Inf}^{\langle Ps\rangle}_{\langle s\rangle} k_\varphi$, and $\tilde{\varphi}$ is the Brauer character of k_φ.

By [5, Proposition 2.7.8] we have another formula

$$F^G_{P,s} = \frac{1}{|C_{N_G(P)}(s)|} \sum_{\substack{\varphi\in\widehat{\langle s\rangle}\\ L\leqslant P\\ L^s=L}} \tilde{\varphi}(s^{-1})|C_L(s)|\mu\big((L, P)^s\big)\mathrm{Ind}^G_{\langle Ls\rangle} k^{\langle Ps\rangle}_{\langle Ls\rangle,\varphi}.$$

Here $\mu\big((-, -)^s\big)$ is the Möbius function of the poset of s-stable subgroups of P.

Lemma 2.5. *For finite groups G and H, the set $\{F_{P,s}^{G\times H} : (P,s) \in [\mathcal{Q}_{G\times H,p}^{\Delta}]\}$ of primitive idempotents form an $\mathbb{F}$-basis for the split semisimple algebra $\mathbb{F}T^{\Delta}(G,H)$.*

Proof. First we will show that we have $F_{P,s}^{G\times H} \in \mathbb{F}T^{\Delta}(G,H)$ whenever $(P,s) \in [\mathcal{Q}_{G\times H,p}^{\Delta}]$. Let $\varphi \in \widehat{\langle s\rangle}$ and $L \leqslant \langle Ps\rangle$. It suffices to show that $\mathrm{Ind}_L^G k_{L,\varphi}^{\langle Ps\rangle} \in \mathbb{F}T^{\Delta}(G,H)$. Since P acts trivially on $\mathrm{Inf}_{\langle s\rangle}^{\langle Ps\rangle} k_\varphi$, the subgroup P is contained in a vertex of k_φ considered as a $k\langle Ps\rangle$-module. But since P is the Sylow p-subgroup of $\langle Ps\rangle$, it follows that P is the vertex of k_φ. Therefore the module $k_{L,\varphi}^{\langle Ps\rangle} = \mathrm{Res}_L^{\langle Ps\rangle}\mathrm{Inf}_{\langle s\rangle}^{\langle Ps\rangle} k_\varphi$ has a vertex contained in $L\cap {}^xP \leqslant P$ for some $x\in\langle Ps\rangle$. Since a subgroup of twisted diagonal subgroup is again twisted diagonal, this means that $k_{L,\varphi}^{\langle Ps\rangle}$ has twisted diagonal vertices. This shows that $\mathrm{Ind}_L^G k_{L,\varphi}^{\langle Ps\rangle} \in \mathbb{F}T^{\Delta}(G,H)$ as desired. Now since the $\mathbb{F}$-dimension of $\mathbb{F}T^{\Delta}(G,H)$ is equal to the cardinality of $[\mathcal{P}_{G\times H,p}^{\Delta}]$, which is equal to the cardinality of $[\mathcal{Q}_{G\times H,p}^{\Delta}]$, it follows that the set $\{F_{P,s}^{G\times H} : (P,s) \in [\mathcal{Q}_{G\times H,p}^{\Delta}]\}$ of primitive idempotents form an $\mathbb{F}$-basis for $\mathbb{F}T^{\Delta}(G,H)$. □

Let G, H and L be finite groups. If X is a (kG,kH)-bimodule and Y is a (kH,kL)-bimodule, then $X\circ Y := X\otimes_{kH} Y$ is a (kG,kL)-bimodule. Extending this product by $\mathbb{F}$-bilinearity, we get a map

$$\mathbb{F}T(G,H)\circ\mathbb{F}T(H,L)\to\mathbb{F}T(G,L).$$

Note that this induces a map

$$\mathbb{F}T^{\Delta}(G,H)\circ\mathbb{F}T^{\Delta}(H,L)\to\mathbb{F}T^{\Delta}(G,L)$$

which is used to define the composition of morphisms in the following category.

Definition 2.6. *Let $\mathbb{F}pp_k^{\Delta}$ be the category with*

- *objects: finite groups*
- $\mathrm{Mor}_{\mathbb{F}pp_k^{\Delta}}(G,H) = \mathbb{F}\otimes_{\mathbb{Z}} T^{\Delta}(H,G) = \mathbb{F}T^{\Delta}(H,G)$.

An $\mathbb{F}$-linear functor from $\mathbb{F}pp_k^{\Delta}$ to $\mathbb{F}$-Mod is called a *diagonal p-permutation functor.* Diagonal p-permutation functors form an abelian category $\mathcal{F}_{pp_k}^{\Delta}$.

3. The Essential Algebra

For a finite group G, the quotient algebra

$$\mathcal{E}^{\Delta}(G) := \mathbb{F}T^{\Delta}(G,G)/\Big(\sum_{|H|<|G|} \mathbb{F}T^{\Delta}(G,H) \circ \mathbb{F}T^{\Delta}(H,G)\Big)$$

is called the *essential algebra* of G.

By [5, Proposition 4.1.2 and Theorem 4.1.12] the algebra

$$\mathcal{E}(G) := \mathbb{F}T(G,G)/\Big(\sum_{|H|<|G|} \mathbb{F}T(G,H) \circ \mathbb{F}T(H,G)\Big)$$

is non-zero if and only if there exists a pair (P,s) in G such that $G = \langle Ps \rangle$ and $C_{\langle s \rangle}(P) = 1$. In that case, we also have an algebra isomorphism

$$\mathcal{E}(G) \cong \big(\mathbb{F}[X]/\Phi_n[X]\big) \rtimes \mathrm{Out}(G)$$

where n is the order of s [5, Theorem 4.1.12].

Note that the inclusion map $\mathbb{F}T^{\Delta}(G,G) \hookrightarrow \mathbb{F}T(G,G)$ induces a map

$$\Theta : \mathcal{E}^{\Delta}(G) \to \mathcal{E}(G).$$

We will show that this map is an algebra isomorphism.

Let $\varphi \in \mathrm{Aut}(G)$ be an automorphism and $\lambda : G/O_p(G) \to k^{\times}$ be a character, where $O_p(G)$ denotes the largest normal p-subgroup of G. We define a (kG,kG)-bimodule structure on kG, denoted by $kG_{\varphi,\lambda}$, via

$$a \cdot g \cdot b := \lambda(b) a g \varphi(b)$$

for $a,b,g \in G$.

Let $\langle Rt \rangle$ be a twisted diagonal subgroup of $G \times G$ with $p_1(\langle Rt \rangle) = G$ and $p_2(\langle Rt \rangle) = G$. Let also $\eta : p_1(\langle Rt \rangle) \to p_2(\langle Rt \rangle)$ be the canonical isomorphism. Then by [5, Section 4.1.2] we have an isomorphism

$$\mathrm{Ind}_{\langle Rt \rangle}^{G\times G} k_{\langle Rt \rangle,\varphi}^{\langle Rt \rangle} \cong kG_{\eta^{-1},\varphi^{-1}}$$

of (kG,kG)-bimodules. Again by [5, Section 4.1.2] the algebra $\mathcal{E}(G)$ is generated by the images of $kG_{\varphi,\lambda}$.

Proposition 3.1. *If the essential algebra $\mathcal{E}^{\Delta}(G)$ of a finite group G is non-zero, then there exists a pair (P,s) in G such that $G = \langle Ps \rangle$ and $C_{\langle s \rangle}(P) = 1$.*

Proof. Let (Q,t) be a pair contained in $G\times G$ such that Q is a twisted diagonal subgroup and recall the idempotent formula

$$F_{Q,t}^{G\times G}=\frac{1}{|C_{N_{G\times G(Q)}(t)}|}\sum_{\substack{\varphi\in\widehat{\langle t\rangle}\\ L\leqslant Q\\ L^t=L}}\tilde{\varphi}(t^{-1})|C_L(t)|\mu((L,Q)^t)\mathrm{Ind}_{\langle Lt\rangle}^{G\times G}k_{L,\varphi}^{\langle Qt\rangle}.$$

By [5, Lemma 2.5.9] we have an isomorphism

$$\begin{aligned}\mathrm{Ind}_{\langle Lt\rangle}^{G\times G}k_{L,\varphi}^{\langle Qt\rangle}&\cong \mathrm{Ind}_{p_1(\langle Lt\rangle)}^{G}\otimes_{p_1(\langle Lt\rangle)}\mathrm{Ind}_{\langle Lt\rangle}^{p_1(\langle Lt\rangle)\times p_2(\langle Lt\rangle)}(k_{L,\varphi}^{\langle Qt\rangle})\otimes_{p_2(\langle Lt\rangle)}\mathrm{Res}_{p_2(\langle Lt\rangle)}^{G}\\ &\cong kG\otimes_{p_1(\langle Lt\rangle)}\mathrm{Ind}_{\langle Lt\rangle}^{p_1(\langle Lt\rangle)\times p_2(\langle Lt\rangle)}(k_{L,\varphi}^{\langle Qt\rangle})\otimes_{p_2(\langle Lt\rangle)}kG\end{aligned}$$

of (kG,kG)-bimodules. As (kG,kG)-bimodule, we have the isomorphism $kG\cong\mathrm{Ind}_{\Delta G}^{G\times G}k$. Thus as $(kG,kp_1(\langle Lt\rangle))$-bimodule we have,

$$\mathrm{Res}_{G\times p_1(\langle Lt\rangle)}^{G\times G}kG\cong\mathrm{Res}_{G\times p_1(\langle Lt\rangle)}^{G\times G}\mathrm{Ind}_{\Delta G}^{G\times G}k\cong\mathrm{Ind}_{\Delta(p_1(\langle Lt\rangle))}^{G\times p_1(\langle Lt\rangle)}\mathrm{Res}_{\Delta(p_1(\langle Lt\rangle))}^{\Delta(G)}k.$$

Therefore as $(kG\times p_1(\langle Lt\rangle))$-module, the indecomposable direct summands of kG have vertices contained in $\Delta(p_1(\langle Lt\rangle))$. Similary, one can show that the indecomposable direct summands of kG as $k(p_2(\langle Lt\rangle)\times G)$-module, have vertices contained in $\Delta(p_2(\langle Lt\rangle))$. We also know that the module $k_{L,\varphi}^{\langle Qt\rangle}$, and hence the indecomposable direct summands of $\mathrm{Ind}_{\langle Lt\rangle}^{p_1(\langle Lt\rangle)\times p_2(\langle Lt\rangle)}(k_{L,\varphi}^{\langle Qt\rangle})$, have twisted diagonal vertices. Now suppose $\mathcal{E}^{\Delta}(G)$ is non-zero. Then there is an idempotent $F_{Q,t}^{G\times G}$ whose image in $\mathcal{E}^{\Delta}(G)$ is non-zero. Therefore the argument above shows that there is a pair (Q,t) in $G\times G$ such that $p_1(\langle Qt\rangle)=G$ and $p_2(\langle Qt\rangle)=G$. This implies that there is a p-subgroup P of G and a p'-element s of G that normalises P such that $G=\langle Ps\rangle$. Now we will show that in that case we have $C_{\langle s\rangle}(P)=1$.
Let $\overline{G}:=G/C_{\langle s\rangle}(P)$, $Q:=\{(u,\overline{u}:u\in P\}\leqslant G\times\overline{G}$ and $Q':=\{(\overline{u},u):u\in P\}\leqslant\overline{G}\times G$. Then by [5, Proof of Proposition 4.1.2] we have an isomorphism between kG and

$$\bigoplus_i \mathrm{Indinf}_{\overline{N}_{G\times\overline{G}}(Q)}^{G\times\overline{G}}\big(kC_G(P)/C_{\langle s\rangle}(P)\otimes_k k_{\alpha_i}\big)\otimes_{k\overline{G}}\mathrm{Indinf}_{\overline{N}_{\overline{G}\times G}(Q')}^{\overline{G}\times G}\big(kC_G(P)/C_{\langle s\rangle}(P)\otimes_k k_{\alpha'_i}\big)$$

as (kG,kG)-bimodules, where $\mathrm{Indinf}_{\overline{N}_{G\times\overline{G}}(Q)}^{G\times\overline{G}}=\mathrm{Ind}_{N_{G\times\overline{G}}(Q)}^{G\times\overline{G}}\circ\mathrm{Inf}_{\overline{N}_{G\times\overline{G}}(Q)}^{N_{G\times\overline{G}}(Q)}$. Here α_i and α'_i run over the irreducible characters of $\langle s\rangle$. Again by [5, Proof of Proposition 4.1.2] for each i, the modules $kC_G(P)/C_{\langle s\rangle}(P)\otimes_k k_{\alpha_i}$ and $kC_G(P)/C_{\langle s\rangle}(P)\otimes_k k_{\alpha'_i}$

are projective indecomposable $k\overline{N}_{G\times\overline{G}}(Q)$-modules and $k\overline{N}_{\overline{G}\times G}(Q')$-modules respectively. Now since $kC_G(P)/C_{\langle s\rangle}(P)\otimes_k k_{\alpha_i}$ is projective indecomposable, it has the trivial group as vertex. So $\mathrm{Inf}_{\overline{N}_{G\times\overline{G}}(Q)}^{N_{G\times\overline{G}}(Q)}\big(kC_G(P)/C_{\langle s\rangle}(P)\otimes_k k_{\alpha_i}\big)$ has the group Q as a vertex. Note that the group Q is twisted diagonal. Therefore indecomposable direct summands of $\mathrm{Indinf}_{\overline{N}_{G\times\overline{G}}(Q)}^{G\times\overline{G}}\big(kC_G(P)/C_{\langle s\rangle}(P)\otimes_k k_{\alpha_i}\big)$ have twisted diagonal vertices, i.e. $\mathrm{Indinf}_{\overline{N}_{G\times\overline{G}}(Q)}^{G\times\overline{G}}\big(kC_G(P)/C_{\langle s\rangle}(P)\otimes_k k_{\alpha_i}\big)\in\mathbb{F}T^{\Delta}(G,\overline{G})$. Similarly, we have $\mathrm{Indinf}_{\overline{N}_{\overline{G}\times G}(Q')}^{\overline{G}\times G}\big(kC_G(P)/C_{\langle s\rangle}(P)\otimes_k k_{\alpha_i'}\big)\in\mathbb{F}T^{\Delta}(\overline{G},G)$. Now since $\mathcal{E}^{\Delta}(G)\neq 0$, the image of identity element $kG\in\mathbb{F}T^{\Delta}(G,G)$ in $\mathcal{E}^{\Delta}(G)$ is non-zero. Hence we have $\overline{G}=G$, i.e. $C_{\langle s\rangle}(P)=1$. □

Suppose we have $G=\langle Ps\rangle$ and $C_{\langle s\rangle}(P)=1$. The essential algebra $\mathcal{E}^{\Delta}(G)$ is generated by the images of the primitive idempotents

$$F_{Q,t}^{G\times G}=\frac{1}{\mid C_{N_{G\times G(Q)}(t)}\mid}\sum_{\substack{\varphi\in\widehat{\langle t\rangle}\\ L\leqslant Q\\ L^t=L}}\tilde{\varphi}(t^{-1})|C_L(t)|\mu((L,Q)^t)\mathrm{Ind}_{\langle Lt\rangle}^{G\times G}k_{L,\varphi}^{\langle Qt\rangle}$$

where Q is a twisted diagonal subgroup of $G\times G$. By [5, Lemma 2.5.9], if the image of $\mathrm{Ind}_{\langle Lt\rangle}^{G\times G}k_{L,\varphi}^{\langle Qt\rangle}$ is non-zero, then we must have that $p_1(\langle Lt\rangle)=G=p_2(\langle Lt\rangle)$. Write $t=(u,v)$. Then $p_1(\langle Lt\rangle)=\langle p_1(L)u\rangle$ and $p_2(\langle Lt\rangle)=\langle p_2(L)v\rangle$. Therefore we have $|u|=|v|=|s|$. Being a subgroup of twisted diagonal subgroup Q, the group L itself is also twisted diagonal. Since $k_1(L)=k_2(L)=1$ and $|u|=|v|=|s|$, we have $k_1(\langle Lt\rangle)=k_2(\langle Lt\rangle)=1$. This shows that the subgroup $\langle Lt\rangle$ is twisted diagonal and $p_1(\langle Lt\rangle)=G=p_2(\langle Lt\rangle)$. Since the images of $\mathrm{Ind}_{\langle Lt\rangle}^{G\times G}k_{L,\varphi}^{\langle Qt\rangle}$ in $\mathcal{E}(G)$ with $\langle Lt\rangle$ satisfying these properties, generate the non-zero algebra $\mathcal{E}(G)$, this shows that the algebra $\mathcal{E}^{\Delta}(G)$ is also non-zero and the map $\Theta:\mathcal{E}^{\Delta}(G)\to\mathcal{E}(G)$ is surjective. Thus we have proved the following:

Proposition 3.2. *The essential algebra $\mathcal{E}^{\Delta}(G)$ is non-zero if and only if there is a pair (P,s) in G such that $G=\langle Ps\rangle$ and $C_{\langle s\rangle}(P)=1$. Moreover the map Θ : $\mathcal{E}^{\Delta}(G)\to\mathcal{E}(G)$ is surjective.*

Suppose we have $G=\langle Ps\rangle$ for some pair and $C_{\langle s\rangle}(P)=1$. We will show that the map $\Theta:\mathcal{E}^{\Delta}(G)\to\mathcal{E}(G)$ is also injective.
Suppose an element $\sum\overline{r_{\varphi,\alpha}kG_{\varphi,\alpha}}\in\mathcal{E}^{\Delta}(G)$ is mapped to zero by Θ. Then the element $\sum\overline{r_{\varphi,\alpha}kG_{\varphi,\alpha}}$ of $\mathcal{E}(G)$ is zero. Write

$$\sum r_{\varphi,\alpha}kG_{\varphi,\alpha}=\sum_{|H|<|G|}t_{H,U_H,V_H}U_H\otimes_{kH}V_H$$

for some (kG, kH)-bimodule U_H and (kH, kG)-bimodule V_H and some constants $t_{H,U_H,V_H} \in \mathbb{F}$. Suppose the coefficient t_{H,U_H,V_H} is non-zero for some group H. Then as in [5] we can assume that $H = \langle Rt \rangle$ for some pair (R, t) and that the modules U_H and V_H are indecomposable. By [5, Section 4.1] one has

$$U_H \otimes_{kH} V_H \cong \mathrm{Indinf}_{\overline{N}_{G\times G}(\Delta(P))}^{G\times G} \bigoplus_i \big(kZ(P) \otimes k_{\lambda_i}\big)^{n_i}$$

where λ_i is a character of $\langle s \rangle$ and $n_i \in \mathbb{N}$. Again by [5, Section 4.1] each summand $kZ(P) \otimes k_{\lambda_i}$ is a projective indecomposable $k\overline{N}_{G\times G}(\Delta(P))$-module. This shows that if the the coefficient t_{H,U_H,V_H} is non-zero, then the indecomposable direct summands of the bimodule $U_H \otimes_{kH} V_H$ have twisted diagonal vertices. Therefore the element $\sum \overline{r_{\varphi,\alpha} kG_{\varphi,\alpha}}$ is zero in $\mathcal{E}^\Delta(G)$. This proves that the map $\Theta : \mathcal{E}^\Delta(G) \to \mathcal{E}(G)$ is injective. We summarise our results as a theorem below.

Theorem 3.3. *The essential algebra $\mathcal{E}^\Delta(G)$ is non-zero if and only if there is a pair (P, s) in G such that $G = \langle Ps \rangle$ and $C_{\langle s \rangle}(P) = 1$. In that case, the algebra $\mathcal{E}^\Delta(G)$ is isomorphic to the algebra $\big(\mathbb{F}[X]/\Phi_n[X]\big) \rtimes \mathrm{Out}(G)$ where n is the order of s.*

4. D^Δ-pairs

Let $H \leqslant G$ be a subgroup. The (kG, kH)-bimodule kG is denoted by Ind_H^G and (kH, kG)-bimodule kG is denoted by Res_H^G. Similarly, if $N \trianglelefteq G$ is a normal subgroup, the $(kG/N, kG)$-bimodule kG/N is denoted by $\mathrm{Def}_{G/N}^G$ and $(kG, kG/N)$-bimodule kG/N is denoted by $\mathrm{Inf}_{G/N}^G$. This notation is consistent with our previous use of induction, restriction, inflation and deflation symbols, in the sense that for example, if M is a kH-module, then the induced module $\mathrm{Ind}_H^G M$ is isomorphic to $\mathrm{Ind}_H^G \otimes_{kH} M$.

We have the following lemma due to [3] and [5].

Lemma 4.1. *(i) Let $(P, s) \in \mathcal{Q}_{G,p}$ be a pair and $H \leqslant G$ be a subgroup. Then we have*

$$\mathrm{Res}_H^G F_{P,s}^G = \sum_{Q,t} F_{Q,t}^H$$

where (Q, t) runs over a set of representatives of H-conjugacy classes of G-conjugates of (P, s) contained in H.

(ii) *Let* $(Q,t) \in \mathcal{Q}_{H,p}$ *be a pair and* $H \leqslant G$ *be a subgroup. Then we have*

$$\mathrm{Ind}_H^G F_{Q,t}^H = |N_G(Q,t) : N_H(Q,t)| F_{Q,t}^G.$$

(iii) *Let* $N \trianglelefteq G$ *and* $(P,s) \in \mathcal{Q}_{G/N,p}$. *Then*

$$\mathrm{Inf}_{G/N}^G F_{P,s}^{G/N} = \sum_{Q,t} F_{Q,t}^G$$

where (Q,t) *runs over a set of representatives of* G*-conjugacy classes of pairs in* $\mathcal{Q}_{G,p}$ *such that* $QN/N = {}^{\overline{g}}P$ *and* $\overline{t} = {}^{g}s$ *for some* $\overline{g} \in G/N$.

(iv) *Let* $N \trianglelefteq G$ *and* $(P,s) \in \mathcal{Q}_{G,p}$. *Then*

$$\mathrm{Def}_{G/N}^G F_{P,s}^G = m_{P,s,N} \cdot F_{Q,t}^{G/N}$$

for some pair $(Q,t) \in \mathcal{Q}_{G/N,p}$ *and a constant* $m_{P,s,N} \in \mathbb{F}$.
If $G = \langle Ps \rangle$ *then*

$$\mathrm{Def}_{G/N}^G F_{P,s}^G = m_{P,s,N} \cdot F_{PN/N,\overline{s}}^{G/N}.$$

Proof. See [3, Proposition 3.1. and Proposition 3.2.] for (i) and (ii), [5, Proposition 3.1.3] for (iii) and [5, Lemma 3.1.4 and Proposition 3.1.5] for (iv). □

Lemma 4.2. *Let* $N \trianglelefteq G$ *be a normal subgroup of* G.

(i) *We have* $\mathrm{Def}_{G/N}^G \in \mathbb{F}T^\Delta(G/N, G)$ *if and only if* N *is a* p'*-group.*

(ii) *We have* $\mathrm{Inf}_{G/N}^G \in \mathbb{F}T^\Delta(G, G/N)$ *if and only if* N *is a* p'*-group.*

Proof. (i) Let $Q \leqslant (G/N) \times G$ be a maximal vertex of an indecomposable direct summand of the $(kG/N, kG)$-bimodule kG/N. Equivalently Q is a maximal p-subgroup having a fixed point on the set G/N. Suppose $(aN, b) \in Q$ stabilises a basis element gN of kG/N. Then we have $(aN)gNb^{-1} = gN$ which implies that $a^g \cdot b^{-1} \in N$. Since the vertices of an indecomposable module are conjugate, we may assume that $g = 1$. Thus, up to conjugacy, Q is a Sylow p-subgroup of

$$H = \{(aN, b) : ab^{-1} \in N\} \leqslant (G/N) \times G.$$

Note that $k_1(Q) = k_1(H) = 1$ and $k_2(Q)$ is a Sylow p-subgroup of N. Hence Q is twisted diagonal if and only if N is a p'-group. The result follows.
(ii) Similar. □

Let (P, s) be a pair and suppose $G = \langle Ps\rangle$. Then by [5, Corollary 3.1.9] for any normal subgroup N of G, we have the following formula for the constant $m_{P,s,N}$:

$$m_{P,s,N} = \frac{|s|}{|N \cap \langle s\rangle||C_G(s)|} \sum_{\substack{Q \leqslant P \\ Q^s = Q \\ \langle Qs\rangle N = G}} |C_Q(s)|\mu\big((Q, P)^s\big).$$

Lemma 4.3. *Let (P, s) be a pair and suppose $G = \langle Ps\rangle$. Then for any normal p'-subgroup N of G we have*

$$m_{P,s,N} = \frac{1}{|N|}.$$

Proof. First observe that since N is a p'-group, we have $N \leqslant C_{\langle s\rangle}(P)$. For any subgroup Q of P the condition $\langle Qs\rangle N = \langle Ps\rangle$ implies that $|Q| = |P|$ and hence $Q = P$. Therefore the formula above becomes

$$m_{P,s,N} = \frac{|s||C_P(s)|}{|N||C_G(s)|} = \frac{1}{|N|}.$$

□

Definition 4.4. *A pair (P, s) is called D^Δ-pair if $\mathrm{Def}^{\langle Ps\rangle}_{\langle Ps\rangle/N} F^{\langle Ps\rangle}_{P,s} = 0$ for any nontrivial normal p'-subgroup N of $\langle Ps\rangle$.*

Lemma 4.5. *Let (P, s) be a pair. Then (P, s) is a D^Δ-pair if and only if the group $\langle Ps\rangle$ does not have any nontrivial normal p'-subgroup, that is, if and only if $C_{\langle s\rangle}(P) = 1$.*

Proof. By Lemma 4.3, for any normal p'-subgroup $N \trianglelefteq \langle Ps\rangle$ we have $m_{P,s,N} = 1/|N|$. Therefore (P, s) is a D^Δ-pair if and only if the group $\langle Ps\rangle$ does not have any nontrivial normal p'-subgroup. The result follows. □

5. The functor $\mathbb{F}T^\Delta$

By [2], the simple diagonal p-permutation functors are parametrized by the pairs (G, V) where G is a finite group and V is a simple $\mathcal{E}^\Delta(G)$-module. Note that this implies $\mathcal{E}^\Delta(G) \neq 0$.

For a simple $\mathcal{E}^\Delta(G)$-module V, we define two functors in $\mathbb{F}pp^\Delta_k$ by:

$$L_{G,V}(H) := \mathbb{F}T^\Delta(H, G) \otimes_{\mathcal{E}^\Delta(G)} V$$

and

$$J_{G,V}(H) := \left\{ \sum_i \phi_i \otimes v_i \in L_{G,V} : \forall \psi \in \mathbb{F}T^{\Delta}(G,H), \sum_i (\psi \circ \phi_i) \cdot v_i = 0 \right\},$$

for any finite group H. The action of morphisms in $\mathbb{F}pp_k^{\Delta}$ on these evaluations is given by left composition. The functor $J_{G,V}$ is the unique maximal subfunctor of $L_{G,V}$, so the quotient

$$S_{G,V} := L_{G,V}/J_{G,V}$$

is a simple functor [2].

Let $\mathbb{F}T^{\Delta} : \mathbb{F}pp_k^{\Delta} \to \mathbb{F}\text{-Mod}$ be the functor given by

- $\mathbb{F}T^{\Delta}(G) := \mathbb{F} \otimes_{\mathbb{Z}} T(G) = \mathbb{F}T(G)$,
- $\mathbb{F}T^{\Delta}(X) : \mathbb{F}T(G) \to \mathbb{F}T(H), M \mapsto X \otimes_{kH} M$ for any $X \in \mathbb{F}T^{\Delta}(H,G)$.

For any kG-module X, we denote by $\widetilde{X}$ the (kG,kG)-bimodule $k(G \times X)$ where the action of kG-kG is given by

$$a \cdot (g,x) \cdot b^{-1} := (agb, b^{-1}x)$$

for all $a,b,g \in G$ and $x \in X$. We have an isomorphism of (kG,kG)-bimodules

$$\widetilde{X} \cong \operatorname{Ind}_{\delta(G)}^{G \times G^{op}} \operatorname{Iso}(\delta)(X)$$

where $\delta : G \to G \times G^{op}$, $g \mapsto (g, g^{-1})$. See [5, Definition 2.5.17]. Note that the image $\delta(G)$ of G in $G \times G^{op}$ is a twisted diagonal subgroup. If X is an indecomposable p-permutation kG-module with a vertex Q, then any vertex of an indecomposable direct summand of $\widetilde{X}$ is contained in $\delta(Q)$, up to conjugation. Therefore for any $X \in \mathbb{F}T(G)$ we have $\widetilde{X} \in \mathbb{F}T^{\Delta}(G,G)$.

Lemma 5.1. *Let F be a subfunctor of $\mathbb{F}T^{\Delta}$. Then for any finite group G, the $\mathbb{F}$-vector space $F(G)$ is an ideal of the algebra $\mathbb{F}T^{\Delta}(G)$ of p-permutation modules.*

Proof. Let $Y \in F(G)$ and assume X is a p-permutation kG-module. By [5, Proposition 2.5.18] we have an isomorphism $X \otimes_k Y \cong \widetilde{X} \otimes_{kG} Y$ of kG-modules. Since $\widetilde{X} \in \mathbb{F}T^{\Delta}(G,G)$ and F is a functor, we have $\widetilde{X} \otimes_{kG} Y \in F(G)$. This shows that $F(G)$ is an ideal of $\mathbb{F}T^{\Delta}(G)$. □

Definition 5.2. *For any pair (P,s) let $\mathbf{e}_{P,s}$ denote the subfunctor of $\mathbb{F}T^{\Delta}$ generated by the idempotent $F_{P,s}^{\langle Ps \rangle} \in \mathbb{F}T^{\Delta}\big(\langle Ps \rangle\big)$.*

Proposition 5.3. *Let F be a subfunctor of $\mathbb{F}T^\Delta$. Then we have*

$$F = \sum_{\mathbf{e}_{P,s} \leqslant F} \mathbf{e}_{P,s}.$$

Proof. Since F is a subfunctor, we have

$$\sum_{\mathbf{e}_{P,s} \leqslant F} \mathbf{e}_{P,s} \leqslant F.$$

Now let G be a finite group, and $u = \sum_{(P,s)} \lambda_{P,s} F_{P,s}^G$, where (P,s) runs in a set of representatives of G-conjugacy classes of $\mathcal{Q}_{G,p}$, and $\lambda_{P,s} \in \mathbb{F}$. Then $F_{P,s}^G \cdot u = \lambda_{P,s} F_{P,s}^G \in F(G)$, since $F(G)$ is an ideal of $\mathbb{F}T^\Delta(G)$. Hence $F_{P,s}^G \in F(G)$ if $\lambda_{P,s} \neq 0$. In this case we have $\mathrm{Res}^G_{\langle Ps \rangle} F_{P,s}^G \in F\big(\langle Ps \rangle\big)$, which implies by Lemma 4.1 that $F_{P,s}^{\langle Ps \rangle} \in F\big(\langle Ps \rangle\big)$. This shows that $\mathbf{e}_{P,s} \leqslant F$. By Lemma 4.1 again, $F_{P,s}^G$ is a non zero scalar multiple of $\mathrm{Ind}^G_{\langle Ps \rangle} F_{P,s}^{\langle Ps \rangle}$, so $F_{P,s}^G \in \mathbf{e}_{P,s}(G)$, which gives finally

$$u \in \sum_{\mathbf{e}_{P,s} \leqslant F} \mathbf{e}_{P,s}(G).$$

Therefore we have

$$F = \sum_{\mathbf{e}_{P,s} \leqslant F} \mathbf{e}_{P,s}$$

as desired. □

Proposition 5.4. *Let $(P_i, s_i)_{i \in I}$ be a set of pairs for an indexing set I. Then for any pair (Q,t) we have $\mathbf{e}_{Q,t} \leqslant \sum_{i \in I} \mathbf{e}_{P_i,s_i}$ if and only if $\mathbf{e}_{Q,t} \leqslant \mathbf{e}_{P_i,s_i}$ for some $i \in I$.*

Proof. If $\mathbf{e}_{Q,t} \leqslant \mathbf{e}_{P_i,s_i}$ for some $i \in I$, then we obviously have $\mathbf{e}_{Q,t} \leqslant \sum_{i \in I} \mathbf{e}_{P_i,s_i}$. Conversely assume we have $\mathbf{e}_{Q,t} \leqslant \sum_{i \in I} \mathbf{e}_{P_i,s_i}$. Then $\mathbf{e}_{Q,t}\big(\langle Qt \rangle\big) \leqslant \sum_{i \in I} \mathbf{e}_{P_i,s_i}\big(\langle Qt \rangle\big)$ and so $F_{Q,t}^{\langle Qt \rangle} \in \sum_{i \in I} \mathbf{e}_{P_i,s_i}\big(\langle Qt \rangle\big)$. Since $F_{Q,t}^{\langle Qt \rangle}$ is a primitive idempotent and since $\mathbf{e}_{P_i,s_i}\big(\langle Qt \rangle\big)$ is an ideal of $\mathbb{F}T^\Delta\big(\langle Qt \rangle\big)$ it follows that we have $F_{Q,t}^{\langle Qt \rangle} \in \mathbf{e}_{P_i,s_i}\big(\langle Qt \rangle\big)$ for some $i \in I$ and hence $\mathbf{e}_{Q,t} \leqslant \mathbf{e}_{P_i,s_i}$. □

Let G be a finite group and $(P,s) \in \mathcal{Q}_{G,p}$ be a pair such that $G = \langle Ps \rangle$. Let also $(Q,t) \in \mathcal{Q}^\Delta_{H \times G,p}$ for a finite group H. Suppose that $\eta : p_1(Q) \to p_2(Q)$ is the canonical isomorphism. Up to conjugation in $H \times G$, we can assume $t = (u, s^j)$. By [5, Section 3.2] if $p_2(\langle Qt \rangle) \neq G$, then the product $F_{Q,t}^{H \times G} \otimes_{kG} F_{P,s}^G$ is zero. So assume that we have $p_2(\langle Qt \rangle) = G$. This implies that we have $p_2(Q) = P$ and $|s^j| = |s|$.

Then since $k_1(Q) = k_2(Q) = 1$, this implies that we have $p_1(Q) \cong P$. Since the group Q is t-stable, the isomorphism $\eta : p_1(Q) \to P$ commutes with conjugations by u and s^j. Now [5, Equation (3.3), Section 3.2] implies that as kH-module the product $F^{H\times G}_{Q,t} \otimes_{kG} F^G_{P,s}$ is equal to

$$\frac{1}{|C_{N_{H\times G}}(Q)(t)||C_G(s)|}\sum_{\substack{\varphi\in\overline{\langle t\rangle}\\ \psi\in\overline{\langle s\rangle}\\ \varphi^{|u|}\psi^{j|u|}=1}} \tilde{\varphi}(t)^{-1}\tilde{\psi}(s)^{-1}|C_Q(t)|\sum_{\substack{J\leqslant p_1(Q)\\ J^u=J}}\sigma(J)\mathrm{Ind}^H_{\langle Ju\rangle}(k^{\langle p_1(Q)u\rangle}_{\langle Ju\rangle,\phi})$$

where $\sigma(J) := \sum_{\substack{L\leqslant P\\ L^s=L\\ \eta(J)=L}} |C_L(s)|\mu\big((L,P)^s\big)$ and $\phi(u) := \varphi(u,s^j)\psi(s)^j$.

Suppose we have $H = \langle P's'\rangle$ for a pair (P',s'). Then by [5, Lemma 2.7.6] if $\tau^H_{P',s'}(F^{H\times G}_{Q,t} \otimes_{kG} F^G_{P,s}) \neq 0$, then we must have $p_1(Q) = P'$ and $|u| = |s'|$. This implies in particular that we must have $P' \cong P$. Moreover again by [5, Lemma 2.7.6] we have $\tau^H_{P',s'}\big(\mathrm{Ind}^H_{\langle Ju\rangle}(k^{\langle p_1(Q)u\rangle}_{\langle Ju\rangle,\phi})\big) = 0$ if $J \neq P'$. Therefore if we have $P' \cong P$ then $\tau^H_{P',s'}(F^{H\times G}_{Q,t} \otimes_{kG} F^G_{P,s})$ is equal to

$$\frac{1}{|C_{N_{H\times G}}(Q)(t)||C_G(s)|}\sum_{\substack{\varphi\in\overline{\langle t\rangle}\\ \psi\in\overline{\langle s\rangle}\\ \varphi^{|u|}\psi^{j|u|}=1}} \tilde{\varphi}(t)^{-1}\tilde{\psi}(s)^{-1}|C_Q(t)||C_P(s)|\tilde{\phi}(s').$$

This shows that if we have $\mathbb{F}T^\Delta\big(\langle P's'\rangle,\langle Ps\rangle\big) \otimes_{k\langle Ps\rangle} F^{\langle Ps\rangle}_{P,s} \neq 0$, then there is an isomorphism $\eta : P' \to P$ and a p'-element $(u,s^j) \in \langle P's'\rangle \times \langle Ps\rangle$ such that $\eta\circ c_u = c_{s^j}\circ\eta$ and $|u| = |s'|$, $|s^j| = |s|$. In that case, assume further that $C_{\langle s\rangle}(P) = 1$. Then we have $|c_s| = |s|$ and $|c_{s^j}| = |s^j|$. Since we have $\eta\circ c_u = c_{s^j}\circ\eta$ it follows that $|c_u| = |c_{s^j}|$. Therefore we have $|s| \mid |s'|$. But then [5, Proposition 2.3.6] implies that there is a surjective group homomorphism $\overline{\eta} : \langle P's'\rangle \to \langle Ps\rangle$ that induces an isomorphism of pairs $\big(P'\ker(\overline{\eta})/\ker(\overline{\eta}), s'\ker(\overline{\eta})\big) \simeq (P,s)$. Note that since $|P'| = |P|$ the order of $\ker(\overline{\eta})$ is coprime to p. We have the following:

Lemma 5.5. *Let (P,s) be a pair with $C_{\langle s\rangle}(P) = 1$ and set $G := \langle Ps\rangle$. Let H be a finite group. The following statements are equivalent:*

(i) $\mathbb{F}T^\Delta(H,G) \otimes_{kG} F^G_{P,s} \neq 0$.

(ii) There exists a pair (P',s') contained in H such that the pair (P,s) is isomorphic to a p'-quotient of the pair (P',s'), that is, there exists a normal p'-subgroup K of $\langle P's'\rangle$ such that $(P,s) \simeq (P'K/K, s'K)$.

Proof. (i) $\Rightarrow$ (ii) Suppose we have $\mathbb{F}T^{\Delta}(H,G)\otimes_{kG}F^G_{P,s}\neq 0$. Then there exists a pair (P',s') in H such that

$$F^H_{P',s'}\in \mathbb{F}T^{\Delta}(H,G)\otimes_{kG}F^G_{P,s}.$$

Via the restriction map this implies that we have

$$F^{\langle P's'\rangle}_{P',s'}\in \mathbb{F}T^{\Delta}\big(\langle P's'\rangle,G\big)\otimes_{kG}F^G_{P,s}.$$

Therefore by the argument above we have an isomorphism $(P'K/K,s'K)\simeq (P,s)$ of pairs where K is a normal p'-subgroup of $\langle P's'\rangle$.
(ii) $\Rightarrow$ (i) Suppose $\Phi:(P'K/K,s'K)\to(P,s)$ is an isomorphism of pairs where K is a normal p'-subgroup of $\langle P's'\rangle$. Then we have

$$\mathrm{Ind}^H_{\langle P's'\rangle}\mathrm{Inf}^{\langle P's'\rangle}_{\langle P's'\rangle/K}\mathrm{Iso}(\Phi)F^G_{P,s}\neq 0.$$

This shows (i). □

Proposition 5.6. *Let (P,s) be a pair. The following are equivalent:*

(i) (P,s) is a D^{Δ}-pair.

(ii) For any finite group H with $|H|<|\langle Ps\rangle|$, we have $\mathbf{e}_{P,s}(H)=\{0\}$.

(iii) If H is a finite group with $\mathbf{e}_{P,s}(H)\neq\{0\}$, then the pair (P,s) is isomorphic to a p'-quotient of a pair (P',s') contained in H.

(iv) For any nontrivial normal p'-subgroup N of $\langle Ps\rangle$, we have $\mathrm{Def}^{\langle Ps\rangle}_{\langle Ps\rangle/N}F^{\langle Ps\rangle}_{P,s}=0$.

(v) The group $\langle Ps\rangle$ does not have any nontrivial normal p'-subgroup.

(vi) We have $C_{\langle s\rangle}(P)=1$.

Proof. (vi)$\Leftrightarrow$(v)$\Leftrightarrow$(i) : This follows from Lemma 4.5.
(iv)$\Leftrightarrow$ (i): This follows from the definition of D^{Δ}-pairs.
(i)$\Rightarrow$ (iii): Since (P,s) is a D^{Δ}-pair, we have $C_{\langle s\rangle}(P)=1$. So (iii) follows from Lemma 5.5.
(iii)$\Rightarrow$ (ii): Assume that (iii) holds and $\mathbf{e}_{P,s}(H)\neq 0$ where H is a finite group with $|H|<|\langle Ps\rangle|$. Then by the assumption, we have $|H|\geq|\langle P's'\rangle|\geq|\langle Ps\rangle|$. Contradiction.
(ii)$\Rightarrow$ (iv): Clear. □

Proposition 5.7. *Let (P,s) and (Q,t) be two pairs.*

(i) If the pair (Q,t) is isomorphic to a p'-quotient of the pair (P,s), then we have $\mathbf{e}_{P,s} = \mathbf{e}_{Q,t}$.

(ii) If (Q,t) is a D^Δ-pair, and if $\mathbf{e}_{P,s} \leqslant \mathbf{e}_{Q,t}$, then (Q,t) is isomorphic to a p'-quotient of (P,s).

Proof. (i) Assume we have an isomorphism $\phi : (PK/K, sK) \to (Q,t)$ of pairs for some normal p'-subgroup K of $\langle Ps\rangle$. Then we have

$$F_{P,s}^{\langle Ps\rangle} \otimes_k \mathrm{Inf}_{\langle Ps\rangle/K}^{\langle Ps\rangle}\mathrm{Iso}(\phi^{-1})F_{Q,t}^{\langle Qt\rangle} \neq 0.$$

Therefore we have $F_{P,s}^{\langle Ps\rangle} \in \mathbf{e}_{Q,t}\big(\langle Ps\rangle\big)$ which implies that $\mathbf{e}_{P,s} \leqslant \mathbf{e}_{Q,t}$.

Now we also have

$$F_{Q,t}^{\langle Qt\rangle} \otimes_k \mathrm{Iso}(\phi)\mathrm{Def}_{\langle Ps\rangle/K}^{\langle Ps\rangle}F_{P,s}^{\langle Ps\rangle} \neq 0$$

which implies that $F_{Q,t}^{\langle Qt\rangle} \in \mathbf{e}_{P,s}\big(\langle Qt\rangle\big)$. Therefore we have $\mathbf{e}_{Q,t} \leqslant \mathbf{e}_{P,s}$ and so $\mathbf{e}_{Q,t} = \mathbf{e}_{P,s}$ as desired.
(ii) Since $\mathbf{e}_{P,s} \leqslant \mathbf{e}_{Q,t}$, we have $F_{P,s}^{\langle Ps\rangle} \in \mathbf{e}_{Q,t}\big(\langle Ps\rangle\big)$. Since (Q,t) is a D^Δ-pair, by the proof of Lemma 5.5, there exists a normal p'-subgroup K of $\langle Ps\rangle$ such that $(Q,t) \simeq (PK/K, sK)$. □

Proposition 5.8. *Let F be a nonzero subfunctor of $\mathbb{F}T^\Delta$. If H is a minimal group of F, then $H = \langle Qt\rangle$ for some D^Δ-pair (Q,t). Moreover we have*

$$F(H) \leqslant \bigoplus_{\substack{(Q',t'),D^\Delta-pair\\ \langle Q't'\rangle = H}} \mathbb{F}F_{Q',t'}^H$$

and $\mathbf{e}_{Q,t} \leqslant F$.

In particular, if $F = \mathbf{e}_{Q,t}$ for some D^Δ-pair (Q,t), then we have

$$\mathbf{e}_{Q,t}\big(\langle Qt\rangle\big) = \bigoplus_{\substack{(Q',t')\simeq(Q,t)\\ \langle Q't'\rangle = \langle Qt\rangle}} \mathbb{F}F_{Q',t'}^H.$$

Proof. Let F be a nonzero subfunctor of $\mathbb{F}T^\Delta$ and assume H is a minimal group of F. Since $F(H) \neq 0$, there exists a pair $(Q,t) \in \mathcal{Q}_{H,p}$ such that $F_{Q,t}^H \in F(H)$. This implies, via the restriction map, that we have $F_{Q,t}^{\langle Qt\rangle} \in F\big(\langle Qt\rangle\big)$. Since H is a minimal group, this implies that we have $H = \langle Qt\rangle$. Now if N is a normal p'-subgroup of

$\langle Qt\rangle$, then $\mathrm{Def}^{\langle Qt\rangle}_{\langle Qt\rangle/N}F^{\langle Qt\rangle}_{Q,t} = \frac{1}{|N|}F^{\langle Qt\rangle/N}_{QN/N,tN} \neq 0$. Again since H is a minimal group this means that N is trivial and hence the pair (Q,t) is a D^Δ-pair. It follows moreover that

$$F(H) \leqslant \bigoplus_{\substack{(Q',t'),D^\Delta-pair\\ \langle Q't'\rangle = H}} \mathbb{F}F^H_{Q',t'}.$$

For the last part, consider the functor $\mathbf{e}_{Q,t}$ for some D^Δ-pair (Q,t). If $F^{\langle Qt\rangle}_{Q',t'} \in \mathbf{e}_{Q,t}\big(\langle Qt\rangle\big)$ for some D^Δ-pair (Q',t'), then by the second part of Proposition 5.7, the pair (Q,t) is isomorphic to a p'-quotient of the pair (Q',t'). But the pair (Q',t') is contained in $\langle Qt\rangle$. Thus we have $(Q',t') \simeq (Q,t)$.
Conversely, if the pairs (Q',t') and (Q,t) are isomorphic via a map ϕ, then we have $F^{\langle Qt\rangle}_{Q',t'} = \mathrm{Iso}(\phi)F^{\langle Qt\rangle}_{Q,t}$. Therefore we have

$$\mathbf{e}_{Q,t}\big(\langle Qt\rangle\big) = \bigoplus_{\substack{(Q',t')\simeq(Q,t)\\ \langle Q't'\rangle = \langle Qt\rangle}} \mathbb{F}F^H_{Q',t'}.$$

□

Let (P,s) be a pair and N a normal p'-subgroup of $\langle Ps\rangle$. Then the pair $(PN/N, sN)$ is a p'-quotient of the pair (P,s) and so by Proposition 5.7 we have $\mathbf{e}_{P,s} = \mathbf{e}_{PN/N,sN}$.

Proposition 5.9. *Let (P,s) be a pair. Then the group $\langle Ps\rangle/C_{\langle s\rangle}(P)$ is the unique, up to isomorphism, minimal group of the functor $\mathbf{e}_{P,s}$. Moreover there is a unique isomorphism class of D^Δ-pairs (P',s') such that $\langle P's'\rangle \cong \langle Ps\rangle/C_{\langle s\rangle}(P)$ and we have $\mathbf{e}_{P',s'} = \mathbf{e}_{P,s}$. Furthermore we have $(P',s') \simeq \big(PC_{\langle s\rangle}(P)/C_{\langle s\rangle}(P), sC_{\langle s\rangle}(P)\big)$.*

Proof. Let (P',s') be a D^Δ-pair such that $\langle P's'\rangle$ is a minimal group of the functor $\mathbf{e}_{P,s}$. By Proposition 5.8, we have $\mathbf{e}_{P',s'} \leqslant \mathbf{e}_{P,s}$. Let $N := C_{\langle s\rangle}(P)$. Then the pair $(PN/N, sN)$ is a D^Δ-pair, and we have $\mathbf{e}_{P,s} = \mathbf{e}_{PN/N,sN}$. Since $(PN/N, sN)$ is a D^Δ-pair, by Proposition 5.7 there exists a normal p'-subgroup K of $\langle P's'\rangle$ such that $(P'K/K, s'K) \simeq (PN/N, sN)$. This means that the idempotent $F^{\langle P's'\rangle/K}_{P'K/K,s'K}$ is in the evaluation at $\langle P's'\rangle/K$ of the functor $\mathbf{e}_{PN/N,sN} = \mathbf{e}_{P,s}$. Since the group $\langle P's'\rangle$ is a minimal group of $\mathbf{e}_{P,s}$ it follows that we must have $K = 1$. Thus we have $(P',s') \simeq (PN/N, sN)$. Therefore we have $\mathbf{e}_{P',s'} = \mathbf{e}_{PN/N,sN} = \mathbf{e}_{P,s}$.

Now we will show the uniqueness of the isomorphism class of the minimal groups of $\mathbf{e}_{P,s}$. Let H be a minimal group of $\mathbf{e}_{P,s}$. It suffices to show that H is isomorphic to $\langle P's'\rangle$. By Proposition 5.8 the group H is of the form $H = \langle Qt\rangle$ for some D^Δ-pair (Q,t). By the first part of the proof we have $\mathbf{e}_{Q,t} = \mathbf{e}_{P,s} = \mathbf{e}_{P',s'}$. Since both (Q,t)

and (P,s) are D^Δ-pairs, the equality $\mathbf{e}_{Q,t} = \mathbf{e}_{P',s'}$ implies that (Q,t) is isomorphic to a p'-quotient of (P,s), and vice versa. Therefore we have $(Q,t) \simeq (P',s')$ which implies that $H = \langle Qt\rangle \cong \langle P's'\rangle$ as desired. □

For any pair (P,s) we denote by $(\tilde{P},\tilde{s})$ a representative of the isomorphism class of the pair $(PC_{\langle s\rangle}(P)/C_{\langle s\rangle}(P), sC_{\langle s\rangle}(P))$.

Theorem 5.10. *Let (P,s) be a pair.*

(i) If (Q,t) is isomorphic to a p'-quotient of (P,s) and if (Q,t) is a D^Δ-pair, then (Q,t) is isomorphic to the pair $(\tilde{P},\tilde{s})$. In particular, for any normal p'-subgroup $N \trianglelefteq \langle Ps\rangle$, we have $(PN/N, sN) \simeq (\tilde{P},\tilde{s})$ if and only if $(PN/N, sN)$ is a D^Δ-pair.

(ii) Let $N \trianglelefteq \langle Ps\rangle$ be a normal p'-subgroup. Then the pair $(\tilde{P},\tilde{s})$ is isomorphic to a p'-quotient of $(PN/N, sN)$ and we have $(\tilde{P},\tilde{s}) \simeq (\widetilde{PN/N}, \widetilde{sN})$.

Proof. (i) Since the pair (Q,t) is isomorphic to a p'-quotient of the pair (P,s), by Proposition 5.7, we have $\mathbf{e}_{\tilde{P},\tilde{s}} = \mathbf{e}_{P,s} \leqslant \mathbf{e}_{Q,t}$. Since (Q,t) is a D^Δ-pair, again by Proposition 5.7, the pair (Q,t) is isomorphic to a p'-quotient of $(\tilde{P},\tilde{s})$. But since the pair $(\tilde{P},\tilde{s})$ is a D^Δ-pair, it follows that the pair (Q,t) is isomorphic to the pair $(\tilde{P},\tilde{s})$.

(ii) Since the constant $m_{P,s,N}$ is non-zero, we have $F^{\langle Ps\rangle/N}_{PN/N,sN} \in \mathbf{e}_{P,s}\big(\langle Ps\rangle/N\big) = \mathbf{e}_{\tilde{P},\tilde{s}}\big(\langle Ps\rangle/N\big)$. Therefore we have $\mathbf{e}_{PN/N,sN} \leqslant \mathbf{e}_{\tilde{P},\tilde{s}}$ and since $(\tilde{P},\tilde{s})$ is a D^Δ-pair, by Proposition 5.7, $(\tilde{P},\tilde{s})$ is isomorphic to a p'-quotient of $(PN/N, sN)$. Again since the pair $(\tilde{P},\tilde{s})$ is a D^Δ-pair, by part (i), it is isomorphic to the pair $(\widetilde{PN/N}, \widetilde{sN})$. □

Let [D^Δ*-pair*] denote a set of isomorphism classes of D^Δ-pairs. Then the subfunctor lattice of the functor $\mathbb{F}T^\Delta$ is isomorphic to the lattice of subsets of the set [D^Δ*-pair*] ordered by inclusion.

Theorem 5.11. *Let $\mathcal{S}$ be the lattice of subfunctors of $\mathbb{F}T^\Delta$ ordered by inclusion of subfunctors. Let $\mathcal{T}$ be the lattice of subsets of [D^Δ-pair] ordered by inclusion of subsets. Then the map*

$$\Theta : \mathcal{S} \to \mathcal{T}$$

that sends a subfunctor F to the set $\{(P,s) \in [D^\Delta\text{-}pair] : \mathbf{e}_{P,s} \leqslant F\}$, is an isomorphism of lattices with inverse

$$\Psi : \mathcal{T} \to \mathcal{S}$$

that sends a subset A to the functor $\sum_{(P,s)\in A} \mathbf{e}_{P,s}$.

Proof. We need to show that the maps Θ and Ψ are inverse of each other. Let $F \in \mathcal{S}$ be a subfunctor. By Proposition 5.3 we have

$$F = \sum_{\substack{(P,s)\in\Gamma \\ \mathbf{e}_{P,s}\leqslant F}} \mathbf{e}_{P,s}$$

where Γ is a set of representatives of the isomorphism classes of pairs. But for any pair (P,s) we have $\mathbf{e}_{P,s} = \mathbf{e}_{\tilde{P},\tilde{s}}$ and $(\tilde{P},\tilde{s})$ is a D^{Δ}-pair. Therefore we have

$$F = \sum_{\substack{(P,s)\in[D^{\Delta}\text{-}pair] \\ \mathbf{e}_{P,s}\leqslant F}} \mathbf{e}_{P,s}.$$

This shows that $\Psi(\Theta(F)) = F$.

Now let $A \in \mathcal{T}$ be a subset and let $(Q,t) \in \Theta(\Psi(A))$ be a D^{Δ}-pair. Then we have $\mathbf{e}_{Q,t} \leqslant \sum_{(P,s)\in A} \mathbf{e}_{P,s}$ and so by Proposition 5.4 this implies that we have $\mathbf{e}_{Q,t} \leqslant \mathbf{e}_{P,s}$ for some $(P,s) \in A$. Since both (P,s) and (Q,t) are D^{Δ}-pairs, it follows that $(P,s) \simeq (Q,t)$ and hence $(Q,t) \in A$. This shows that $\Theta(\Psi(A)) \subseteq A$. The inclusion $A \subseteq \Theta(\Psi(A))$ is trivial. Therefore we have $\Theta(\Psi(A)) = A$. □

The following corollary follows immediately from Theorem 5.11.

Corollary 5.12. *We have* $\mathbb{F}T^{\Delta} = \bigoplus_{(P,s)\in[D^{\Delta}\text{-}pair]} \mathbf{e}_{P,s}$.

The first statement of Proposition 5.8 can also be made stronger.

Corollary 5.13. *Let F be a nonzero subfunctor of $\mathbb{F}T^{\Delta}$. If H is a minimal group of F, then $H = \langle Qt \rangle$ for some D^{Δ}-pair (Q,t) and we have*

$$F(H) = \bigoplus_{\substack{(Q',t')\simeq(Q,t) \\ \langle Q't'\rangle=\langle Qt\rangle}} \mathbb{F}F^{H}_{Q',t'}.$$

Proof. Since H is a minimal group of F, by Proposition 5.8 it follows that $H = \langle Qt \rangle$ for some D^{Δ}-pair with the property that $\mathbf{e}_{Q,t} \leqslant F$. By Theorem 5.11 we have

$$F = \sum_{\substack{(Q,t)\in[D^{\Delta}\text{-}pair] \\ \mathbf{e}_{Q,t}\leqslant F}} \mathbf{e}_{Q,t}.$$

Therefore by Proposition 5.8 again we have

$$F(H) = \mathbf{e}_{Q,t}(H) = \bigoplus_{\substack{(Q',t')\simeq(Q,t) \\ \langle Q't'\rangle=\langle Qt\rangle}} \mathbb{F}F^{H}_{Q',t'}$$

as desired. □

Corollary 5.14. *Let (P,s) be a D^Δ-pair. Then the subfunctor $\mathbf{e}_{P,s}$ of $\mathbb{F}T^\Delta$ is isomorphic to the simple functor $S_{\langle Ps\rangle,W_{P,s}}$ where $W_{P,s} = \oplus_{\substack{(Q,t)\simeq(P,s)\\ \langle Qt\rangle=\langle Ps\rangle}} \mathbb{F}F_{P,s}^{\langle Ps\rangle}$.*

Proof. By Theorem 5.11 the lattice of subfunctors of $\mathbf{e}_{P,s}$ is isomorphic to the lattice of subsets of the set $\Theta(\mathbf{e}_{P,s}) = \{(Q,t) \in [D^\Delta\text{-}pair] : \mathbf{e}_{Q,t} \leqslant \mathbf{e}_{P,s}\} = \{(P,s)\}$. Therefore the subfunctor $\mathbf{e}_{P,s}$ is simple. By Proposition 5.9 the group $\langle Ps\rangle$ is a minimal group of the functor $\mathbf{e}_{P,s}$. By Proposition 5.8 we have $\mathbf{e}_{P,s}\big(\langle Ps\rangle\big) = W_{P,s}$. Moreover, by [5, Theorem 4.2.5], the module $W_{P,s}$ is a simple module for the essential algebra $\mathcal{E}^\Delta\big(\langle Ps\rangle\big)$. Thus we have $\mathbf{e}_{P,s} \simeq S_{\langle Ps\rangle,W_{P,s}}$ as desired. □

Proposition 5.15. *If $F \leqslant F'$ are subfunctors of $\mathbb{F}T^\Delta$ such that F'/F is simple, then there exists a unique D^Δ-pair $(P,s) \in [D^\Delta\text{-}pair]$ such that $\mathbf{e}_{P,s} \leqslant F'$ and $\mathbf{e}_{P,s} \nleqslant F$. In particular, we have $\mathbf{e}_{P,s} + F = F'$, $\mathbf{e}_{P,s} \cap F = \{0\}$, and $F'/F \simeq S_{\langle Ps\rangle,W_{P,s}}$*

Proof. The existence of a pair (P,s) with the property that $\mathbf{e}_{P,s} \leqslant F'$ and $\mathbf{e}_{P,s} \nleqslant F$ is clear. Suppose (P',s') is another pair with these properties. Since F'/F is simple, we have

$$(P',s') \in \Theta(F) \cup \{(P,s)\}.$$

Thus $(P',s') \simeq (P,s)$ as $(P',s') \notin \Theta(F)$. Now since $\mathbf{e}_{P,s} \nleqslant F$ and F'/F is simple, we have $\mathbf{e}_{P,s} + F = F'$. Thus the quotient $\mathbf{e}_{P,s}/(\mathbf{e}_{P,s} \cap F) \simeq F'/F$ is simple and so $\mathbf{e}_{P,s} \cap F = \{0\}$. Therefore we have $F'/F \simeq S_{\langle Ps\rangle,W_{P,s}}$. □

Proposition 5.16. *Let $F \leqslant F'$ be subfunctors of $\mathbb{F}T^\Delta$ such that F'/F is simple. Let H (respectively H') be a finite group and W (respectively W') be a simple $\mathcal{E}^\Delta(H)$-mod (respectively $\mathcal{E}^\Delta(H')$-mod) such that $S_{H,W} \simeq S_{H',W'} \simeq F'/F$. Then $H \cong H'$. Moreover $W \cong W'$, after identification of H and H' via the previous isomorphism.*

Proof. By Proposition 5.15 there exists a unique D^Δ-pair (P,s) such that $F'/F \simeq S_{\langle Ps\rangle,W_{P,s}}$. Therefore it suffices to prove that $H \cong \langle Ps\rangle$. Since $(F'/F)(H) \neq 0$ there exists a pair (Q,t) contained in H such that $F_{Q,t}^H \in F'(H) \setminus F(H)$. Since H is a minimal group of F'/F, it follows that $H = \langle Qt\rangle$ and (Q,t) is a D^Δ-pair. Moreover we have $\mathbf{e}_{Q,t} \leqslant F'$ and $\mathbf{e}_{Q,t} \nleqslant F$. But the pair (P,s) is the unique D^Δ-pair with these properties. Therefore we have $(Q,t) \simeq (P,s)$. Thus $H \cong \langle Ps\rangle$ as desired. The last assertion follows from the fact that $S_{H,W}(H) \cong W$. □

Proposition 5.17. *Let (P,s) be a pair. Then for any finite group H, the $\mathbb{F}$-vector space $\mathbf{e}_{P,s}(H)$ is the subspace of $\mathbb{F}T(H)$ generated by the set of primitive idempotents $F_{Q,t}^H$ where (Q,t) runs over a set of conjugacy classes of pairs in H with the property that (P,s) is isomorphic to a p'-quotient of (Q,t).*

Proof. Since the pair $(\tilde{P}, \tilde{s})$ is isomorphic to a p'-quotient of the pair (P, s) and since $\mathbf{e}_{P,s} = \mathbf{e}_{\tilde{P},\tilde{s}}$, we may assume that the pair (P, s) is a D^Δ-pair. Since $\mathbf{e}_{P,s}(H)$ is an ideal of $\mathbb{F}T(H)$, it has a $\mathbb{F}$-basis consisting of a set of primitive idempotents $F^H_{Q,t}$. If $F^H_{Q,t} \in \mathbf{e}_{P,s}(H)$, then $F^{\langle Qt\rangle}_{Q,t} \in \mathbf{e}_{P,s}\big(\langle Qt\rangle\big)$ and so $\mathbf{e}_{Q,t} \leqslant \mathbf{e}_{P,s}$. Since (P, s) is a D^Δ-pair, by Proposition 5.7, it is isomorphic to a p'-quotient of the pair (Q, t). Conversely, if (P, s) is isomorphic to a p'-quotient of the pair (Q, t), then again by Proposition 5.7, we have $\mathbf{e}_{Q,t} \leqslant \mathbf{e}_{P,s}$. So we have $F^{\langle Qt\rangle}_{Q,t} \in \mathbf{e}_{P,s}\big(\langle Qt\rangle\big)$ and hence $F^H_{Q,t} \in \mathbf{e}_{P,s}(H)$. The result follows. □

Theorem 5.18. *Let (P, s) be a D^Δ-pair. Then for any finite group H, the $\mathbb{F}$-dimension of $S_{\langle Ps\rangle,W_{P,s}}(H)$ is equal to the number of conjugacy classes of pairs (Q, t) in H such that $(\tilde{Q}, \tilde{t}) \simeq (P, s)$.*

Proof. By Proposition 5.17 $\mathbf{e}_{P,s}(H)$ is generated by the idempotents $F^H_{Q,t}$ where (Q, t) is a pair in H with the property that the pair $(\tilde{P}, \tilde{s}) \simeq (P, s)$ is isomorphic to a p'-quotient of the pair (Q, t). Since (P, s) is a D^Δ-pair, Theorem 5.10 implies that $(\tilde{Q}, \tilde{t}) \simeq (P, s)$. The result follows. □

Corollary 5.19. *Let H be a finite group. The $\mathbb{F}$-dimension of $S_{1,\mathbb{F}}(H)$ is equal to the number of isomorphism classes of simple kH-modules.*

Proof. By Theorem 5.18, $\dim_{\mathbb{F}} S_{1,\mathbb{F}}(H)$ is equal to the number of conjugacy classes of pairs (Q, t) in H such that $(\tilde{Q}, \tilde{t}) \simeq (1, 1)$. Suppose (Q, t) is a pair with $(\tilde{Q}, \tilde{t}) \simeq (1, 1)$. Then we have $\tilde{Q} = 1$ and $\tilde{t} = 1$. So there exists a normal p'-subgroup N of $\langle Qt\rangle$ such that $(QN/N, tN) \simeq (1, 1)$. Since $|Q|$ and $|N|$ are coprime, this implies that $Q = 1$. We also have $t \in N$. But then $N \trianglelefteq \langle t\rangle$ implies that $N = \langle t\rangle$. Therefore the number of conjugacy classes of pairs (Q, t) in H such that $(\tilde{Q}, \tilde{t}) \simeq (1, 1)$ is equal to the number of conjugacy classes of p'-elements in H. The result follows. □

Theorem 5.20. *The functor $S_{1,\mathbb{F}}$ is isomorphic to the functor that sends a finite group H to the subspace $\mathbb{F}K_0(kH)$ of $\mathbb{F}T^\Delta(H)$ generated by the projective indecomposable kH-modules.*

Proof. Let H be a finite group. We have

$$S_{1,\mathbb{F}}(H) = (\mathbb{F}T^\Delta(H, 1) \otimes_{\mathbb{F}} \mathbb{F})/J_{1,\mathbb{F}}(H) \cong \mathbb{F}T^\Delta(H, 1)/J_{1,\mathbb{F}}(H)$$

where $J_{1,\mathbb{F}}(H) = \{\phi \in \mathbb{F}T^\Delta(H, 1) : \forall \psi \in \mathbb{F}T^\Delta(1, H), (\psi \circ \phi) \cdot 1 = 0\}$. Now $\mathbb{F}T^\Delta(H, 1)$ is isomorphic to the subspace $\mathbb{F}K_0(kH)$ of $\mathbb{F}T(H)$ generated by the isomorphism classes of projective indecomposable kH-modules. Similarly any $W \in \mathbb{F}T^\Delta(1, H)$

can be identified with $W^* \in \mathbb{F}K_0(kH)$. As in [6] we have the following:
For any p-permutation kH-modules V and W we have

$$(W^* \otimes_{kH} V) \cdot 1 = \dim_k(W^* \otimes_{kH} V) = \dim_k(\mathrm{Hom}_{kH}(W, V)).$$

Therefore $J_{1,\mathbb{F}}(H)$ is the right kernel of the bilinear form

$$< -, - >: \mathbb{F}K_0(kH) \to \mathbb{F}$$

defined as $< W, V >:= \dim_k(\mathrm{Hom}_{kH}(W, V))$. But the matrix that represents this bilinear form is the Cartan matrix of kH. Since the Cartan matrix of a group algebra is non-degenerate, it follows that $J_{1,\mathbb{F}}(H) = 0$. Therefore we have

$$S_{1,\mathbb{F}}(H) = \mathbb{F}T^{\Delta}(H, 1) \otimes_{\mathbb{F}} \mathbb{F} \cong \mathbb{F}T^{\Delta}(H, 1) \cong \mathbb{F}K_0(kH).$$

Note that both of these isomorphisms are functorial in H. The result follows. □

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