CORRESPONDENCE FUNCTORS AND LATTICES

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Abstract. A correspondence functor is a functor from the category of finite sets and correspondences to the category of k-modules, where k is a commutative ring. A main tool for this study is the construction of a correspondence functor associated to any finite lattice T. We prove for instance that this functor is projective if and only if the lattice T is distributive. Moreover, it has quotients which play a crucial role in the analysis of simple functors. The special case of total orders yields some more specific and complete results.

1. Introduction

The present paper is the second in a series which develops the theory of correspondence functors, namely functors from the category of finite sets and correspondences to the category of k-modules, where k is a commutative ring. In the first paper [BT2], we showed that the category of finitely generated correspondence functors is artinian when k is a field. In representation theory, simple modules, or simple functors, are the most basic and important objects to understand. We showed in [BT2] how to parametrize the simple correspondence functors $S_{E,R,V}$ by means of a finite set E, an order relation R on E, and a simple $k\,\mathrm{Aut}(E,R)$-module V (up to isomorphism).

The present paper establishes a connection between finite lattices and correspondence functors. Associated to any finite lattice T, we construct a correspondence functor F_T (Section 4). This is the second indication of the importance of posets and lattices in our work and we describe the interplay between lattices and functors. For instance, one of our first results asserts that the functor F_T is projective if and only if the lattice T is distributive (Theorem 4.12).

The second main purpose of this paper is to introduce a *fundamental functor* $\mathbb{S}_{E,R}$ associated to any finite poset (E,R). This is a precursor of each of the simple correspondence functors $S_{E,R,V}$ and it turns out that understanding $\mathbb{S}_{E,R}$ is the key for understanding those simple functors. In particular, the fundamental functors $\mathbb{S}_{E,R}$ will play a crucial role for the determination of $\dim(S_{E,R,V}(X))$ for any finite set X, which will appear in our next paper [BT3]. Actually, the formula for this dimension involves a new invariant associated to lattices which will be introduced in [BT3] and which will give another important motivation for studying the link between finite lattices and correspondence functors.

The fundamental functors can be analyzed by using lattices. If (E,R) is the subposet of irreducible elements in a finite lattice T, then the functor F_T has a fundamental functor as a quotient, which turns out to be $\mathbb{S}_{E,R^{op}}$ where R^{op} denotes the opposite order relation (Theorem 6.5). The kernel of the morphism $F_T \to \mathbb{S}_{E,R^{op}}$ can be described by a system of linear equations.

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We show that there is a duality between F_T and $F_{T^{op}}$ over any commutative ring k (Theorem 8.9). Moreover, the fundamental functor $\mathbb{S}_{E,R}$ also appears as a subfunctor of $F_{T^{op}}$ (Theorem 9.5). In Section 10, some endomorphisms and idempotents of a lattice T are defined, associated with all possible quotients of T which are total orders. In Section 11, those idempotents are used to describe completely the functor F_T in the special case where T is totally ordered and they are also used to find all direct summands associated to total orders in a functor F_T corresponding to an arbitrary lattice T.

2. Correspondence functors

In this introductory section, we recall the basic facts we need about correspondence functors (which also appear in [BT2]). We denote by $\mathcal{C}$ the category of finite sets and correspondences. Its objects are the finite sets and the set $\mathcal{C}(Y,X)$ of morphisms from X to Y is the set of all correspondences from X to Y, namely all subsets of $Y\times X$ (using a reverse notation which is convenient for left actions). Given two correspondences $R\subseteq Z\times Y$ and $S\subseteq Y\times X$, their composition RS is defined by

$$RS := \{\, (z,x)\in Z\times X \mid \exists\, y\in Y \;\text{ such that }\; (z,y)\in R \;\text{ and }\; (y,x)\in S\,\}\;.$$

A correspondence from X to X is also called a relation on X.

Let Σ_X be the symmetric group of all permutations of X. Associated with a permutation $\sigma\in\Sigma_X$, there is a relation on X which we write

$$\Delta_\sigma := \{(\sigma(x),x)\in X\times X \mid x\in X\}\;.$$

In particular, $\Delta_X := \Delta_{\mathrm{id}}$ is the identity morphism of the object X. If $\sigma,\tau\in\Sigma_X$, then $\Delta_{\sigma\tau}=\Delta_\sigma\Delta_\tau$. The symmetric group Σ_X acts on relations by conjugation and we write ${}^{\sigma}R=\Delta_\sigma R\Delta_{\sigma^{-1}}$.

For any commutative ring k, we let $k\mathcal{C}$ be the k-linearization of $\mathcal{C}$. The objects are again the finite sets and $k\mathcal{C}(Y,X)$ is the free k-module with basis $\mathcal{C}(Y,X)$. A correspondence functor is a k-linear functor from $k\mathcal{C}$ to k-Mod. We let $\mathcal{F}_k$ be the category of all correspondence functors (for some fixed commutative ring k). This category has the following feature :

2.1. Lemma. *Let E and F be finite sets with $|E|\leq|F|$. Let M be a correspondence functor. If $M(F)=0$, then $M(E)=0$.*

Proof : Since $|E|\leq|F|$, there exists an injective map $i:E\hookrightarrow F$. Let $i_*\subseteq F\times E$ denote the correspondence

$$i_*=\big\{\big(i(e),e\big)\mid e\in E\big\}\;,$$

and $i^*\subseteq E\times F$ denote the correspondence

$$i^*=\big\{\big(e,i(e)\big)\mid e\in E\big\}\;.$$

As i is injective, one checks easily that $i^*i_*=\Delta_E$, that is, $i^*i_*=\mathrm{id}_E$. For any $m\in M(E)$, we have $m=i^*i_*\cdot m$. But $i_*\cdot m\in M(F)$, so $i_*\cdot m=0$. Therefore $m=0$. ∎

We define a *minimal set* for a correspondence functor F to be a finite set X of minimal cardinality such that $F(X) \neq 0$. For a nonzero functor, such a minimal set always exists and is unique up to bijection.

The first instances of correspondence functors are the representable functors $k\mathcal{C}(-,E)$, where E is a finite set, and the functors

$$L_{E,W} := k\mathcal{C}(-,E) \otimes_{k\mathcal{C}(E,E)} W$$

where W is a left $k\mathcal{C}(E,E)$-module. The proof of the following result is easy and is sketched in Lemma 2.3 of [BST] in the special case of biset functors for finite groups, but it extends without change to representations of an arbitrary category.

2.2. Lemma. *Let $\mathcal{F}_k$ be the category of all correspondence functors and let E be a finite set. The functor*

$$k\mathcal{C}(E,E)-\mathrm{Mod} \longrightarrow \mathcal{F}_k \ , \quad W \mapsto L_{E,W}$$

is left adjoint of the evaluation functor

$$\mathcal{F}_k \longrightarrow k\mathcal{C}(E,E)-\mathrm{Mod} \ , \quad F \mapsto F(E) \ .$$

The functor $L_{E,W}$ has a subfunctor $J_{E,W}$ defined on any finite set X by

$$J_{E,W}(X) := \Big\{ \sum_i \phi_i \otimes w_i \in L_{E,W}(X) \mid \forall \psi \in k\mathcal{C}(E,X), \sum_i (\psi\phi_i)\cdot w_i = 0 \Big\} \ .$$

2.3. Lemma. *Let E be a finite set and let W be a $k\mathcal{C}(E,E)$-module.*

(a) $J_{E,W}$ is the unique subfunctor of $L_{E,W}$ which is maximal with respect to the condition that it vanishes at E.

(b) If W is a simple $k\mathcal{C}(E,E)$-module, then $J_{E,W}$ is the unique maximal subfunctor of $L_{E,W}$ and $L_{E,W}/J_{E,W}$ is a simple functor.

Proof : The result is a slight extension of the first lemma of [Bo1]. The proof is also sketched in Lemma 2.3 of [BST] in the special case of biset functors for finite groups, but it extends without change to the representation theory of an arbitrary category. □

Now we want to consider the functor $L_{E,W}/J_{E,W}$ for some specific choices of $k\mathcal{C}(E,E)$-modules. The algebra $k\mathcal{C}(E,E)$ of all relations on E was studied in [BT1] and we need a few facts from this approach. A relation R on E is called *essential* if it does not factor through a set of cardinality strictly smaller than $|E|$. The k-submodule generated by the set of inessential relations is a two-sided ideal

$$I_E := \sum_{|Y|<|E|} k\mathcal{C}(E,Y)k\mathcal{C}(Y,E)$$

and the quotient

$$\mathcal{E}_E := k\mathcal{C}(E,E)/I_E$$

is called the *essential algebra*. A large part of its structure has been elucidated in [BT1]. There is a quotient algebra $\mathcal{P}_E = \mathcal{E}_E/N$, where N is a nilpotent two-sided ideal defined in [BT1]. We call $\mathcal{P}_E$ the *algebra of permuted orders*, because it has a k-basis consisting of all relations on E of the form $\Delta_\sigma R$, where σ runs through the symmetric group Σ_E of all permutations of E, and R is an order on E. By an order, we always mean a partial order relation. The product of two orders R and S in $\mathcal{P}_E$ is the transitive closure of $R \cup S$ if this closure is an order, and zero otherwise. This describes completely the algebra structure of $\mathcal{P}_E$.

Among the $k\mathcal{C}(E,E)$-modules, there is the *fundamental module* $\mathcal{P}_E f_R$, associated to any poset (E,R), where E is a finite set and R denotes the order relation on E which defines the poset structure. Here f_R is a suitable idempotent in $\mathcal{P}_E$, depending on R, and $\mathcal{P}_E f_R$ is the left ideal generated by f_R. Actually, the fundamental module $\mathcal{P}_E f_R$ only depends on the isomorphism type of the poset (E,R), or in other words, for a fixed set E, on the Σ_E-conjugacy class of R. More explicitly, if $\sigma \in \Sigma_E$, then conjugation by σ induces an isomorphism of posets $(E,R) \cong (E, {}^{\sigma}R)$ and we also have an isomorphism of $\mathcal{P}_E$-modules $\mathcal{P}_E f_R \cong \mathcal{P}_E f_{{}^{\sigma}R}$, because $f_{{}^{\sigma}R} = {}^{\sigma}f_R$ (see Lemma 7.1 in [BT1] for details).

The only thing we really need to know about the fundamental module $\mathcal{P}_E f_R$ is its structure as a $k\mathcal{C}(E,E)$-module. This is described in the next result, which combines Corollary 7.3 and Proposition 8.5 of [BT1] (see also Proposition 4.5 of [BT2] and its use).

2.4. Proposition. *Let E be a finite set and R an order on E.*

(a) The fundamental module $\mathcal{P}_E f_R$ is a left module for the algebra $\mathcal{P}_E$, hence also a left module for the essential algebra $\mathcal{E}_E$ and for the algebra of relations $k\mathcal{C}(E,E)$.

(b) $\mathcal{P}_E f_R$ is a free k-module with a k-basis consisting of the elements $\Delta_\sigma f_R$, where σ runs through the group Σ_E of all permutations of E.

(c) $\mathcal{P}_E f_R$ is a $(\mathcal{P}_E, k\,\mathrm{Aut}(E,R))$-bimodule and the right action of $k\,\mathrm{Aut}(E,R)$ is free.

(d) The action of the algebra of relations $k\mathcal{C}(E,E)$ on the module $\mathcal{P}_E f_R$ is given as follows. For any relation $Q \in \mathcal{C}(E,E)$,

$$Q \cdot \Delta_\sigma f_R = \begin{cases} \Delta_{\tau\sigma} f_R & \text{if } \exists \tau \in \Sigma_E \text{ such that } \Delta_E \subseteq \Delta_{\tau^{-1}} Q \subseteq {}^{\sigma}R, \\ 0 & \text{otherwise}, \end{cases}$$

where ${}^{\sigma}R = \big\{\big(\sigma(e),\sigma(f)\big) \mid (e,f) \in R\big\}$ (or equivalently ${}^{\sigma}R = \Delta_\sigma R \Delta_{\sigma^{-1}}$). (Moreover, τ is unique in the first case.)

Using the $(\mathcal{P}_E, k\,\mathrm{Aut}(E,R))$-bimodule structure on $\mathcal{P}_E f_R$, we define

$$T_{R,V} := \mathcal{P}_E f_R \otimes_{k\,\mathrm{Aut}(E,R)} V \;,$$

where V is any $k\,\mathrm{Aut}(E,R)$-module. Then $T_{R,V}$ is a left $\mathcal{P}_E$-module, hence also a $k\mathcal{C}(E,E)$-module since $\mathcal{P}_E$ is a quotient of $k\mathcal{C}(E,E)$. The left action of $k\mathcal{C}(E,E)$ on $T_{R,V}$ is induced from the action on $\mathcal{P}_E f_R$ described in Proposition 2.4 above. Again, the module $T_{R,V}$ is invariant under Σ_E-conjugacy, that is, for a fixed set E and for $\sigma \in \Sigma_E$, we have an isomorphism of $\mathcal{P}_E$-modules $T_{{}^{\sigma}R,{}^{\sigma}V} \cong T_{R,V}$, where ${}^{\sigma}V$ denotes the conjugate module, namely a module for the group $\mathrm{Aut}(E,{}^{\sigma}R) = \sigma\,\mathrm{Aut}(E,R)\sigma^{-1}$ (see Theorem 8.1 in [BT1] for details).

The main thing we need to know about $T_{R,V}$ is the following result, which is part of Theorem 8.1 in [BT1].

2.5. Proposition. *Assume that k is a field. Let E be a finite set, R an order on E, and V a simple $k\,\mathrm{Aut}(E,R)$-module. Then $T_{R,V}$ is a simple $\mathcal{P}_E$-module (hence also a simple $\mathcal{E}_E$-module).*

Actually, Theorem 8.1 in [BT1] asserts that every simple $\mathcal{E}_E$-module is isomorphic to some module $T_{R,V}$ and that, consequently, the set of isomorphism classes of simple $\mathcal{E}_E$-modules is parametrized by the set of conjugacy classes of pairs (R,V) where R is an order on E and V is a simple $k\,\mathrm{Aut}(E,R)$-module.

Associated with the above $k\mathcal{C}(E,E)$-modules, we can now define some specific correspondence functors. Using the fundamental module $\mathcal{P}_E f_R$, we define

$$\mathbb{S}_{E,R} := L_{E,\mathcal{P}_E f_R}/J_{E,\mathcal{P}_E f_R}$$

and we call it the *fundamental functor* associated with the poset (E,R). Using the module $T_{R,V}$, we define

$$S_{E,R,V} := L_{E,T_{R,V}}/J_{E,T_{R,V}}\ .$$

Note that $\mathbb{S}_{E,{}^\sigma R} \cong \mathbb{S}_{E,R}$ and $S_{E,{}^\sigma R,{}^\sigma V} \cong S_{E,R,V}$, for any permutation $\sigma \in \Sigma_E$.

2.6. Proposition.

(a) *The set E is a minimal set for $\mathbb{S}_{E,R}$ and $\mathbb{S}_{E,R}(E) \cong \mathcal{P}_E f_R$ as left $k\mathcal{C}(E,E)$-modules.*
(b) *The set E is a minimal set for $S_{E,R,V}$ and $S_{E,R,V}(E) \cong T_{R,V}$ as left $k\mathcal{C}(E,E)$-modules.*
(c) *If k is a field and V is a simple $k\,\mathrm{Aut}(E,R)$-module, then $S_{E,R,V}$ is a simple correspondence functor.*

Proof : Let W be any $\mathcal{P}_E$-module and set $S = L_{E,W}/J_{E,W}$. Suppose that Y is a finite set such that $S(Y) \neq \{0\}$. Then $L_{E,W}(Y) \neq J_{E,W}(Y)$, so there exists a correspondence $\phi \in \mathcal{C}(Y,E)$ and $v \in W$ such that $\phi \otimes v \in L_{E,W}(Y) - J_{E,W}(Y)$. By definition of $J_{E,W}$, this means that there exists a correspondence $\psi \in \mathcal{C}(E,Y)$ such that $\psi\phi \cdot v \neq 0$. Since $\mathcal{P}_E$ is a quotient of the essential algebra $\mathcal{E}_E$, it follows that W is a module for $\mathcal{E}_E = k\mathcal{C}(E,E)/I_E$, so that the ideal I_E acts by zero on W. Therefore $\psi\phi \notin I_E$. But $\psi\phi$ factorizes through Y, so we must have $|Y| \geq |E|$. Thus E is a minimal set for S. In particular, this holds for $S = \mathbb{S}_{E,R}$ (taking $W = \mathcal{P}_E f_R$) and also for $S = S_{E,R,V}$ (taking $W = T_{R,V}$).

Since $J_{E,W}$ vanishes at E by Lemma 2.3, evaluation at E yields

$$S(E) = L_{E,W}(E)/J_{E,W}(E) = L_{E,W}(E) = k\mathcal{C}(E,E) \otimes_{k\mathcal{C}(E,E)} W \cong W\ ,$$

and therefore $\mathbb{S}_{E,R}(E) \cong \mathcal{P}_E f_R$ and $S_{E,R,V}(E) \cong T_{R,V}$.

For the proof of (c), notice that $T_{R,V}$ is a simple $k\mathcal{C}(E,E)$-module by Proposition 2.5 and therefore

$$S_{E,R,V} = L_{E,T_{R,V}}/J_{E,T_{R,V}}$$

is a simple correspondence functor by Lemma 2.3. □

Although we do not need it here, let us mention that more is known about simple correspondence functors, when k is a field. It is proved in [BT2] that any simple functor has the form $S_{E,R,V}$ for some triple (E,R,V) and that the set of isomorphism classes of simple correspondence functors is parametrized by the set of isomorphism classes of triples (E,R,V) where E is a finite set, R is an order on E, and V is a simple $k\,\mathrm{Aut}(E,R)$-module.

We note that the fundamental functor $\mathbb{S}_{E,R}$ is a precursor of $S_{E,R,V}$, in the sense of the following lemma.

2.7. Lemma. *Suppose that V is a $k\,\mathrm{Aut}(E,R)$-module generated by a single element v (e.g. a simple module). Consider the surjective morphism of correspondence functors*

$$\Phi : L_{E,\mathcal{P}_E f_R} \longrightarrow L_{E,T_{R,V}}$$

induced by the surjective homomorphism of $\mathcal{P}_E$-modules

$$\Phi_E : \mathcal{P}_E f_R \longrightarrow \mathcal{P}_E f_R \otimes_{k\,\mathrm{Aut}(E,R)} V = T_{R,V}\ , \qquad a \mapsto a \otimes v\ .$$

Then Φ induces a surjective morphism of correspondence functors

$$\mathbb{S}_{E,R} \longrightarrow S_{E,R,V}\ .$$

Proof : Let $\varphi \otimes f_R \in k\mathcal{C}(X,E) \otimes_{k\mathcal{C}(E,E)} \mathcal{P}_E f_R = L_{E,\mathcal{P}_E f_R}(X)$. If $\varphi \otimes f_R \in J_{E,\mathcal{P}_E f_R}(X)$, then for every $\psi \in \mathcal{C}(E,X)$, we have $\psi\varphi \cdot f_R = 0$. Then $\Phi_X(\varphi \otimes f_R) = \varphi \otimes f_R \otimes v$ and we have

$$\psi\varphi \cdot (f_R \otimes v) = (\psi\varphi \cdot f_R) \otimes v = 0 \ .$$

This shows that $\Phi_X(\varphi \otimes f_R) \in J_{E,T_{R,V}}(X)$, so that $\Phi_X(J_{E,\mathcal{P}_E f_R}(X)) \subseteq J_{E,T_{R,V}}(X)$. Therefore Φ induces a morphism of correspondence functors $\mathbb{S}_{E,R} \longrightarrow S_{E,R,V}$ which remains surjective. ∎

By means of a very detailed analysis of $\mathbb{S}_{E,R}$ which will be carried out in [BT3], we shall show that it is possible to recover $S_{E,R,V}$ from $\mathbb{S}_{E,R}$ by simply tensoring with V. Consequently, mastering $\mathbb{S}_{E,R}$ will be the key for obtaining information about the simple functors $S_{E,R,V}$ and this explains why the fundamental functors play a crucial role throughout our work.

3. Posets and lattices

In this section, we give some definitions, fix some notation, and prove some basic lemmas, which will be used throughout.

By an *order* R on a finite set E, we mean a partial order relation on E. In other words, (E,R) is a finite poset.

3.1. Notation and definitions. *Let (E,R) be a finite poset.*

(a) *We write $\leq_R$ for the order relation, so that $(a,b) \in R$ if and only if $a \leq_R b$. Moreover $a <_R b$ means that $a \leq_R b$ and $a \neq b$.*

(b) *If $a,b \in E$ with $a \leq_R b$, we define intervals*

$$[a,b]_E := \{x \in E \mid a \leq_R x \leq_R b\}\,, \qquad]a,b[_E := \{x \in E \mid a <_R x <_R b\}\,,$$
$$[a,b[_E := \{x \in E \mid a \leq_R x <_R b\}\,, \qquad]a,b]_E := \{x \in E \mid A <_R x \leq_R b\}\,,$$
$$[a,\cdot[_E := \{x \in E \mid a \leq_R x\}\,, \qquad]\cdot,b]_E := \{x \in E \mid x \leq_R b\}\ .$$

When the context is clear, we write $[a,b]$ instead of $[a,b]_E$.

(c) *A subset A of E is a* lower R-ideal*, or simply a* lower ideal*, if, whenever $a \in A$ and $x \leq_R a$, we have $x \in A$. Similarly, a subset A of E is an* upper R-ideal*, or simply an* upper ideal*, if, whenever $a \in A$ and $a \leq_R x$, we have $x \in A$.*

(d) *A* principal lower ideal*, or simply* principal ideal*, is a subset of the form $]\cdot,a]_E$, where $a \in E$. A* principal upper ideal *is defined similarly.*

(e) *The* opposite order relation R^{op} *is defined by the property that $a \leq_{R^{op}} b$ if and only if $b \leq_R a$.*

3.2. Notation and definitions. *Let T be a finite lattice.*

(a) *We write $\leq_T$, or sometimes simply $\leq$, for the order relation, $\vee$ for the join (least upper bound), $\wedge$ for the meet (greatest lower bound), $\hat{0}$ for the least element and $\hat{1}$ for the greatest element.*

(b) *An element $e \in T$ is called* join-irreducible*, or simply* irreducible*, if, whenever $e = \bigvee\limits_{a\in A} a$ for some subset A of T, then $e \in A$. In case $A = \emptyset$, the join is $\hat{0}$ and it follows that $\hat{0}$ is not irreducible. An element $e \neq \hat{0}$ is irreducible if and only if the equality $e = s \vee t$, for $s,t \in T$, implies $e = s$ or $e = t$. In other words, if $e \neq \hat{0}$, then e is irreducible if and only if $[\hat{0},e[$ has a unique maximal element.*

(c) *Let (E, R) be a subposet of T. We say that it is a* full *subposet of T if for all $e, f \in E$ we have :*

$$e \leq_R f \Longleftrightarrow e \leq_T f \ .$$

Note that if (E, R) is the poset of irreducible elements in a finite lattice T, then T is *generated* by E in the sense that any element $x \in T$ is a join of elements of E. To see this, define the height of $t \in T$ to be the maximal length of a chain in $[\hat{0}, t]_T$. If x is not irreducible and $x \neq \hat{0}$, then $x = t_1 \vee t_2$ with t_1 and t_2 of smaller height than x. By induction on the height, both t_1 and t_2 are joins of elements of E. Therefore $x = t_1 \vee t_2$ is also a join of elements of E.

3.3. Notation. *Let (E, R) be a finite poset.*

(a) *Let $I_\downarrow(E, R)$ denote the set of lower R-ideals of E. Then $I_\downarrow(E, R)$, ordered by inclusion of subsets, is a lattice : the join operation is union of subsets, and the meet operation is intersection.*

(b) *Similarly, $I^\uparrow(E, R)$ denotes the set of upper R-ideals of E, which is also a lattice. If R^{op} is the relation opposite to R, then clearly $I^\uparrow(E, R) = I_\downarrow(E, R^{op})$.*

3.4. Remark. Let R be a preorder on a finite set E, that is, a relation which is reflexive and transitive. There is an equivalence relation $\sim$ associated with R, defined by

$$x \sim y \iff (x, y) \in R \ \text{ and } \ (y, x) \in R \ .$$

Then R induces an order relation $\overline{R}$ on the quotient set $\overline{E} = E/\!\sim$ such that

$$(x, y) \in R \iff (\overline{x}, \overline{y}) \in \overline{R} \ ,$$

where $\overline{x}$ denotes the equivalence class of x under $\sim$. It is easy to see that the quotient map $E \to \overline{E}$ induces an isomorphism of lattices $I_\downarrow(E, R) \cong I_\downarrow(\overline{E}, \overline{R})$.

Note that it is proved in Lemma 3.9 of [BT2] that the representable functors $k\mathcal{C}(-, E)R$ and $k\mathcal{C}(-, \overline{E})\overline{R}$ are isomorphic, but actually we will view them in a new way in Proposition 4.5. These remarks show that, for our purposes, it is enough to consider orders rather than preorders, and we shall do so in the rest of this paper, without loss of generality.

3.5. Lemma. *Let (E, R) be a finite poset.*

(a) *The irreducible elements in the lattice $I_\downarrow(E, R)$ are the principal ideals $]\cdot, e]_E$, where $e \in E$. Thus the poset E is isomorphic to the poset of all irreducible elements in $I_\downarrow(E, R)$ by mapping $e \in E$ to the principal ideal $]\cdot, e]_E$.*

(b) *$I_\downarrow(E, R)$ is a distributive lattice.*

(c) *If T is a distributive lattice and (E, R) is its subposet of irreducible elements, then T is isomorphic to $I_\downarrow(E, R)$.*

(d) *For any finite lattice T having (E, R) as poset of irreducible elements, there is a join-preserving surjective map $f : I_\downarrow(E, R) \longrightarrow T$ which sends any lower ideal $A \in I_\downarrow(E, R)$ to the join $\bigvee_{e \in A} e$ in T.*

Proof : This is not difficult and well-known. For details, see Theorem 3.4.1 and Proposition 3.4.2 in [St]. □

3.6. Convention. *In the situation of Lemma 3.5, we shall identify E with its image via the map*

$$E \longrightarrow I_{\downarrow}(E,R)\,, \qquad e \mapsto\,]\cdot, e]_E\ .$$

Thus we view E as a full subposet of $I_{\downarrow}(E,R)$. This abusive convention is a conceptual simplification and has many advantages for the rest of this paper.

Given a poset (E,R), the map

$$E \longrightarrow I^{\uparrow}(E,R)\,, \qquad e \mapsto\ [e,\cdot[_E$$

is order-reversing, so it is in fact (E,R^{op}) which is identified with the poset of irreducible elements in $I^{\uparrow}(E,R)$. Since $I^{\uparrow}(E,R)=I_{\downarrow}(E,R^{op})$, this is actually just Convention 3.6 applied to R^{op}.

We now introduce a notation which will play an important role in our work (and which was already used in the proof of Theorem 9.2 in [BT2]).

3.7. Notation. *Let T be a finite lattice and let (E,R) be the full subposet of its irreducible elements. For any finite set X and any map $\varphi : X \to T$, we associate the correspondence*

$$\Gamma_\varphi := \{(x,e)\in X\times E \mid e \leq_T \varphi(x)\} \subseteq X\times E\ .$$

In the special case where $T=I_{\downarrow}(E,R)$ and in view of Convention 3.6, we obtain

$$\Gamma_\varphi = \{(x,e)\in X\times E \mid e \in \varphi(x)\}\ .$$

3.8. Lemma. *Let T be a finite lattice and let (E,R) be the full subposet of its irreducible elements.*

(a) For any map $\varphi : X\to T$, we have $\Gamma_\varphi R^{op}=\Gamma_\varphi$.
(b) If $T=I_{\downarrow}(E,R)$, then a correspondence $S\subseteq X\times E$ has the form $S=\Gamma_\varphi$ for some map $\varphi : X\to I_{\downarrow}(E,R)$ if and only if $SR^{op}=S$.
(c) If $T=I^{\uparrow}(E,R)$, then a correspondence $S\subseteq X\times E$ has the form $S=\Gamma_\varphi$ for some map $\varphi : X\to I^{\uparrow}(E,R)$ if and only if $SR=S$.

Proof : (a) Since $\Delta_E\subseteq R^{op}$, we always have $\Gamma_\varphi=\Gamma_\varphi\Delta_E\subseteq\Gamma_\varphi R^{op}$. Conversely, if $(x,f)\in\Gamma_\varphi R^{op}$, then there exists $e\in E$ such that $(x,e)\in\Gamma_\varphi$ and $(e,f)\in R^{op}$, that is, $e\leq_T\varphi(x)$ and $f\leq_R e$. But $f\leq_R e$ if and only if $f\leq_T e$, because (E,R) is a full subposet of T. It follows that $f\leq_T\varphi(x)$, that is, $(x,f)\in\Gamma_\varphi$. Thus $\Gamma_\varphi R^{op}\subseteq\Gamma_\varphi$ and equality follows.

(b) One direction follows from (a). For the other direction, let $S\in\mathcal{C}(X,E)$ be such that $SR^{op}=S$, or equivalently $S\in\mathcal{C}(X,E)R^{op}$ (because R^{op} is idempotent by reflexivity and transitivity). Then the set

$$\phi(x)=\big\{e\in E \mid (x,e)\in S\big\}$$

is a lower R-ideal in E, thus ϕ is a function $X\to I_{\downarrow}(E,R)$. Clearly $\Gamma_\phi=S$.

(c) This follows from (b) applied to R^{op}, because $I^{\uparrow}(E,R)=I_{\downarrow}(E,R^{op})$ and (E,R^{op}) is its poset of irreducible elements. □

4. Functors associated to lattices

A fundamental construction associates a correspondence functor F_T to any finite lattice T. This is one of our main tools for the analysis of correspondence functors. Throughout this section, k is an arbitrary commutative ring.

4.1. Definition. *Let T be a finite lattice. For a finite set X, we define $F_T(X)$ to be the free k-module with basis the set T^X of all functions from X to T :*

$$F_T(X) := k(T^X) \ .$$

For two finite sets X and Y and a correspondence $R \subseteq Y \times X$, we define a map $F_T(R) : F_T(X) \to F_T(Y)$ as follows : to a function $\varphi : X \to T$, we associate the function $F_T(R)(\varphi) : Y \to T$, also simply denoted by $R\varphi$, defined by

$$(R\varphi)(y) := \bigvee_{\substack{x \in X \\ (y,x) \in R}} \varphi(x) \ ,$$

with the usual rule that a join over the empty set is equal to $\hat{0}$. The map

$$F_T(R) : F_T(X) \to F_T(Y)$$

is the unique k-linear extension of this construction. More generally, for every element $\alpha = \sum\limits_{R \in \mathcal{C}(Y,X)} \alpha_R R \in k\mathcal{C}(Y,X)$, where $\alpha_R \in k$, we set

$$F_T(\alpha) = \sum_{R \in \mathcal{C}(Y,X)} \alpha_R F_T(R) \ .$$

4.2. Proposition. *The assignment sending a finite set X to the k-module $F_T(X)$ and a morphism $\alpha \in k\mathcal{C}(Y,X)$ to the k-linear map $F_T(\alpha) : F_T(X) \to F_T(Y)$ is a correspondence functor.*

Proof : First it is clear that if X is a finite set and $\Delta_X \in \mathcal{C}(X,X)$ is the identity correspondence, then for any $\varphi : X \to T$ and any $y \in X$

$$(\Delta_X\varphi)(y) = \bigvee_{(y,x) \in \Delta_X} \varphi(x) = \varphi(y) \ ,$$

hence $\Delta_X\varphi = \varphi$ and $F_T(\Delta_X)$ is the identity map of $F_T(X)$.

Now if X, Y, and Z are finite sets, if $R \in \mathcal{C}(Y,X)$ and $S \in \mathcal{C}(Z,Y)$, then for any $\varphi : X \to T$ and any $z \in Z$, we have

$$\begin{aligned} \big(S(R\varphi)\big)(z) &= \bigvee_{(z,y) \in S} (R\varphi)(y) \\ &= \bigvee_{(z,y) \in S} \bigvee_{(y,x) \in R} \varphi(x) \\ &= \bigvee_{(z,x) \in SR} \varphi(x) \\ &= (SR\varphi)(z) \ . \end{aligned}$$

By linearity, it follows that $F_T(\beta) \circ F_T(\alpha) = F_T(\beta\,\alpha)$, for any $\beta \in k\mathcal{C}(Z,Y)$ and any $\alpha \in k\mathcal{C}(Y,X)$. ∎

4.3. Remark. The definition of F_T only uses the join operation in the lattice T. It follows that the definition would work for a join semi-lattice, but it is actually well-known that a finite join semi-lattice has automatically a structure of lattice (the meet operation being uniquely determined from the sole join). This explains our choice of working with lattices. Such a choice will also be useful in Section 8 when we shall work with opposite lattices.

We now establish the link between the action of correspondences on functions $\varphi : X \to T$ (as in Definition 4.1 above) and the correspondences Γ_φ defined in Notation 3.7.

4.4. Lemma. *Let T be a finite lattice and let (E,R) be the full subposet of its irreducible elements. Assume that T is distributive, so that $T \cong I_\downarrow(E,R)$. Then, for any finite sets X, Y, any correspondence $S \in \mathcal{C}(Y,X)$, and any function $\varphi : X \to T$, we have*

$$\Gamma_{S\varphi} = S\Gamma_\varphi \;,$$

where Γ_φ is defined in Notation 3.7.

Proof : Let $y \in Y$ and $e \in E$. Then

$$\begin{aligned}(y,e) \in \Gamma_{S\varphi} \quad &\iff e \leq_T (S\varphi)(y) \iff e \leq_T \bigvee_{(y,x)\in S} \varphi(x)\\ &\iff e = e \wedge \big(\bigvee_{(y,x)\in S} \varphi(x)\big) \;.\end{aligned}$$

But, since T is distributive, the latter equality is equivalent to $e = \bigvee\limits_{(y,x)\in S} \big(e \wedge \varphi(x)\big)$. Now, since e is irreducible, this is in turn equivalent to

$$\begin{aligned}&\exists x \in X\,,\ (y,x) \in S \text{ and } e \wedge \varphi(x) = e \iff \exists x \in X\,,\ (y,x) \in S \text{ and } e \leq_T \varphi(x)\\ &\iff \exists x \in X\,,\ (y,x) \in S \text{ and } (x,e) \in \Gamma_\varphi \iff (y,e) \in S\Gamma_\varphi \;.\end{aligned}$$

This completes the proof. ∎

Now we can give another description of the correspondence functor associated to a distributive lattice.

4.5. Proposition. *Let (E,R) be a finite poset.*

(a) For any finite set X

$$\big\{\Gamma_\varphi \mid \varphi : X \to I^\uparrow(E,R)\big\} = \big\{S \in \mathcal{C}(X,E) \mid SR = S\big\} = \mathcal{C}(X,E)R \;.$$

(b) The correspondence functor $F_{I^\uparrow(E,R)}$ is isomorphic to $k\mathcal{C}(-,E)R$. In particular $F_{I^\uparrow(E,R)}$ is a projective object of $\mathcal{F}_k$.

(c) The correspondence functor $F_{I_\downarrow(E,R)}$ is isomorphic to $k\mathcal{C}(-,E)R^{op}$. In particular $F_{I_\downarrow(E,R)}$ is a projective object of $\mathcal{F}_k$.

Proof : (a) This is a restatement of Lemma 3.8.

(b) The map

$$F_{I^\uparrow(E,R)}(X) \longrightarrow k\mathcal{C}(X,E)R\,, \qquad \varphi \mapsto \Gamma_\varphi$$

is an isomorphism of correspondence functors, by (a) and Lemma 4.4. Moreover $k\mathcal{C}(-,E)$ is a projective functor by Yoneda's lemma and $k\mathcal{C}(-,E)R$ is a direct summand of $k\mathcal{C}(-,E)$ because R is idempotent. Therefore $k\mathcal{C}(-,E)R$ is projective.

(c) follows from (b) and the obvious equality $I_\downarrow(E,R) = I^\uparrow(E,R^{op})$. ∎

We now introduce a suitable category $\mathcal{L}$ of lattices, as well as its k-linearization $k\mathcal{L}$. Our aim is to show that the assignment $T \mapsto F_T$ becomes a k-linear functor from $k\mathcal{L}$ to $\mathcal{F}_k$, which will have the remarkable property of being full and faithful.

4.6. Definition. *Let $\mathcal{L}$ and $k\mathcal{L}$ denote the following categories :*

- *The objects of $\mathcal{L}$ and $k\mathcal{L}$ are the finite lattices.*
- *For any two lattices T and T', the set $\mathrm{Hom}_{\mathcal{L}}(T,T')$ is the set of all maps $f : T \to T'$ which commute with joins, i.e. such that*
$$f(\bigvee_{a\in A} a) = \bigvee_{a\in A} f(a) \;,$$
for any subset A of T.
- *For any two lattices T and T', the set $\mathrm{Hom}_{k\mathcal{L}}(T,T')$ is the free k-module $k\,\mathrm{Hom}_{\mathcal{L}}(T,T')$ with basis $\mathrm{Hom}_{\mathcal{L}}(T,T')$.*
- *The composition of morphisms in $\mathcal{L}$ is the composition of maps.*
- *The composition of morphisms in $k\mathcal{L}$ is the k-bilinear extension of the composition in $\mathcal{L}$.*

It is easy to see that a morphism in $\mathcal{L}$ is order-preserving, by considering the join $t_1 \vee t_2$ in the case where $t_1 \leq_T t_2$ in the lattice T.

4.7. Remark. The case $A = \emptyset$ in Definition 4.6 shows that a morphism $f : T \to T'$ in $\mathcal{L}$ always maps $\hat{0} \in T$ to $\hat{0} \in T'$. Conversely, if $f : T \to T'$ satisfies $f(\hat{0}) = \hat{0}$ and $f(a \vee b) = f(a) \vee f(b)$ for all $a, b \in T$, then f is a morphism in $\mathcal{L}$.

Morphisms in $\mathcal{L}$ are morphisms of join semi-lattices (see Remark 4.3), but they are generally not morphisms of lattices in the sense that they need not commute with the meet operation. The choice of not using the meet operation turns out to be important for the next main theorem.

For our next theorem, we need some notation. Let $f : T \to T'$ be a morphism in the category $\mathcal{L}$. For a finite set X, let $F_{f,X} : F_T(X) \to F_{T'}(X)$ be the k-linear map sending the function $\varphi : X \to T$ to the function $f \circ \varphi : X \to T'$.

4.8. Theorem.

(a) Let $f : T \to T'$ be a morphism in the category $\mathcal{L}$. Then the collection of maps $F_{f,X} : F_T(X) \to F_{T'}(X)$, for all finite sets X, yields a natural transformation $F_f : F_T \to F_{T'}$ of correspondence functors.

(b) The assignment sending a lattice T to F_T, and a morphism $f : T \to T'$ in $\mathcal{L}$ to $F_f : F_T \to F_{T'}$, yields a functor $\mathcal{L} \to \mathcal{F}_k$. This functor extends uniquely to a k-linear functor
$$F_? : k\mathcal{L} \longrightarrow \mathcal{F}_k \;.$$

(c) The functor $F_?$ is fully faithful.

Proof : (a) Let X and Y be finite sets, let $\varphi : X \to T$ be a function, and let $U \in \mathcal{C}(Y,X)$ be a correspondence. Then $F_{T'}(U)\big(F_{f,X}(\varphi)\big) = F_{T'}(U)(f \circ \varphi) = U(f \circ \varphi)$ and $F_{f,Y}\big(F_T(U)(\varphi)\big) = F_{f,Y}(U\varphi) = f \circ U\varphi$. We show that they are equal by evaluating at any $y \in Y$:
$$\begin{aligned} U(f\circ\varphi)(y) &= \bigvee_{(y,x)\in U} (f\circ\varphi)(x) \\ &= \bigvee_{(y,x)\in U} f\big(\varphi(x)\big) \\ &= f\big(\bigvee_{(y,x)\in U} \varphi(x)\big) \\ &= (f\circ U\varphi)(y) \;, \end{aligned}$$
hence $U(f \circ \varphi) = f \circ U\varphi$, which proves (a).

(b) It follows that the assignment $T \mapsto F_T$ is a functor $\mathcal{L} \to \mathcal{F}_k$. Since $k\mathcal{L}$ is the k-linearization of $\mathcal{L}$, this functor extends uniquely to a k-linear functor $F_? : k\mathcal{L} \to \mathcal{F}_k$.

(c) Let S and T be finite lattices, and $\Phi : F_S \to F_T$ be a morphism of functors. Thus, for any finite set X, we have a morphism of k-modules $\Phi_X : F_S(X) \to F_T(X)$ such that for any finite set Y and any correspondence $R \subseteq (Y \times X)$, the diagram

$$\begin{tikzcd} F_S(X) \arrow[r, "\Phi_X"] \arrow[d, "F_S(R)"'] & F_T(X) \arrow[d, "F_T(R)"] \\ F_S(Y) \arrow[r, "\Phi_Y"] & F_T(Y) \end{tikzcd}$$

is commutative. In other words, for any function $\alpha : X \to S$

$$R\Phi_X(\alpha) = \Phi_Y(R\alpha) \ . \tag{4.9}$$

Taking $X = S$ and $\alpha = \mathrm{id}_S$ in this relation, and setting

$$\varphi = \Phi_S(\mathrm{id}_S) = \sum_{\lambda : S \to T} u_\lambda \lambda \ ,$$

where $u_\lambda \in k$, this gives

$$R\varphi = \Phi_Y(R\,\mathrm{id}_S) \ ,$$

for any Y and any $R \subseteq (Y \times S)$.

Given a function $\beta : Y \to S$ and taking $R = \Omega_\beta := \{(y, \beta(y)) \mid y \in Y\}$, one can check easily that $\Omega_\beta \,\mathrm{id}_S = \beta$. It follows that

$$\Phi_Y(\beta) = \Omega_\beta \, \varphi \ . \tag{4.10}$$

Hence Φ is entirely determined by φ. Now Condition (4.9) is fulfilled if and only if, for any finite sets X and Y, any correspondence $R \subseteq (Y \times X)$, and any function $\alpha : X \to S$, we have

$$R\Omega_\alpha(\varphi) = \Omega_{R\alpha}(\varphi) \ .$$

In other words

$$\sum_\lambda u_\lambda R\Omega_\alpha(\lambda) = \sum_\lambda u_\lambda \Omega_{R\alpha}(\lambda) \ .$$

Hence Condition (4.9) is satisfied if and only if, for any finite sets X and Y, any correspondence $R \subseteq (Y \times X)$, any function $\alpha : X \to S$, and any function $\psi : Y \to T$, we have

$$\sum_{R\Omega_\alpha(\lambda) = \psi} u_\lambda = \sum_{\Omega_{R\alpha}(\lambda) = \psi} u_\lambda \ . \tag{4.11}$$

But for $y \in Y$

$$\begin{aligned} R\Omega_\alpha(\lambda)(y) &= \bigvee_{(y,s) \in R\Omega_\alpha} \lambda(s) \\ &= \bigvee_{(y,x) \in R} \lambda\alpha(x) \ . \end{aligned}$$

On the other hand

$$\begin{aligned} \Omega_{R\alpha}(\lambda)(y) &= \bigvee_{(y,s) \in \Omega_{R\alpha}} \lambda(s) \\ &= \lambda\big(R\alpha(y)\big) \\ &= \lambda\big(\bigvee_{(y,x) \in R} \alpha(x)\big) \ . \end{aligned}$$

Now take $X = S$ and $\alpha = \mathrm{id}_S$ in (4.11). Then let $Y = \mathcal{B}(S)$ be the set of subsets of S and let $R \subseteq (Y \times S)$ be the set of pairs (A, s), where $A \subseteq S$ and $s \in A$.

Then for a given map $\lambda : S \to T$, let us *define* $\psi : Y \to T$ by $\psi = \Omega_{R\,\mathrm{id}_S}(\lambda)$, in other words

$$\forall A \subseteq S,\ \psi(A) = \lambda(\bigvee_{s\in A} s)\ .$$

Suppose that there exists $\lambda' : S \to T$ such that $\Omega_{R\,\mathrm{id}_S}(\lambda') = \psi$. Then for $A \subseteq S$

$$\psi(A) = \lambda'(\bigvee_{s\in A} s)\ .$$

Taking $A = \{s\}$, it follows that $\lambda' = \lambda$. Hence in (4.11) with our specific choices, the right hand side is simply equal to u_λ.

On the other hand the left hand side is equal to the sum of $u_{\lambda'}$, for all λ' such that $R\lambda' = \psi$, that is, satisfying

$$\forall A \subseteq S,\ \psi(A) = \bigvee_{s\in A} \lambda'(s)\ .$$

Again, taking $A = \{s\}$, it follows that $\lambda' = \lambda$. With our specific choices, the left hand side of (4.11) is then equal to u_λ if and only if $R\lambda = \Omega_{R\,\mathrm{id}_S}(\lambda)$, that is, for any $A \subseteq S$,

$$\bigvee_{s\in A} \lambda(s) = \lambda(\bigvee_{s\in A} s)\ .$$

If this condition is not satisfied, then the left hand side of (4.11) is zero (empty sum). In other words $u_\lambda = 0$ if λ is not a morphism in the category $\mathcal{L}$ (and this is where we see the relevance of the definition of morphisms in $\mathcal{L}$).

It follows that $\varphi = \sum\limits_\lambda u_\lambda \lambda$ is a morphism in $k\mathcal{L}$, from S to T. We claim that the image of this morphism via the functor $F_?$ is equal to Φ and this will prove that the functor $F_? : k\mathcal{L} \to \mathcal{F}_k$ is full. To prove the claim, notice that, for any function $\beta : Y \to S$, we have

$$F_{\varphi,Y}(\beta) = \sum_\lambda u_\lambda F_{\lambda,Y}(\beta) = \sum_\lambda u_\lambda(\lambda \circ \beta) = \sum_\lambda u_\lambda \Omega_\beta\, \lambda = \Omega_\beta\, \varphi = \Phi_Y(\beta)\ ,$$

using the equation (4.10). This proves the claim and completes the proof that $F_?$ is full.

It remains to show that the functor $F_?$ is faithful. So let φ and ψ be two linear combinations of morphisms $S \to T$ in $\mathcal{L}$, which induce the same morphism $\theta = F_\varphi = F_\psi : F_S \to F_T$. Evaluating this morphism at the set S gives a map $\theta_S : F_S(S) \to F_T(S)$, and moreover

$$\theta_S(\mathrm{id}_S) = F_{\varphi,S}(\mathrm{id}_S) = \varphi \circ \mathrm{id}_S = \varphi \in F_T(S) = k(T^S)\ .$$

For the same reason, $\theta_S(\mathrm{id}_S) = F_{\psi,S}(\mathrm{id}_S) = \psi$, hence $\varphi = \psi$. This completes the proof of Theorem 4.8. $\square$

The connection between finite lattices and correspondence functors also has the following rather remarkable feature.

4.12. Theorem. *Let T be a finite lattice. The functor F_T is projective in $\mathcal{F}_k$ if and only if T is distributive.*

Proof : Let $\mathcal{B}(T)$ be the lattice of subsets of T. Let $\upsilon : \mathcal{B}(T) \to T$ be the morphism in the category $\mathcal{L}$ defined by

$$\forall A \subseteq T,\ \upsilon(A) = \bigvee_{t\in A} t\ .$$

This morphism induces a morphism of functors $F_\upsilon : F_{\mathcal{B}(T)} \to F_T$, and F_υ is surjective : indeed, if X is a finite set and $\alpha : X \to T$ is a function, and if we define $\hat{\alpha} : X \to \mathcal{B}(T)$ by

$$\forall x \in X,\ \hat{\alpha}(x) = \{\alpha(x)\}\ ,$$

then, for any $x \in X$

$$F_\upsilon(\hat{\alpha})(x) = (\upsilon \circ \hat{\alpha})(x) = \bigvee_{t \in \hat{\alpha}(x)} t = \alpha(x)\ ,$$

thus $F_\upsilon(\hat{\alpha}) = \alpha$, so F_υ is surjective.

Now if F_T is projective, then the morphism F_υ splits and there exists a morphism $\Phi : F_T \to F_{\mathcal{B}(T)}$ such that $F_\upsilon \circ \Phi$ is the identity morphism of F_T. It follows from Theorem 4.8 that Φ is of the form $\sum\limits_{\sigma \in M} u_\sigma F_\sigma$, where M is a finite set of morphisms $\sigma : T \to \mathcal{B}(T)$ in $\mathcal{L}$, and $u_\sigma \in k$. Moreover $F_\upsilon \circ \Phi$ is then equal to $\sum\limits_{\sigma \in M} u_\sigma F_{\upsilon\circ\sigma}$, hence there exists at least one such $\sigma \in M$ such that $\upsilon \circ \sigma$ is equal to the identity of T. This means that

$$\forall t \in T,\ t = \bigvee_{x \in \sigma(t)} x\ .$$

In particular $\sigma(t) \subseteq [\hat{0}, t]_T$ for any $t \in T$. Then for $r, s \in T$

$$[\hat{0}, r \wedge s]_T = [\hat{0}, r]_T \cap [\hat{0}, s]_T \supseteq \sigma(r) \cap \sigma(s) \supseteq \sigma(r \wedge s)\ ,$$

because σ is order-preserving. It follows that

$$r \wedge s \ \geq \bigvee_{x \in \sigma(r) \cap \sigma(s)} x \ \geq \bigvee_{x \in \sigma(r \wedge s)} x = r \wedge s\ ,$$

hence

$$r \wedge s = \bigvee_{x \in \sigma(r) \cap \sigma(s)} x\ .$$

Now, since σ preserves joins, we obtain, for all $r, s, t \in T$,

$$\begin{aligned}
t \wedge (r \vee s) &= \bigvee_{x \in \sigma(t) \cap \sigma(r \vee s)} x \\
&= \bigvee_{x \in \sigma(t) \cap \big(\sigma(r) \cup \sigma(s)\big)} x \\
&= \bigvee_{x \in \big(\sigma(t) \cap \sigma(r)\big) \cup \big(\sigma(t) \cap \sigma(s)\big)} x \\
&= \Big(\bigvee_{x \in \sigma(t) \cap \sigma(r)} x\Big) \vee \Big(\bigvee_{x \in \sigma(t) \cap \sigma(s)} x\Big) \\
&= (t \wedge r) \vee (t \wedge s)\ .
\end{aligned}$$

In other words the lattice T is distributive.

Conversely, by Lemma 3.5, any finite distributive lattice T is isomorphic to the lattice $I_\downarrow(E, R)$ of lower ideals of a finite poset (E, R). By Proposition 4.5, the associated functor F_T is projective in $\mathcal{F}_k$. This completes the proof of Theorem 4.12□

5. Quotients of functors associated to lattices

We now introduce, for any finite lattice T, a subfunctor of F_T naturally associated with the set of irreducible elements of T.

5.1. Notation. *Let T be a finite lattice and let (E,R) be the full subposet of its irreducible elements. For a finite set X, let $H_T(X)$ denote the k-submodule of $F_T(X)=k(T^X)$ generated by all functions $\varphi:X\to T$ such that $E\nsubseteq\varphi(X)$.*

5.2. Proposition. *Let T be a finite lattice and let (E,R) be the full subposet of its irreducible elements.*

(a) The assignment sending a finite set X to $H_T(X)\subseteq F_T(X)$ is a subfunctor H_T of F_T.
(b) The evaluation $(F_T/H_T)(X)$ has a k-basis consisting of (the classes of) all functions $\varphi:X\to T$ such that $E\subseteq\varphi(X)$.
(c) The set E is a minimal set for the functor F_T/H_T .

Proof : (a) Let X and Y be finite sets, let $Q\in\mathcal{C}(Y,X)$ be a correspondence, and let $\varphi:X\to T$ be a function. Then

$$(Q\varphi)(Y)\cap E\subseteq\varphi(X)\cap E\ . \tag{5.3}$$

Indeed, if $e\in E$ and $e=(Q\varphi)(y)$, for $y\in Y$, then

$$e=\bigvee_{(y,x)\in Q}\varphi(x)\ .$$

As e is irreducible in T, there exists $x\in X$ such that $(y,x)\in Q$ and $e=\varphi(x)$, and (5.3) follows.

In particular, if $\varphi(X)\cap E$ is a proper subset of E, then $(Q\varphi)(Y)\cap E$ is a proper subset of E. Hence H_T is a subfunctor of F_T.

(b) This follows from the definitions of F_T and H_T.

(c) If $|X|<|E|$, then $|\varphi(X)|<|E|$ and therefore there is no map $\varphi:X\to T$ such that $E\subseteq\varphi(X)$. Consequently, the k-basis of (b) is empty in that case, so that $(F_T/H_T)(X)=\{0\}$. Now if $X=E$, then the k-basis of (b) consists of all bijections $E\to E$ (followed by the inclusion map $E\to T$), so that $(F_T/H_T)(E)\cong k\Sigma_E$. This shows that E is a minimal set for F_T/H_T. □

The quotient functor F_T/H_T plays a important role in our work, in particular in Theorem 6.5 and for the description of the fundamental functors and the simple functors in [BT3]. We now give another characterization of $H_T(X)$.

5.4. Proposition. *Let $T=I^{\uparrow}(E,R)$ for a finite poset (E,R) and let X be a finite set.*

(a) Under the isomorphism $F_T\to k\mathcal{C}(-,E)R$ of Proposition 4.5, $H_T(X)$ is isomorphic to the k-submodule of $k\mathcal{C}(X,E)R$ generated by the correspondences S which have no retraction, that is, for which there is no $U\in\mathcal{C}(E,X)$ such that $US=R$.
(b) Under the isomorphism $F_T\to k\mathcal{C}(-,E)R$ of Proposition 4.5, the image of $F_T(X)/H_T(X)$ is a free k-module with basis consisting of all the correspondences $S\in\mathcal{C}(X,E)R$ which have a retraction $U\in\mathcal{C}(E,X)$.

Proof : By Proposition 4.5, the functor F_T is isomorphic to the functor $k\mathcal{C}(-,E)R$ by sending, for a finite set X, a function $\varphi:X\to I^{\uparrow}(E,R)$ to the correspondence $\Gamma_\varphi=\{(x,e)\in X\times E\mid e\in\varphi(x)\}$.

(a) The set $E^{\uparrow}$ of irreducible elements of the lattice $I^{\uparrow}(E,R)$ is the set of principal upper ideals

$$[e,\cdot[_R=\big\{f\in E\mid (e,f)\in R\big\}\ ,$$

for $e \in E$. Let $\varphi : X \to I^{\uparrow}(E,R)$ be such that $\varphi \notin H_T(X)$, that is, $\varphi(X) \supseteq E^{\uparrow}$. Then, for each $e \in E$, there exists $x_e \in X$ such that $\varphi(x_e) = [e, \cdot[_R$. Let $U \in \mathcal{C}(E,X)$ be defined by

$$U = \big\{(e, x_e) \mid e \in E\big\} \subseteq E \times X \ .$$

Then for any $e \in E$

$$(U\varphi)(e) = \bigcup_{(e,x)\in U} \varphi(x) = \varphi(x_e) = [e, \cdot[_R \ .$$

By Lemma 4.4, it follows that

$$U\Gamma_\varphi = \Gamma_{U\varphi} = \big\{(e,f) \in E \times E \mid f \in [e, \cdot[_R\big\} = R \ ,$$

so Γ_φ has a retraction.

Conversely, let $S \in \mathcal{C}(X,E)R$ be a correspondence such that there exists a correspondence $U \in \mathcal{C}(X,E)$ with $US = R$. Then $S = \Gamma_\varphi$, where $\varphi : X \to I^{\uparrow}(E,R)$ is the function defined by $\varphi(x) = \big\{e \in E \mid (x,e) \in S\big\}$, for any $x \in X$. It follows that $US = \Gamma_{U\varphi} = R$, or in other words

$$\forall e, f \in E, \ (e,f) \in R \iff \exists x \in X, \ (e,x) \in U, \ (x,e) \in S \ .$$

As $\Delta_E \subseteq R$, for any $e \in E$, there exists $x_e \in X$ such that $(e, x_e) \in U$ and $(x_e, e) \in S$. Moreover if $(x_e, f) \in S$, then $(e,f) \in R$, and conversely, if $(e,f) \in R$, then $(x_e, f) \in SR = S$. In other words, $f \in \varphi(x_e)$ if and only if $(e,f) \in R$. It follows that $\varphi(x_e) = [e, \cdot[_R$, hence $\varphi(X) \supseteq E^{\uparrow}$. This proves that $\varphi \notin H_T(X)$.

(b) This follows from (a). □

5.5. Remark. In the special case when $R = \Delta_E$ is the equality relation, then $\mathcal{C}(X,E)R = \mathcal{C}(X,E)$ and a retraction of $S \in \mathcal{C}(X,E)$ is a correspondence $U \in \mathcal{C}(E,X)$ such that $US = \mathrm{id}_E$ (a retraction in the usual sense). Moreover, if $S \in \mathcal{C}(X,E)$ has a retraction, then S is a monomorphism in the category $\mathcal{C}$. It can be shown conversely that any monomorphism in the category $\mathcal{C}$ has a retraction. Thus in this case, the evaluation $F_T(X)/H_T(X)$ of the quotient functor F_T/H_T has a k-basis consisting of all the monomorphisms in $\mathcal{C}(X,E)$.

In order to deal with quotients of the functor F_T, we need information on morphisms starting from F_T. We first need a lemma.

5.6. Lemma. *Let T be a finite lattice, let (E,R) be the full subposet of its irreducible elements, and let $\iota : E \to T$ denote the inclusion map.*

(a) If $\varphi : X \to T$ is a function, then $\Gamma_\varphi \, \iota = \varphi$ and $\Gamma_\varphi R^{op} = \Gamma_\varphi$, where Γ_φ is defined in Notation 3.7.
(b) $\Gamma_\iota = R^{op}$.
(c) The correspondence functor F_T is generated by $\iota \in F_T(E)$.

Proof : (a) By definition, the map $\Gamma_\varphi \, \iota : X \to T$ satisfies

$$\forall x \in X, \ \ (\Gamma_\varphi \, \iota)(x) = \bigvee_{(x,e)\in\Gamma_\varphi} \iota(e) = \bigvee_{e \leq_T \varphi(x)} e \ = \varphi(x) \ ,$$

as any element t of T is equal to the join of the irreducible elements of T smaller than t. Thus we have $\Gamma_\varphi \, \iota = \varphi$. The equality $\Gamma_\varphi R^{op} = \Gamma_\varphi$ was proved in Lemma 3.8.

(b) $\Gamma_\iota = \{(x,e) \in E \times E \mid e \leq_T \iota(x)\} = \{(x,e) \in E \times E \mid e \leq_R x\} = R^{op}$.

(c) For every function $\varphi : X \to T$, we have $\varphi = \Gamma_\varphi \iota$ by (a). Therefore F_T is generated by $\iota \in F_T(E)$. □

5.7. Proposition. *Let T be a finite lattice, let (E,R) be the full subposet of its irreducible elements, and let $\iota : E \to T$ denote the inclusion map. Let M be a correspondence functor.*

(a) *The k-linear map*
$$\mathrm{Hom}_{\mathcal{F}_k}(F_T, M) \longrightarrow M(E)\,, \quad \Phi \mapsto \Phi_E(\iota)$$
is injective. Its image is contained in the k-submodule
$$R^{op}M(E) = \{m \in M(E) \mid R^{op}m = m\}\ .$$

(b) *If T is distributive, so that $T \cong I_\downarrow(E,R)$, then the image of the above map is equal to $R^{op}M(E)$, so that $\mathrm{Hom}_{\mathcal{F}_k}(F_T, M) \cong R^{op}M(E)$ as k-modules.*

Proof : (a) By Lemma 5.6, for any $\Phi \in \mathrm{Hom}_{\mathcal{F}_k}(F_T, M)$ and any map $\varphi : X \to T$, we have
$$\Phi_X(\varphi) = \Phi_X(\Gamma_\varphi\, \iota) = \Gamma_\varphi\, \Phi_E(\iota)\ .$$
This shows that Φ is entirely determined by $\Phi_E(\iota)$, proving the injectivity of the map $\Phi \mapsto \Phi_E(\iota)$.

Moreover, $\Phi_E(\iota) = \Gamma_\iota \Phi_E(\iota) = R^{op}\Phi_E(\iota)$, because $\Gamma_\iota = R^{op}$ by Lemma 5.6. Therefore $\Phi_E(\iota)$ is contained in $R^{op}M(E)$.

(b) Since T is distributive, we have
$$\Gamma_{Q\varphi} = Q\Gamma_\varphi$$
by Lemma 4.4. Now given $m \in R^{op}M(E)$, we can define $\Phi : F_T \to M$ by setting
$$\Phi_X(\varphi) = \Gamma_\varphi m\ , \quad \forall \varphi : X \to T\ .$$
This is indeed a natural transformation of functors since
$$\Phi_Y(Q\varphi) = \Gamma_{Q\varphi} m = Q\Gamma_\varphi m = Q\Phi_X(\varphi)$$
for any correspondence $Q \subseteq Y \times X$. Moreover,
$$\Phi_E(\iota) = \Gamma_\iota m = R^{op} m = m\ ,$$
because $m \in R^{op}M(E)$ by assumption and R^{op} is idempotent. Thus m is indeed in the image of the map $\Phi \mapsto \Phi_E(\iota)$. □

When k is a field, we wish to give some information on simple functors $S_{F,Q,V}$ appearing as quotients of F_T. We prove a more general result over an arbitrary commutative ring k, involving the not necessarily simple functors $S_{F,Q,V}$ introduced in Section 2.

5.8. Theorem. *Let T be a finite lattice and let (E,R) be the full subposet of its irreducible elements. Let (F,Q) be a poset and let V be a $k\,\mathrm{Aut}(F,Q)$-module generated by a single element (e.g. a simple module).*

(a) *If $S_{F,Q,V}$ is isomorphic to a quotient of F_T , then $|F| \leq |E|$.*
(b) *Assume that $F = E$. If $S_{E,Q,V}$ is isomorphic to a quotient of F_T, then there exists a permutation $\sigma \in \Sigma_E$ such that $R^{op} \subseteq {}^\sigma Q$.*
(c) *Assume that $F = E$ and that T is distributive, so that $T \cong I_\downarrow(E,R)$. Then $S_{E,Q,V}$ is isomorphic to a quotient of F_T if and only if there exists a permutation $\sigma \in \Sigma_E$ such that $R^{op} \subseteq {}^\sigma Q$.*

Proof : (a) If $S_{F,Q,V}$ is isomorphic to a quotient of F_T, then $\mathrm{Hom}_{\mathcal{F}_k}(F_T, S_{F,Q,V}) \neq \{0\}$, so we have $S_{F,Q,V}(E) \neq \{0\}$ by Proposition 5.7. But F is a minimal set for $S_{F,Q,V}$ by Proposition 2.6, so $|F| \leq |E|$.

(b) If $S_{E,Q,V}$ is isomorphic to a quotient of F_T, there exists a nonzero morphism $\Phi : F_T \to S_{E,Q,V}$. By Proposition 5.7, $\Phi_E(\iota) = m \neq 0 \in R^{op}S_{E,Q,V}(E)$. By Proposition 2.6, we know that

$$S_{E,Q,V}(E) \cong T_{Q,V} = \mathcal{P}_E f_Q \otimes_{k\,\mathrm{Aut}(E,Q)} V$$

and $\mathcal{P}_E f_Q$ is a free k-module with basis $\{\Delta_\sigma f_Q \mid \sigma \in \Sigma_E\}$, by Proposition 2.4. Thus we can write

$$m = \sum_{\sigma\in\Sigma_E} \lambda_\sigma \Delta_\sigma f_Q \otimes v \qquad (\lambda_\sigma \in k) \; .$$

Since $m \in R^{op}S_{E,Q,V}(E)$, we have $R^{op}m = m$ and so there exists $\sigma \in \Sigma_E$ such that $R^{op}\Delta_\sigma f_Q \neq 0$. Hence $R^{op}\Delta_\sigma f_Q \Delta_\sigma^{-1} \neq 0$, that is, $R^{op} \subseteq {}^\sigma Q$, by Theorem 6.2 of [BT1].

(c) One implication follows from (b). Assume now that there exists a permutation $\sigma \in \Sigma_E$ such that $R^{op} \subseteq {}^\sigma Q$. We first note that $S_{E,Q,V}$ is generated by $f_Q \otimes v \in S_{E,Q,V}(E) \cong \mathcal{P}_E f_Q \otimes_{k\,\mathrm{Aut}(E,Q)} V$, where v is a generator of V. This follows from the definition of $S_{E,Q,V}$ as a quotient of $L_{E,T_{Q,V}}$ and the fact that any functor $L_{E,W}$ is generated by $L_{E,W}(E) = W$ by definition.

It is easy to see that $S_{E,Q,V} \cong S_{E,{}^\sigma Q,{}^\sigma V}$ for any $\sigma \in \Sigma_E$, because $T_{Q,V} \cong T_{{}^\sigma Q,{}^\sigma V}$ by construction (see Theorem 8.1 in [BT1] for more details). Since R^{op} is contained in a conjugate of Q, we can assume that $R^{op} \subseteq Q$. This is equivalent to $R^{op}f_Q = f_Q$, by Theorem 6.2 of [BT1].

Thus $m = f_Q \otimes v \in S_{E,Q,V}(E)$ is invariant under left multiplication by R^{op}. By Proposition 5.7 and the assumption that T is distributive, there exists a morphism $\Phi : F_T \to S_{E,Q,V}$ such that $\Phi_E(\iota) = f_Q \otimes v$. Since this is nonzero and generates $S_{E,Q,V}$, this functor is isomorphic to a quotient of F_T. □

The similar question of finding fundamental functors appearing as quotients of F_T will be considered later in Theorem 6.9.

6. The fundamental functor associated to a poset

The fundamental functor $\mathbb{S}_{E,R}$ associated to a poset (E,R) was introduced in Section 2. One of our important goals is to give a precise description of its evaluations and use it to deduce a precise description of the evaluations of simple functors, but this will be fully achieved only in [BT3]. We prepare the ground by proving several main results about $\mathbb{S}_{E,R}$. Recall from Proposition 2.6 that E is a minimal set for $\mathbb{S}_{E,R}$ and that $\mathbb{S}_{E,R}(E)$ is isomorphic to the fundamental module $\mathcal{P}_E f_R$, which is described in Proposition 2.4.

Since $\mathbb{S}_{E,R} = L_{E,\mathcal{P}_E f_R}/J_{E,\mathcal{P}_E f_R}$ by definition, it is important to know when an element of $L_{E,\mathcal{P}_E f_R}(X)$ belongs to $J_{E,\mathcal{P}_E f_R}(X)$, where X is a finite set. For this analysis, we note that an element of $L_{E,\mathcal{P}_E f_R}(X)$ is written

$$\sum_{S\in\mathcal{C}(X,E)} \lambda_S S \otimes f_R \in k\mathcal{C}(X,E) \otimes_{k\mathcal{C}(E,E)} \mathcal{P}_E f_R = L_{E,\mathcal{P}_E f_R}(X) \; ,$$

where $\lambda_S \in k$ for every S, because the tensor product is over $k\mathcal{C}(E,E)$ and $\mathcal{P}_E$ is a quotient algebra of $k\mathcal{C}(E,E)$. But since $Rf_R = f_R$ (by Proposition 2.4) and the tensor product is over $k\mathcal{C}(E,E)$, we can replace S by SR and obtain a sum running only over $S \in \mathcal{C}(X,E)R$.

6.1. Lemma. *Consider an element*

$$\sum_{S\in\mathcal{C}(X,E)R} \lambda_S S\otimes f_R \in L_{E,\mathcal{P}_E f_R}(X)\ .$$

This element belongs to $J_{E,\mathcal{P}_E f_R}(X)$ *if and only if*

$$\forall U\in R\mathcal{C}(E,X)\,,\qquad \sum_{\substack{S\in\mathcal{C}(X,E)R\\ US=R}} \lambda_S = 0\ .$$

Proof : By the definition of $J_{E,\mathcal{P}_E f_R}(X)$, we have

$$\begin{aligned}\sum_{S\in\mathcal{C}(X,E)R} \lambda_S S\otimes f_R \ &\in\ J_{E,\mathcal{P}_E f_R}(X) \iff \\ &\forall U\in \mathcal{C}(E,X)\,,\qquad \sum_{S\in\mathcal{C}(X,E)R} \lambda_S US\cdot f_R = 0\ .\end{aligned}$$

By Proposition 2.4, the action of the relation $US\in\mathcal{C}(E,E)$ on f_R is given by

$$US\cdot f_R = \begin{cases} \Delta_\tau f_R & \text{if } \exists \tau\in\Sigma_E \text{ such that } \Delta_E\subseteq \Delta_{\tau^{-1}}US\subseteq R,\\ 0 & \text{otherwise .}\end{cases}$$

We claim that

$$\Delta_E\subseteq \Delta_{\tau^{-1}}US\subseteq R \iff US=\Delta_\tau R\ .$$

If the left hand side holds, then multiply on the right by R and use the fact that $SR=S$ and $R^2=R$ (by transitivity and reflexivity of R) to obtain $\Delta_{\tau^{-1}}US=R$, hence $US=\Delta_\tau R$. Conversely, if the right hand side holds, then $\Delta_{\tau^{-1}}US=R$, hence

$$R\Delta_{\tau^{-1}}US=R^2=R$$

by transitivity and reflexivity of R. In particular, by reflexivity again,

$$\Delta_E\subseteq R\Delta_{\tau^{-1}}US$$

so that, for any $(a,a)\in\Delta_E$, there exists $b\in E$ with $(a,b)\in R$ and $(b,a)\in\Delta_{\tau^{-1}}US=R$. By antisymmetry of R, it follows that $b=a$ and therefore $(a,a)\in\Delta_{\tau^{-1}}US$, so that $\Delta_E\subseteq\Delta_{\tau^{-1}}US$. This shows that the left hand side holds, proving the claim.

It follows that our given element belongs to $J_{E,\mathcal{P}_E f_R}(X)$ if and only if

$$\textbf{(6.2)}\qquad \forall U\in\mathcal{C}(E,X)\,,\qquad \sum_{\substack{S\in\mathcal{C}(X,E)R\\ \tau\in\Sigma_E\\ US=\Delta_\tau R}} \lambda_S\Delta_\tau f_R = 0\ .$$

But by Proposition 2.4, $\mathcal{P}_E f_R$ is a free k-module with basis $\{\Delta_\tau f_R \mid \tau\in\Sigma_E\}$. Therefore (6.2) is equivalent to

$$\forall U\in\mathcal{C}(E,X)\,,\quad \forall\tau\in\Sigma_E\,,\qquad \sum_{\substack{S\in\mathcal{C}(X,E)R\\ US=\Delta_\tau R}} \lambda_S = 0\ .$$

Replacing U by $\Delta_{\tau^{-1}}U$, this is equivalent to

$$\textbf{(6.3)}\qquad \forall U\in\mathcal{C}(E,X)\,,\qquad \sum_{\substack{S\in\mathcal{C}(X,E)R\\ US=R}} \lambda_S = 0\ .$$

Now we claim that

$$US=R\iff RUS=R\ .$$

If the left hand side holds, then multiply on the left by R and use the fact that $R^2=R$. Conversely, if the right hand side holds, then $US=\Delta_E US\subseteq RUS=R$. Moreover, $\Delta_E\subseteq R=RUS$, so that, for any $(a,a)\in\Delta_E$, there exists $b\in E$ with

$(a, b) \in R$ and $(b, a) \in US \subseteq R$. By antisymmetry of R, it follows that $b = a$ and therefore $(a, a) \in US$, so that $\Delta_E \subseteq US$. Multiplying on the right by R and using the fact that $SR = S$, we obtain $R \subseteq US$. This shows that $R = US$, as required, proving the claim.

It now follows that in the family of equations (6.3), the equation for U is the same as the equation for RU. Therefore we can assume that $U \in R\mathcal{C}(E, X)$ and (6.3) is equivalent to

$$\forall\, U \in R\mathcal{C}(E, X)\,, \qquad \sum_{\substack{S \in \mathcal{C}(X,E)R \\ US = R}} \lambda_S = 0\,,$$

proving the lemma. □

Before stating the main theorem, we need another lemma.

6.4. Lemma. *Let T be a finite lattice, let (E, R) be the full subposet of its irreducible elements, and let $\iota : E \to T$ denote the inclusion map. For any correspondence $W \in \mathcal{C}(E, X)$ and any map $\varphi : X \to T$, the following two statements are equivalent :*

$$W\Gamma_\varphi = R^{op} \iff W\varphi = \mathrm{id}_E\ ,$$

where $\Gamma_\varphi \in \mathcal{C}(X, E)$ is defined in Notation 3.7.

Proof : Let $e, f \in E$. Since (E, R) is a full subposet of T, the relation $(f, e) \in R^{op}$ is equivalent to $e \leq f$, where we write $\leq$ instead of $\leq_T$ for simplicity. Suppose that $W\Gamma_\varphi = R^{op}$. Then $e \leq f$ if and only if

$$\exists\, x \in X \ \text{ with } \ (f, x) \in W\,,\ (x, e) \in \Gamma_\varphi \iff \exists\, x \in X \ \text{ with } \ (f, x) \in W\,,\ e \leq \varphi(x)\ .$$

Now we can write

$$\varphi(x) = \bigvee_{\substack{e \in E \\ e \leq \varphi(x)}} e$$

because any element of T is a join of irreducible elements. It follows that

$$(W\varphi)(f) = \bigvee_{\substack{x \in X \\ (f,x) \in W}} \varphi(x) = \bigvee_{\substack{x \in X \\ (f,x) \in W}} \ \bigvee_{\substack{e \in E \\ e \leq \varphi(x)}} e = \bigvee_{e \leq f} e\ ,$$

by the equivalence above. Thus $(W\varphi)(f) = f$, so that $W\varphi = \mathrm{id}_E$.

Conversely, suppose that $W\varphi = \mathrm{id}_E$. Since $\varphi = \Gamma_\varphi \iota$ by Lemma 5.6, we have $W\Gamma_\varphi \iota = \mathrm{id}_E$ and we obtain, for every $e \in E$,

$$e = (W\Gamma_\varphi \iota)(e) = \bigvee_{\substack{f \in E \\ (e,f) \in W\Gamma_\varphi}} \iota(f) = \bigvee_{\substack{f \in E \\ (e,f) \in W\Gamma_\varphi}} f\ .$$

Since e is irreducible, it follows that $e = f$ for some $f \in E$ such that $(e, f) \in W\Gamma_\varphi$, and so $(e, e) \in W\Gamma_\varphi$, showing that $\Delta_E \subseteq W\Gamma_\varphi$. On the other hand, we also obtain $f \leq e$ for every $f \in E$ such that $(e, f) \in W\Gamma_\varphi$, showing that $W\Gamma_\varphi \subseteq R^{op}$. Therefore $\Delta_E \subseteq W\Gamma_\varphi \subseteq R^{op}$. Multiplying on the right by R^{op}, we deduce that $W\Gamma_\varphi = R^{op}$, thanks to the fact that $\Gamma_\varphi R^{op} = \Gamma_\varphi$ by Lemma 3.8. □

The following theorem establishes the link between the functor F_T associated with a lattice T and the fundamental correspondence functors.

6.5. Theorem. *Let T be a finite lattice, let (E,R) be the full subposet of its irreducible elements, and let $\iota : E \to T$ denote the inclusion map.*

(a) There exists a unique morphism of correspondence functors
$$\Theta_T : F_T \longrightarrow \mathbb{S}_{E,R^{op}}$$
mapping $\iota \in F_T(E)$ to $f_{R^{op}} \in \mathbb{S}_{E,R^{op}}(E) \cong \mathcal{P}_E f_{R^{op}}$. Moreover, Θ_T is surjective.

(b) The subfunctor H_T of F_T is contained in the kernel of Θ_T. In particular, Θ_T induces a surjective morphism $\overline{\Theta}_T : F_T/H_T \longrightarrow \mathbb{S}_{E,R^{op}}$.

Proof : (a) For any finite set X and any function $\varphi : X \to T$, we let
$$\widetilde{\Theta}_{T,X}(\varphi) = \Gamma_\varphi \otimes f_{R^{op}} \in k\mathcal{C}(X,E) \otimes_{k\mathcal{C}(E,E)} f_{R^{op}} = L_{E,\mathcal{P}_E f_{R^{op}}}(X) ,$$
where $\Gamma_\varphi \in \mathcal{C}(X,E)$ is defined in Notation 3.7. This extends to a k-linear map
$$\widetilde{\Theta}_{T,X} : F_T(X) \longrightarrow L_{E,\mathcal{P}_E f_{R^{op}}}(X)$$
which we compose with the canonical surjection
$$\pi_X : L_{E,\mathcal{P}_E f_{R^{op}}}(X) \longrightarrow L_{E,\mathcal{P}_E f_{R^{op}}}(X)/J_{E,\mathcal{P}_E f_{R^{op}}}(X) = \mathbb{S}_{E,R^{op}}(X)$$
to obtain a k-linear map
$$\Theta_{T,X} : F_T(X) \longrightarrow \mathbb{S}_{E,R^{op}}(X) .$$
The family of maps $\widetilde{\Theta}_{T,X}$ is not a morphism of functors, but we are going to prove that it becomes so after composition with $\pi : L_{E,\mathcal{P}_E f_{R^{op}}} \to \mathbb{S}_{E,R^{op}}$. We have to show that, for any correspondence $V \in \mathcal{C}(Y,X)$ and any map $\varphi : X \to T$, we have
$$\Theta_{T,Y}(V\varphi) = V\Theta_{T,X}(\varphi) , \quad \text{that is,} \quad \widetilde{\Theta}_{T,Y}(V\varphi) - V\widetilde{\Theta}_{T,X}(\varphi) \in J_{E,\mathcal{P}_E f_{R^{op}}}(Y) .$$
In other words, we need to prove that
$$(\Gamma_{V\varphi} - V\Gamma_\varphi) \otimes f_{R^{op}} \in J_{E,\mathcal{P}_E f_{R^{op}}}(Y) .$$
Since $\Gamma_{V\varphi}R^{op} = \Gamma_{V\varphi}$ and $\Gamma_\varphi R^{op} = \Gamma_\varphi$ by Lemma 3.8, we can apply Lemma 6.1 to the element $(\Gamma_{V\varphi} - V\Gamma_\varphi) \otimes f_{R^{op}}$ (with R^{op} instead of R). Since we have a difference of two terms, we only have to show that the additional conditions in the equations occur simultaneously, that is,
$$\forall U \in R^{op}\mathcal{C}(E,Y) , \quad U\Gamma_{V\varphi} = R^{op} \iff UV\Gamma_\varphi = R^{op} . \tag{6.6}$$
We emphasize that the equation $V\Gamma_\varphi = \Gamma_{V\varphi}$ does not hold in general (it holds if the lattice T is distributive by Lemma 6.6), but it will become correct after left multiplication by $U \in R^{op}\mathcal{C}(E,Y)$. Using Lemma 6.4, we have to show that, for all $U \in R^{op}\mathcal{C}(E,Y)$,
$$U(V\varphi) = \mathrm{id}_E \iff (UV)\varphi = \mathrm{id}_E .$$
But this is obvious in view of the action of correspondences on $\varphi \in F_T(X)$. Therefore Θ_T is a morphism of correspondence functors, as was to be shown.

By Lemma 5.6, the image of $\iota \in F_T(E)$ is
$$\Theta_{T,E}(\iota) = \Gamma_\iota \otimes f_{R^{op}} = R^{op} \otimes f_{R^{op}} = \Delta_E \otimes R^{op} f_{R^{op}} = \Delta_E \otimes f_{R^{op}} ,$$
because the tensor product is over $k\mathcal{C}(E,E)$ and because $R^{op} f_{R^{op}} = f_{R^{op}}$ by the action of relations on $f_{R^{op}}$ (see Proposition 2.4). Now $\Delta_E \otimes f_{R^{op}}$ corresponds to $f_{R^{op}}$ under the isomorphism $\mathbb{S}_{E,R^{op}}(E) \cong \mathcal{P}_E f_{R^{op}}$. Therefore Θ_T maps ι to $f_{R^{op}}$.

Since $R^{op} f_{R^{op}} = f_{R^{op}}$, any generator $S \otimes f_{R^{op}}$ of $L_{E,\mathcal{P}_e f_{R^{op}}}(X)$, where $S \in \mathcal{C}(X,E)$, can be written
$$S \otimes f_{R^{op}} = S \otimes R^{op} f_{R^{op}} = SR^{op} \otimes f_{R^{op}} = \Gamma_\varphi \otimes f_{R^{op}}$$
for some $\varphi : X \to T$, in view of Lemma 3.8. This shows that Θ_T is surjective.

(b) A generator of $H_T(X)$ is a map $\varphi : X \to T$ such that $E \not\subseteq \varphi(X)$. We need to prove that $\varphi \in \operatorname{Ker}\Theta_T$, or in other words $\Gamma_\varphi \otimes f_{R^{op}} \in J_{E,\mathcal{P}_e f_{R^{op}}}(X)$. In order to prove this, we apply Lemma 6.1 and we note that we have here a sum with a single term, so we have to check that the additional condition $U\Gamma_\varphi = R^{op}$ never holds, where $U \in R^{op}\mathcal{C}(E,X)$. By Lemma 6.4, $U\Gamma_\varphi \neq R^{op}$ is equivalent to $U\varphi \neq \mathrm{id}_E$. But $U\varphi \in H_T(E)$ because H_T is a subfunctor and obviously $\mathrm{id}_E \notin H_T(E)$, so $U\varphi \neq \mathrm{id}_E$, as required. □

6.7. Remark. By Lemma 3.5, any lattice T having (E,R) as its full subposet of irreducible elements is isomorphic to a quotient of the distributive lattice $I_\downarrow(E,R)$ via the map

$$\pi : I_\downarrow(E,R) \longrightarrow T\,, \qquad A \mapsto \bigvee_{e\in A} e$$

where the join is taken in the lattice T. Since $I_\downarrow(E,R)$ also has (E,R) as its full subposet of irreducible elements, there is a surjective morphism

$$\Theta_{I_\downarrow(E,R)} : F_{I_\downarrow(E,R)} \longrightarrow \mathbb{S}_{E,R^{op}}$$

which is easily seen to correspond, via the isomorphism $F_{I_\downarrow(E,R)} \cong k\mathcal{C}(-,E)R^{op}$ of Proposition 4.5, to the morphism

$$\omega : k\mathcal{C}(-,E)R^{op} \longrightarrow \mathbb{S}_{E,R^{op}}\,, \qquad R^{op} \mapsto f_{R^{op}}\ .$$

Moreover, it is also elementary to check that the composite

$$k\mathcal{C}(-,E)R^{op} \cong F_{I_\downarrow(E,R)} \xrightarrow{F_\pi} F_T \xrightarrow{\Theta_T} \mathbb{S}_{E,R^{op}}$$

is equal to ω.

For our next result, we need a well-known result of algebraic K-theory.

6.8. Lemma. *Let L and L' be finitely generated free k-modules of the same rank, where k is a commutative ring. Then any surjective homomorphism of k-modules $f : L \to L'$ is an isomorphism.*

Proof : The homomorphism f splits, because L' is free. Hence there exists a homomorphism $g : L' \to L$ such that $f \circ g = \mathrm{id}_{L'}$. Let n be the common rank of the free k-modules L and L'. We can view f and g as square matrices of size n with coefficients in k, such that $fg = \mathrm{id}_n$. Taking determinants (which makes sense as k is commutative), we get that $\det(f)\det(g) = \det(\mathrm{id}_n) = 1$. Hence $\det(f)$ is invertible in k, and f is invertible as well. □

We now show that the relationship between F_T and $\mathbb{S}_{E,R^{op}}$ is very strong, in the sense that $\mathbb{S}_{E,R^{op}}$ is the only fundamental functor appearing as a quotient of F_T/H_T, where H_T is defined in Notation 5.1. Recall that E is a minimal set for the functor F_T/H_T, by Proposition 5.2.

6.9. Theorem. *Let T be a finite lattice and let (E,R) be the full subposet of its irreducible elements. Let (D,Q) be a finite poset such that there exists a surjective morphism $\Phi : F_T/H_T \to \mathbb{S}_{D,Q}$.*

- *(a) $|D| = |E|$ (so we can assume that $D = E$).*
- *(b) Assuming that $D = E$, there exists $\sigma \in \Sigma_E$ such that ${}^\sigma Q = R^{op}$. In other words, the posets (E,Q) and (E,R^{op}) are isomorphic.*
- *(c) The kernel of $\overline{\Theta}_T : F_T/H_T \to \mathbb{S}_{E,R^{op}}$ vanishes at E and*

$$\overline{\Theta}_{T,E} : F_T(E)/H_T(E) \longrightarrow \mathbb{S}_{E,R^{op}}(E) \cong \mathcal{P}_E f_{R^{op}}$$

is an isomorphism.

Proof : (a) By Proposition 5.7, any morphism $F_T/H_T \to \mathbb{S}_{D,Q}$ is determined by an element of $\mathbb{S}_{D,Q}(E)$. Since our given morphism Φ is surjective, it is nonzero, and therefore $\mathbb{S}_{D,Q}(E) \neq \{0\}$. Since D is a minimal set for $\mathbb{S}_{D,Q}$, we have $|D| \leq |E|$. Now there is a surjective morphism

$$\Phi_D : F_T(D)/H_T(D) \to S_{D,Q}(D) \neq \{0\} ,$$

so F_T/H_T does not vanish at D. Since E is a minimal set for F_T/H_T by Proposition 5.2, we have $|E| \leq |D|$.

(b) We prove that $\Phi_E : F_T(E)/H_T(E) \to S_{E,Q}(E)$ is an isomorphism. By Proposition 5.1, $F_T(E)/H_T(E)$ is a free k-module with basis the set of all bijections $E \to E$ (followed by the inclusion map $E \to T$). By Proposition 2.4, $S_{E,Q}(E) \cong \mathcal{P}_E f_Q$ is a free k-module with a basis consisting of the elements $\Delta_\sigma f_Q$, where σ runs through the group Σ_E of all permutations of E. Thus the two modules under consideration are finitely generated free k-modules with the same rank and we know that the map Φ_E is surjective. Therefore Φ_E is an isomorphism by Lemma 6.8.

The same argument applies to the surjective morphism $\overline{\Theta}_{T,E} : F_T(E)/H_T(E) \to \mathbb{S}_{E,R^{op}}(E)$, which is therefore also an isomorphism. It follows that there are isomorphisms of $k\mathcal{C}(E,E)$-modules

$$\mathcal{P}_E f_Q \cong S_{E,Q}(E) \cong F_T(E)/H_T(E) \cong \mathbb{S}_{E,R^{op}}(E) \cong \mathcal{P}_E f_{R^{op}} .$$

This isomorphism maps f_Q to $af_{R^{op}}$ for some $a \in \mathcal{P}_E$. Therefore it maps $f_Q = Qf_Q$ to $af_{R^{op}} = Qaf_{R^{op}}$ and in particular $Qaf_{R^{op}} \neq 0$. By Proposition 2.4, this is possible only if there exists $\sigma, \tau \in \Sigma_E$ such that

$$\Delta_E \subseteq \Delta_{\tau^{-1}} Q \subseteq {}^{\sigma}(R^{op}) .$$

In particular, $\Delta_\tau \subseteq Q$, forcing $\tau = \mathrm{id}$ because Q is an order relation. Thus $Q \subseteq {}^{\sigma}(R^{op})$. Swapping the role of Q and R^{op}, the same argument shows that R^{op} is contained in a conjugate of Q. Therefore Q and R^{op} are conjugate, as was to be shown.

(c) We have just seen that $\overline{\Theta}_{T,E} : F_T(E)/H_T(E) \longrightarrow \mathbb{S}_{E,R^{op}}(E)$ is an isomorphism. Therefore $\operatorname{Ker} \Theta_{T,E} = \{0\}$. In other words, the subfunctor $\operatorname{Ker} \Theta_T$ vanishes at E. □

7. The kernel of Θ_T

Theorem 6.5 shows that $\mathbb{S}_{E,R^{op}}(X)$ is isomorphic to a quotient of F_T and we want to understand the kernel. We do this in the following result in terms of a system of equations. The solution of this system of equations is quite hard and will only be obtained in [BT3], when we will compute the dimension of the evaluations of fundamental functors and simple functors.

7.1. Theorem. *Let T be a finite lattice, let (E,R) be the full subposet of its irreducible elements, and let X be a finite set. The kernel of the map*

$$\Theta_{T,X} : F_T(X) \to \mathbb{S}_{E,R^{op}}(X)$$

is equal to the set of linear combinations $\sum\limits_{\varphi:X\to T} \lambda_\varphi \varphi$, where $\lambda_\varphi \in k$, such that for any map $\psi : X \to I^{\uparrow}(E,R)$

$$\sum_{\substack{\varphi:X\to T \\ \Gamma_\psi^{op}\Gamma_\varphi = R^{op}}} \lambda_\varphi = 0 .$$

Recall from Notation 3.7 that $\Gamma_\varphi = \{(x,e) \in X \times E \mid e \in \varphi(x)\} \subseteq \mathcal{C}(X,E)$ *and* $\Gamma_\psi = \{(x,e) \in X \times E \mid e \in \psi(x)\} \subseteq \mathcal{C}(X,E)$.

Proof : The image of $\varphi : X \to T$ under the map $\Theta_{T,X}$ is equal to the class of $\Gamma_\varphi \otimes f_{R^{op}} \in L_{E,\mathcal{P}_E f_{R^{op}}}(X)$ in the quotient

$$\mathbb{S}_{E,R^{op}}(X) = L_{E,\mathcal{P}_E f_{R^{op}}}(X)/J_{E,\mathcal{P}_E f_{R^{op}}}(X) \; .$$

Therefore a linear combination $u = \sum\limits_{\varphi:X\to T} \lambda_\varphi \varphi$ lies in $\operatorname{Ker}\Theta_{T,X}$ if and only if $\sum\limits_{\varphi:X\to T} \lambda_\varphi \Gamma_\varphi \otimes f_{R^{op}}$ belongs to $J_{E,\mathcal{P}_E f_{R^{op}}}(X)$. We apply Lemma 6.1, using the fact that $\Gamma_\varphi R^{op} = \Gamma_\varphi$. It follows that $u \in \operatorname{Ker}\Theta_{T,X}$ if and only if

$$\forall\, U \in R^{op}\mathcal{C}(E,X)\,, \qquad \sum_{\substack{\varphi:X\to T\\ U\Gamma_\varphi = R^{op}}} \lambda_\varphi = 0 \; .$$

Now $U \in R^{op}\mathcal{C}(E,X)$ if and only if $U^{op} \in \mathcal{C}(X,E)R$ if and only if $U = \Gamma_\psi$ for some map $\psi : X \to I^{\uparrow}(E,R)$ (by Lemma 3.8). Thus the condition becomes

$$\forall\, \psi : X \to I^{\uparrow}(E,R)\,, \qquad \sum_{\substack{\varphi:X\to T\\ \Gamma_\psi^{op}\Gamma_\varphi = R^{op}}} \lambda_\varphi = 0 \; ,$$

as was to be shown. ∎

The condition $\Gamma_\psi^{op}\Gamma_\varphi = R^{op}$ which appears in the system of equations in Theorem 7.1 is the key for the description of the fundamental functor $\mathbb{S}_{E,R^{op}}$, and consequently for understanding the simple functors $S_{E,R,V}$. We need to characterize this condition in various useful ways.

We first introduce the following notation.

7.2. Notation. *Let T be a finite lattice, let (E,R) be the full subposet of its irreducible elements, let $\psi : X \to I^{\uparrow}(E,R)$ be any map, and let $\varphi : X \to T$ be any map. We define the function $\wedge\psi : X \to T$ by*

$$\forall x \in X, \;\; \wedge\psi(x) = \bigwedge_{e\in\psi(x)} e \; ,$$

where $\bigwedge$ is the meet in the lattice T.

Moreover, the notation $\varphi \leq \wedge\psi$ means that $\varphi(x) \leq_T \wedge\psi(x)$ for all $x \in X$.

We can now state the various characterizations of the condition which we need.

7.3. Theorem. *Let T be a finite lattice, let (E,R) be the full subposet of its irreducible elements, and let X be a finite set. Let $\varphi : X \to T$ be a map and let*

$$\Gamma_\varphi = \{(x,e) \in X \times E \mid e \leq_T \varphi(x)\}$$

be the associated correspondence. Let $\psi : X \to I^{\uparrow}(E,R)$ be a map and let

$$\Gamma_\psi^{op} = \big\{(e,x) \in E \times X \mid e \in \psi(x)\big\}$$

be the associated correspondence. The following conditions are equivalent.

(a) $\Gamma_\psi^{op}\varphi = \iota$.
(b) $\Gamma_\psi^{op}\Gamma_\varphi\iota = \iota$.
(c) $\Delta_E \subseteq \Gamma_\psi^{op}\Gamma_\varphi \subseteq R^{op}$.
(d) $\Gamma_\psi^{op}\Gamma_\varphi = R^{op}$.
(e) $\varphi \leq \wedge\psi$ *and* $\forall e \in E,\ \exists x \in X$ *such that* $\varphi(x) = e$ *and* $\psi(x) = [e,\cdot[_E$.
(f) $\forall t \in T,\ \psi\big(\varphi^{-1}(t)\big) \subseteq [t,\cdot[_T \cap E$ *and* $\forall e \in E,\ \psi\big(\varphi^{-1}(e)\big) = [e,\cdot[_E$.

Proof : (a) $\Leftrightarrow$ (b). By Lemma 5.6, we have $\varphi = \Gamma_\varphi \iota$.

(b) $\Leftrightarrow$ (c). We prove more generally that $Q\iota = \iota$ if and only if $\Delta_E \subseteq Q \subseteq R^{op}$, for any relation $Q \in \mathcal{C}(E, E)$. Suppose first that $\Delta_E \subseteq Q \subseteq R^{op}$. Then for any $e \in E$,

$$(Q\iota)(e) = \bigvee_{(e,f)\in Q} \iota(f) = \bigvee_{(e,f)\in Q} f \ .$$

Since $(e, e) \in Q$, it follows that $(Q\iota)(e) \geq_T e$. On the other hand since $Q \subseteq R^{op}$, if $(e, f) \in Q$, then $f \leq_R e$, hence $f \leq_T e$. Thus $(Q\iota)(e) \leq_T e$. It follows that $(Q\iota)(e) = e$ for any $e \in E$, i.e. $Q\iota = \iota$.

Conversely, if $Q\iota = \iota$, then

$$\forall e \in E, \quad \bigvee_{(e,f)\in Q} f = e \ .$$

As e is irreducible, it follows that $(e, e) \in Q$ for any $e \in E$, i.e. $\Delta_E \subseteq Q$. Moreover if $(e, f) \in Q$, then $f \leq_T e$, hence $f \leq_R e$ and so $Q \subseteq R^{op}$. This proves the claim and completes the proof of (b).

(c) $\Leftrightarrow$ (d). If (c) holds, multiply on the right by R^{op} and use the equality $\Gamma_\varphi R^{op} = \Gamma_\varphi$ of Lemma 3.8 to obtain (d). On the other hand, it is clear that (d) implies (c).

(d) $\Rightarrow$ (e). Suppose that $\Gamma_\psi^{op}\Gamma_\varphi = R^{op}$ and let $x \in X$. Then for all $f \leq_T \varphi(x)$ and for all $e \in \psi(x)$, we have $(e, x) \in \Gamma_\psi^{op}$ and $(x, f) \in \Gamma_\varphi$, hence $(e, f) \in R^{op}$, that is, $f \leq_R e$, hence $f \leq_T e$. Therefore $\varphi(x) = \bigvee_{f\leq_T \varphi(x)} f \leq_T e$, whenever $e \in \psi(x)$. Thus

$$\forall x \in X, \ \ \varphi(x) \leq_T \bigwedge_{e\in\psi(x)} e = \wedge\psi(x) \ ,$$

that is, $\varphi \leq \wedge\psi$. This shows that the first property in (e) holds.

Since $(e, e) \in R^{op}$, there exists $x_e \in X$ such that $e \leq_T \varphi(x_e)$ and $e \in \psi(x_e)$. Then for all $f \leq_T \varphi(x_e)$, we have $(e, x_e) \in \Gamma_\psi^{op}$ and $(x_e, f) \in \Gamma_\varphi$, hence $(e, f) \in R^{op}$, that is, $f \leq_R e$, or in other words $f \leq_T e$. Thus again $\varphi(x_e) = \bigvee_{f\leq_T\varphi(x_e)} f \leq_T e$, hence $\varphi(x_e) = e$. Moreover, if $g \in E$ with $g \in \psi(x_e)$, then $(g, x_e) \in \Gamma_\psi^{op}$ and $(x_e, e) \in \Gamma_\varphi$, hence $(g, e) \in R^{op}$, that is, $e \leq_T g$. Therefore $\psi(x_e) \subseteq [e, \cdot[_E$. But we also have $[e, \cdot[_E \subseteq \psi(x_e)$, as $e \in \psi(x_e)$ and $\psi(x_e)$ is an upper ideal of E. Thus $\psi(x_e) = [e, \cdot[_E$. This shows that the second property in (e) holds.

(e) $\Rightarrow$ (d). For any $e \in E$, there exists $x_e \in X$ such that $\varphi(x_e) = e$ and $\psi(x_e) = [e, \cdot[_E$. If now $(f, e) \in R^{op}$, then $e \leq_R f$, hence $f \in \psi(x_e)$. Since we also have $e \leq_R \varphi(x_e)$, we obtain $(f, x_e) \in \Gamma_\psi^{op}$ and $(x_e, e) \in \Gamma_\varphi$. Thus $R^{op} \subseteq \Gamma_\psi^{op}\Gamma_\varphi$.

Moreover if $(f, e) \in \Gamma_\psi^{op}\Gamma_\varphi$, then there exists $x \in X$ such that $f \in \psi(x)$ and $e \leq_T \varphi(x)$. Since $\varphi \leq \wedge\psi$, we have $\varphi(x) \leq_T \bigwedge_{f\in\psi(x)} f$. It follows that $e \leq_T f$, hence $e \leq_R f$, that is, $(f, e) \in R^{op}$. Thus $\Gamma_\psi^{op}\Gamma_\varphi \subseteq R^{op}$. Therefore we obtain $\Gamma_\psi^{op}\Gamma_\varphi = R^{op}$.

(e) $\Leftrightarrow$ (f). We are going to slightly abuse notation by setting, for any subset Y of X, $\psi(Y) = \bigcup_{x\in Y} \psi(x)$. Taking $t = \varphi(x)$, the first condition in (e) is equivalent to

$$\forall t \in T, \ e \in \psi\big(\varphi^{-1}(t)\big) \Longrightarrow t \leq_T e \ ,$$

which in turn is equivalent to

$$\forall t \in T, \ \psi\big(\varphi^{-1}(t)\big) \subseteq [t, \cdot[_T \cap E \ .$$

In particular $\psi\big(\varphi^{-1}(e)\big) \subseteq [e,\cdot[_E$ for all $e \in E$ because $[e,\cdot[_T \cap E = [e,\cdot[_E$. But the second condition in (e) says that e must belong to $\psi\big(\varphi^{-1}(e)\big)$, so we get $\psi\big(\varphi^{-1}(e)\big) = [e,\cdot[_E$. This shows that the second condition in (e) is equivalent to

$$\forall e \in E,\ \ \psi\big(\varphi^{-1}(e)\big) = [e,\cdot[_E\ .$$

This completes the proof of Theorem 7.3. ∎

Condition (d) will play an important role in the proof of Theorem 9.5 below, while conditions (e) and (f) will be main tools used in [BT3].

8. Duality and opposite lattices

In this section, we prove a perfect duality between the functor associated to a lattice T and the functor associated to the opposite lattice T^{op}. We work with an arbitrary commutative base ring k.

Let F be a correspondence functor over k. The *dual* $F^\natural$ of F is the correspondence functor defined on a finite set X by

$$F^\natural(X) := \mathrm{Hom}_k\big(F(X), k\big)\ .$$

If Y is a finite set and $R \subseteq Y \times X$, then the map $F^\natural(R) : F^\natural(X) \to F^\natural(Y)$ is defined by

$$\forall \alpha \in F^\natural(X),\ \ F^\natural(R)(\alpha) := \alpha \circ F(R^{op})\ .$$

Recall that $\mathcal{L}$ denotes the category of finite lattices and $k\mathcal{L}$ its k-linearization (Definition 4.6). For any finite lattice $T = (T,\vee,\wedge)$, denote by $T^{op} = (T,\wedge,\vee)$ the *opposite* lattice, i.e. the set T ordered with the opposite partial order. For simplicity throughout this section, we write $\leq$ for $\leq_T$ and $\leq^{op}$ for $\leq_{T^{op}}$.

8.1. Lemma. *The assignment $T \mapsto T^{op}$ extends to an isomorphism $\mathcal{L} \to \mathcal{L}^{op}$, and to a k-linear isomorphism $k\mathcal{L} \to k\mathcal{L}^{op}$.*

Proof : Let $f : T_1 \to T_2$ be a morphism in the category $\mathcal{L}$. For any $t \in T_2$, let $f^{op}(t)$ denote the join in T_1 of all the elements x such that $f(x) \leq t$, i.e.

$$f^{op}(t) = \bigvee_{f(x)\leq t} x\ . \tag{8.2}$$

Then $f\big(f^{op}(t)\big) = \bigvee\limits_{f(x)\leq t} f(x) \leq t$, so $f^{op}(t)$ is actually the greatest element of $f^{-1}\big([\hat{0},t]_{T_2}\big)$, i.e. $f^{-1}\big([\hat{0},t]_{T_2}\big) = [\hat{0}, f^{op}(t)]_{T_1}$. In other words,

$$\forall t_1 \in T_1,\ \forall t_2 \in T_2,\ \ f(t_1) \leq t_2 \iff t_1 \leq f^{op}(t_2)\ , \tag{8.3}$$

that is, the pair (f, f^{op}) is an adjoint pair of functors between the posets T_1 and T_2, viewed as categories. In those terms, saying that f is a morphism in $\mathcal{L}$ is equivalent to saying that f commutes with colimits in T_1 and T_2. Hence f^{op} commutes with limits, that is, f^{op} commutes with the meet operation, i.e. it is a morphism of lattices $T_2^{op} \to T_1^{op}$.

In more elementary terms, for any subset $A \subseteq T_2$,

$$\begin{aligned}
[\hat{0}, f^{op}(\bigwedge_{t\in A} t)]_{T_1} &= f^{-1}\big([\hat{0}, \bigwedge_{t\in A} t]_{T_2}\big)\\
&= f^{-1}\big(\bigcap_{t\in A}[\hat{0}, t]_{T_2}\big)\\
&= \bigcap_{t\in A} f^{-1}\big([\hat{0}, t]_{T_2}\big)\\
&= \bigcap_{t\in A}[\hat{0}, f^{op}(t)]_{T_1}\\
&= [\hat{0}, \bigwedge_{t\in A} f^{op}(t)]_{T_1} \ .
\end{aligned}$$

It follows that $f^{op}\big(\bigwedge\limits_{t\in A} t\big) = \bigwedge\limits_{t\in A} f^{op}(t)$, i.e. f^{op} is a morphism $T_2^{op} \to T_1^{op}$ in $\mathcal{L}$.

Now denoting by $\leq^{op}$ the opposite order relations on both T_1 and T_2, Equation 8.3 reads

$$\forall t_2 \in T_2,\ \forall t_1 \in T_1,\ \ f^{op}(t_2) \leq^{op} t_1 \iff t_2 \leq^{op} f(t_1)\ ,$$

which shows that the same construction applied to the morphism $f^{op} : T_2^{op} \to T_1^{op}$ yields $(f^{op})^{op} = f$. This proves that the map $f \mapsto f^{op}$ is a bijection from $\mathrm{Hom}_{\mathcal{L}}(T_1, T_2)$ to $\mathrm{Hom}_{\mathcal{L}}(T_2^{op}, T_1^{op})$.

Now if $f : T_1 \to T_2$ and $g : T_2 \to T_3$ are morphisms in $\mathcal{L}$, the adjunction (8.3) easily implies that $(gf)^{op} = f^{op}g^{op}$. It is clear moreover that $(\mathrm{id}_T)^{op} = \mathrm{id}_{T^{op}}$ for any finite lattice T. Hence the assignment $T \mapsto T^{op}$ and $f \mapsto f^{op}$ is an isomorphism $\mathcal{L} \to \mathcal{L}^{op}$, which extends linearly to an isomorphism $k\mathcal{L} \to k\mathcal{L}^{op}$. ∎

8.4. Definition. *Let T be a finite lattice and let X be a finite set. For two functions $\varphi : X \to T$ and $\psi : X \to T^{op}$, set*

$$(\varphi, \psi)_X := \begin{cases} 1 & \text{if } \varphi \leq \psi, \ \text{ i.e. if } \varphi(x) \leq_T \psi(x),\ \forall x \in X,\\ 0 & \text{otherwise.}\end{cases}$$

This definition extends uniquely to a k-bilinear form

$$(-,-)_X : F_T(X) \times F_{T^{op}}(X) \longrightarrow k\ .$$

This bilinear form induces a k-linear map $\Psi_{T,X} : F_{T^{op}}(X) \to (F_T)^\natural(X)$ defined by $\Psi_{T,X}(\psi)(\varphi) = (\varphi, \psi)_X$.

We need some notation.

8.5. Notation. *Let T be a finite lattice, X and Y finite sets, $Q \subseteq Y \times X$ a correspondence, and $\psi : X \to T^{op}$ a map. We denote by $Q \star \psi$ the action of the correspondence Q on ψ. In other words, $Q \star \psi$ is the map $F_{T^{op}}(Q)(\psi) : Y \to T^{op}$. Recall that it is defined by*

$$\forall y \in Y,\ \ (Q \star \psi)(y) = \bigwedge_{(y,x)\in Q} \psi(x)\ ,$$

because the join in T^{op} is the meet in T.

8.6. Lemma.

(a) With the notation 8.5, the family of bilinear forms in Definition 8.4 satisfy

$$(\varphi, Q\star\psi)_Y = (Q^{op}\varphi, \psi)_X \ .$$

(b) The family of maps $\Psi_{T,X} : F_{T^{op}}(X) \to (F_T)^\natural(X)$ form a morphism of correspondence functors $\Psi_T : F_{T^{op}} \to (F_T)^\natural$.

Proof : (a) We have

$$\begin{aligned}
\varphi \leq Q\star\psi &\iff \forall y\in Y,\ \varphi(y) \leq_T Q\star\psi(y)\\
&\iff \forall y\in Y,\ \varphi(y) \leq_T \bigwedge_{(y,x)\in Q}\psi(x)\\
&\iff \forall (y,x)\in Q,\ \varphi(y)\leq_T \psi(x)\\
&\iff \forall x\in X,\ \bigvee_{(x,y)\in Q^{op}}\varphi(y)\leq_T\psi(x)\\
&\iff Q^{op}\varphi\leq\psi \ .
\end{aligned}$$

(b) The equation in part (a) also reads

$$\Psi_{T,X}(\psi)(Q^{op}\varphi) = \Psi_{T,Y}(Q\star\psi)(\varphi) \ ,$$

that is $Q\Psi_{T,X}(\psi) = \Psi_{T,Y}(Q\star\psi)$. □

8.7. Remark. Let $T = I_\downarrow(E, R^{op})$ be the distributive lattice corresponding to a poset (E, R^{op}). Then $T^{op} = I_\downarrow(E, R^{op})^{op}$ is isomorphic, via complementation, to the lattice $I_\downarrow(E, R)$. Using the isomorphisms of Proposition 4.5

$$F_T = F_{I_\downarrow(E,R^{op})} \cong k\mathcal{C}(-,E)R \ , \qquad F_{T^{op}} = F_{I_\downarrow(E,R)} \cong k\mathcal{C}(-,E)R^{op} \ ,$$

we can transport the bilinear forms $(-,-)_X$ defined in (8.4) and obtain a pairing

$$k\mathcal{C}(-,E)R \times k\mathcal{C}(-,E)R^{op} \longrightarrow k \ .$$

It is easy to check, using complementation, that this pairing coincides with the one obtained in Remark 10.5 of [BT2].

8.8. Notation. *Let T be a finite lattice, X a finite set, and $\varphi : X\to T$ a map. We denote by $\varphi^\star$ the element of $F_{T^{op}}(X)$ defined by*

$$\varphi^\star := \sum_{\substack{\rho:X\to T\\ \rho\leq\varphi}} \mu(\rho,\varphi)\rho^\circ \ ,$$

where ρ° is the function ρ, viewed as a map $X\to T^{op}$, and where $\mu(\rho,\varphi)$ is the Möbius function of the poset of maps from X to T, for which $\rho\leq\varphi$ if and only if $\rho(x)\leq\varphi(x)$ in T for any $x\in X$. Recall that $\mu(\rho,\varphi)$ can be computed as follows :

$$\mu(\rho,\varphi) = \prod_{x\in X}\mu_T\big(\rho(x),\varphi(x)\big) \ ,$$

where μ_T is the Möbius function of the poset T.

Now we can prove that we have a perfect duality.

8.9. Theorem. *Let T be a finite lattice.*

(a) Let X be a finite set. The bilinear form (8.4) is nondegenerate, in the strong sense, namely it induces an isomorphism

$$\Psi_{T,X} : F_{T^{op}}(X) \longrightarrow (F_T)^\natural(X) \; .$$

More precisely, $\{\varphi^\star \mid \varphi : X \to T\}$ is the dual basis, in $F_{T^{op}}(X)$, of the k-basis of functions $X \to T$, in $F_T(X)$.

(b) $\Psi_T : F_{T^{op}} \to (F_T)^\natural$ is an isomorphism of correspondence functors.

(c) The functor $T \mapsto F_{T^{op}}$ and the functor $T \mapsto (F_T)^\natural$ are naturally isomorphic functors from $k\mathcal{L}$ to $\mathcal{F}_k^{op}$. More precisely, the family of isomorphisms Ψ_T, for finite lattices T, form a natural transformation Ψ between the functor $T \mapsto F_{T^{op}}$ and the functor $T \mapsto (F_T)^\natural$.

Proof : (a) The set $\{\rho^\circ \mid \rho^\circ : X \to T^{op}\}$ is a k-basis of the free k-module $F_{T^{op}}(X)$. It follows that $\{\varphi^\star \mid \varphi : X \to T\}$ is also a k-basis of $F_{T^{op}}(X)$, because the integral matrix of Möbius coefficients $\mu(\rho,\phi)$ is unitriangular, hence invertible over $\mathbb{Z}$. Actually its inverse is the adjacency matrix of the order relation $\rho \leq \varphi$ on the set of maps $X \to T$.

Now, for any two functions $\varphi, \lambda : X \to T$,

$$\textbf{(8.10)} \qquad (\lambda, \varphi^\star)_X = \sum_{\substack{\rho : X \to T \\ \rho \leq \varphi}} \mu(\rho,\varphi)(\lambda,\rho^\circ)_X = \sum_{\substack{\rho : X \to T \\ \lambda \leq \rho \leq \varphi}} \mu(\rho,\varphi) = \delta_{\lambda,\varphi} \; ,$$

where $\delta_{\lambda,\varphi}$ is the Kronecker symbol (the last equality coming from the definition of the Möbius function). This shows that $\{\varphi^\star \mid \varphi : X \to T\}$ is the dual basis, in $F_{T^{op}}(X)$, of the k-basis of functions $X \to T$, in $F_T(X)$.

(b) This follows immediately from (a). Another way of seeing this is to build an explicit inverse Φ_T of Ψ_T. For each finite set X, we define a linear map $\Phi_{T,X} : (F_T)^\natural(X) \to F_{T^{op}}(X)$ by setting

$$\forall \alpha \in (F_T)^\natural(X), \;\; \Phi_{T,X}(\alpha) = \sum_{\varphi : X \to T} \alpha(\varphi)\varphi^\star \; .$$

Then, for any function $\lambda : X \to T$,

$$\big(\Psi_{T,X}\Phi_{T,X}(\alpha)\big)(\lambda) = \big(\lambda, \Phi_{T,X}(\alpha)\big)_X = \sum_{\varphi : X \to T} \alpha(\varphi)(\lambda,\varphi^\star)_X = \alpha(\lambda) \; ,$$

so $\Psi_{T,X}\Phi_{T,X}$ is the identity map of $(F_T)^\natural(X)$. In particular, $\Psi_{T,X}$ is surjective.

On the other hand, $\Psi_{T,X}$ is injective, because if $\Psi_{T,X}(\beta) = 0$, then we write $\beta = \sum\limits_{\varphi : X \to T} a_\varphi \varphi^\star$, where $a_\varphi \in k$, and then for all $\lambda : X \to T$, we get

$$0 = \Psi_{T,X}\big(\sum_{\varphi : X \to T} a_\varphi \varphi^\star\big)(\lambda) = (\lambda, \sum_{\varphi : X \to T} a_\varphi \varphi^\star)_X = \sum_{\varphi : X \to T} a_\varphi (\lambda,\varphi^\star)_X = a_\lambda \; ,$$

so that $\beta = 0$. Therefore $\Psi_{T,X}$ is an isomorphism and $\Phi_{T,X}$ is its inverse.

(c) Let T' be another finite lattice, and let $\Psi_{T'} : F_{T'^{op}} \to (F_{T'})^\natural$ be the corresponding morphism. Let moreover $f : T \to T'$ be a morphism in $\mathcal{L}$. We claim that for any finite set X, the square

$$\begin{tikzcd}
F_{T^{op}}(X) \arrow[r, "\Psi_{T,X}"] & (F_T)^\natural(X) \\
F_{T'^{op}}(X) \arrow[u, "F_{f^{op}}"] \arrow[r, "\Psi_{T',X}"'] & (F_{T'})^\natural(X) \arrow[u, "(F_f)^\natural"']
\end{tikzcd}$$

is commutative: indeed, for any functions $\psi : X \to T'^{op}$ and $\varphi : X \to T$,

$$\big((F_f)^\natural \Psi_{T',X}(\psi)\big)(\varphi) = \Psi_{T',X}(\psi)(f \circ \varphi) = (f \circ \varphi, \psi)_X \ ,$$

whereas

$$\big(\Psi_{T,X} F_{f^{op}}(\psi)\big)(\varphi) = \big(\varphi, F_{f^{op}}(\psi)\big)_X = (\varphi, f^{op} \circ \psi)_X \ .$$

Now by 8.3, we have that

$$\begin{aligned} f \circ \varphi \leq \psi \quad &\iff \quad \forall x \in X, \;\; f\big(\varphi(x)\big) \leq \psi(x) \\ &\iff \quad \forall x \in X, \;\; \varphi(x) \leq f^{op}\big(\psi(x)\big) \\ &\iff \quad \varphi \leq f^{op} \circ \psi \ , \end{aligned}$$

which proves our claim. This shows that the isomorphisms Ψ_T, for finite lattices T, form a natural transformation Ψ of the functor $T \mapsto F_{T^{op}}$ to the functor $T \mapsto (F_T)^\natural$ from $k\mathcal{L}$ to $\mathcal{F}_k^{op}$. This completes the proof of Theorem 8.9. □

8.11. Corollary. *Let k be a self-injective ring. Then for any distributive lattice T, the functor F_T is projective and injective in $\mathcal{F}_k$.*

Proof : Since T is distributive, the functor F_T is projective by Theorem 4.12, without further assumption on k.

If k is self-injective, the functor sending a k-module A to its k-dual $\mathrm{Hom}_k(A, k)$ is exact. It follows that the functor $M \mapsto M^\natural$ is an exact contravariant endofunctor of the category $\mathcal{F}_k$, where $M^\natural$ denotes the dual correspondence functor.

Let $\alpha : M \to N$ be an injective morphism in $\mathcal{F}_k$, and let $\lambda : M \to F_T$ be any morphism. Then $\alpha^\natural : N^\natural \to M^\natural$ is surjective, and we have the following diagram with exact row in $\mathcal{F}_k$

$$\begin{tikzcd} & (F_T)^\natural \arrow[d, "\lambda^\natural"] & \\ N^\natural \arrow[r, "\alpha^\natural"] & M^\natural \arrow[r] & 0 \end{tikzcd}$$

Now $(F_T)^\natural \cong F_{T^{op}}$ by Theorem 8.9, and T^{op} is distributive. Hence $F_{T^{op}}$ is projective in $\mathcal{F}_k$, and there exists a morphism $\beta : (F_T)^\natural \to N^\natural$ such that $\alpha^\natural \circ \beta = \lambda^\natural$. Dualizing once again the previous diagram yields the commutative diagram

$$\begin{tikzcd} & (F_T)^{\natural\natural} & F_T \arrow[l, "\eta_{F_T}"'] \\ N^{\natural\natural} \arrow[ur, "\beta^\natural"] & M^{\natural\natural} \arrow[l, "\alpha^{\natural\natural}"] \arrow[u, "\lambda^{\natural\natural}"'] & \\ N \arrow[u, "\eta_N"] & M \arrow[l, "\alpha"] \arrow[u, "\eta_M"'] \arrow[uur, bend right, "\lambda"'] & \end{tikzcd}$$

where for any functor M, we denote by η_M the canonical morphism from M to $M^{\natural\natural}$. Now η_{F_T} is an isomorphism, because for any finite set X, the module $F_T(X)$ is a finitely generated free k-module. Let $\varepsilon : N \to F_T$ be defined by $\varepsilon = \eta_{F_T}^{-1} \circ \beta^\natural \circ \eta_N$. Then

$$\varepsilon \circ \alpha = \eta_{F_T}^{-1} \circ \beta^\natural \circ \eta_N \circ \alpha = \eta_{F_T}^{-1} \circ \lambda^{\natural\natural} \circ \eta_M = \eta_{F_T}^{-1} \circ \eta_{F_T} \circ \lambda = \lambda \ .$$

Thus for any injective morphism $\alpha : M \to N$ and any morphism $\lambda : M \to F_T$, there exists a morphism $\varepsilon : N \to F_T$ such that $\varepsilon \circ \alpha = \lambda$. Hence F_T is injective in $\mathcal{F}_k$. □

9. Duality for fundamental functors and simple functors

By Theorem 6.5, any fundamental functor $\mathbb{S}_{E,R}$ is isomorphic to a quotient of some functor associated to a lattice. One of the main purposes of this section is to use duality to realize $\mathbb{S}_{E,R}$ as a subfunctor of some functor associated to another lattice. We also determine what is the dual of a simple functor.

We will study the subfunctor generated by a specific element of $F_{T^{op}}(E)$ which will be defined below. We need some more notation.

9.1. Notation. *Let T be a finite lattice. If $t \in T$, let $r(t)$ denote the join of all the elements of T strictly smaller than t, i.e.*

$$r(t) := \bigvee_{s<t} s$$

Thus $r(t) = t$ if and only if t is not irreducible. If t is irreducible, then $r(t)$ is the unique maximal element of $[\hat{0}, t[$.

9.2. Notation. *Let T be a finite lattice and let (E, R) be the full subposet of its irreducible elements. If $A \subseteq E$, let $\eta_A : E \to T$ be the map defined by*

$$\forall e \in E, \;\; \eta_A(e) := \begin{cases} r(e) & \text{if } e \in A \\ e & \text{if } e \notin A \end{cases} .$$

Moreover, let γ_T denote the element of $F_{T^{op}}(E)$ defined by

$$\gamma_T := \sum_{A \subseteq E} (-1)^{|A|} \eta_A^\circ \,,$$

where η_A° denotes the function η_A, viewed as a map $E \to T^{op}$.

We now show that this element γ_T has another characterization. Recall that we use a star $\star$, as in Notation 8.5, for the action of a correspondence on evaluations of $F_{T^{op}}$.

9.3. Lemma. *Let T be a finite lattice, let (E, R) be the full subposet of its irreducible elements, and let $\iota : E \to T$ be the inclusion map.*

(a) The element γ_T is equal to ι^* (using Notation 8.8).

(b) $R \star \gamma_T = \gamma_T$.

Proof : (a) By definition, $\iota^\star = \sum\limits_{\rho \leq \iota} \mu(\rho, \iota)\rho^\circ$, where ρ° denotes the function ρ, viewed as a map $E \to T^{op}$, and where μ is the Möbius function of the poset of functions from E to T (see Notation 8.8). Furthermore

$$\mu(\rho, \iota) = \prod_{e \in E} \mu_T\big(\rho(e), \iota(e)\big) \,,$$

where μ_T is the Möbius function of the poset T. Now $\mu_T\big(\rho(e), \iota(e)\big) = \mu_T\big(\rho(e), e\big)$ is equal to 0 if $\rho(e) < r(e)$, because in that case the interval $]\rho(e), e[_T$ has a greatest element $r(e)$. Moreover $\mu_T\big(\rho(e), e\big)$ is equal to -1 if $\rho(e) = r(e)$, and to +1 if $\rho(e) = e$. It follows that the only maps ρ appearing in the sum above are of the form $\rho = \eta_A$ for some subset $A \subseteq E$ and $\mu(\eta_A, \iota) = (-1)^{|A|}$. Therefore

$$\iota^\star = \sum_{A \subseteq E} (-1)^{|A|} \eta_A^\circ = \gamma_T \,.$$

(b) For any $A \subseteq E$ and any $e \in E$,

$$(R * \eta_A)(e) = \bigwedge_{(e,e') \in R} \eta_A(e') = \bigwedge_{e \leq e'} \eta_A(e') = \eta_A(e) \,,$$

since $e < e'$ implies $\eta_A(e) \leq e \leq r(e') \leq \eta_A(e')$. Therefore $R \star \gamma_T = \gamma_T$. $\square$

Our aim is to show that the subfunctor $\langle \gamma_T \rangle$ of $F_{T^{op}}$ generated by γ_T is isomorphic to the fundamental correspondence functor $\mathbb{S}_{E,R}$. We first show that $\langle \gamma_T \rangle$ is independent of the choice of T.

9.4. Lemma. *Let $f : T \to T'$ be a morphism in $\mathcal{L}$ and let (E,R), respectively (E',R'), be the full subposet of irreducible elements in T, respectively T'. Suppose that the restriction of f to E is an isomorphism of posets $f_| : (E,R) \xrightarrow{\cong} (E',R')$.*

(a) The map f is surjective and $fr(e) = rf(e)$ for any $e \in E$.
(b) The map $f^{op} : T'^{op} \to T^{op}$ restricts to a bijection $f^{op}_| : E' \xrightarrow{\cong} E$, which is inverse to $f_|$. Moreover $f^{op}r(e') = rf^{op}(e')$ for any $e' \in E'$.
(c) $f^{op} : T'^{op} \to T^{op}$ induces an injective morphism $F_{f^{op}} : F_{T'^{op}} \to F_{T^{op}}$ and an isomorphism $\langle \gamma_{T'} \rangle \cong \langle \gamma_T \rangle$.

Proof : (a) Since any element of T' is a join of irreducible elements, which are in the image of f, and since f commutes with joins, the map f is surjective.

Let $e \in E$. By assumption $f(e) \in E'$. The condition $r(e) < e$ implies $f\big(r(e)\big) \leq f(e)$. Moreover $r(e) = \bigvee\limits_{\substack{e_1 \in E \\ e_1 < e}} e_1$, hence $fr(e) = \bigvee\limits_{\substack{e_1 \in E \\ e_1 < e}} f(e_1)$. Thus if $fr(e) = f(e)$, then there exists $e_1 < e$ such that $f(e_1) = f(e)$, contradicting the assumption on f. It follows that $fr(e) \leq rf(e)$.

Now $rf(e) = \bigvee\limits_{\substack{e' \in E' \\ e' < f(e)}} e'$, and each $e' \in E'$ with $e' < f(e)$ can be written $e' = f(e_1)$, for $e_1 \in E$ with $e_1 < e$. It follows that $rf(e) \leq \bigvee\limits_{\substack{e_1 \in E \\ e_1 < e}} f(e_1) = fr(e)$. Thus $rf(e) = fr(e)$, as was to be shown.

(b) Recall from Equation 8.2 that f^{op} is defined by $f^{op}(t') = \bigvee\limits_{f(t) \leq t'} t$. Let $e' \in E'$. Then there exists $e \in E$ such that $f(e) = e'$. Let $t \in T$ be such that $f(t) \leq e'$ and write $t = \bigvee\limits_{\substack{e_1 \in E \\ e_1 \leq t}} e_1$. For each $e_1 \in E$ with $e_1 \leq t$, we have $f(e_1) \leq f(t) \leq e' = f(e)$, hence $e_1 \leq e$, and $t \leq e$. It follows that $f^{op}(e') = \bigvee\limits_{f(t) \leq e'} t = e$, so $f^{op}_|$ is a bijection $E' \to E$, inverse to $f_|$. This proves the first statement in (b).

Now let $e \in E$, and set $e' = f(e) \in E'$. First we have $r(e') \leq e'$, thus $f^{op}r(e') \leq f^{op}(e') = e$. If $f^{op}r(e') = e$, then $\bigvee\limits_{f(t) \leq r(e')} t = e$, hence

$$f(e) \leq r(e') < e' = f(e) \; ,$$

a contradiction. Thus $f^{op}r(e') \leq r(e) = rf^{op}(e')$. But we also have

$$rf^{op}(e') = r(e) = \bigvee_{\substack{e_1 \in E \\ e_1 < e}} e_1 = \bigvee_{\substack{e_1 \in E \\ f(e_1) < f(e)}} e_1 \leq \bigvee_{\substack{t \in T \\ f(t) \leq rf(e)}} t = f^{op}rf(e) = f^{op}r(e') \; ,$$

so $f^{op}r(e') = rf^{op}(e')$, which proves the second statement in (b).

(c) Since f is surjective by (a), so is the morphism $F_f : F_T \to F_{T'}$. By duality and Theorem 8.9, the morphism $F_{f^{op}} : F_{T'^{op}} \to F_{T^{op}}$ can be identified with the dual of F_f and is therefore injective. This proves the first statement in (c).

Now for any $B \subseteq E'$, consider the map $\eta_B^\circ : E' \to T'^{op}$. Then for any $e' \in E'$

$$f^{op}\eta_B^\circ(e') = \begin{cases} f^{op}(e') & \text{if } e' \notin B, \\ f^{op}r(e') = rf^{op}(e') & \text{if } e' \in B. \end{cases}$$

Hence $f^{op} \circ \eta_B^\circ = \eta_{f_|^{op}(B)}^\circ \circ f_|^{op}$, and therefore $f^{op} \circ \gamma_{T'} = \gamma_T \circ f_|^{op}$. It follows that

$$F_{f^{op}}(\gamma_{T'} \circ f_|) = f^{op} \circ \gamma_{T'} \circ f_| = \gamma_T \circ f_|^{op} \circ f_| = \gamma_T \ .$$

Therefore the injective morphism $F_{f^{op}}$ maps the subfunctor $\langle\, \gamma_{T'} \circ f_|\, \rangle$ isomorphically to the subfunctor $\langle \gamma_T \rangle$. But since $f_| : E \to E'$ is a bijection, the subfunctor $\langle\, \gamma_{T'} \circ f_|\, \rangle$ of $F_{T'^{op}}$ is equal to the subfunctor $\langle \gamma_{T'} \rangle$. This proves the second statement in (c). ∎

Recall that we use a star $\star$, as in Notation 8.5, for the action of a correspondence on evaluations of $F_{T^{op}}$. We now come to our main result.

9.5. Theorem. *Let T be a finite lattice and let (E,R) be the full subposet of its irreducible elements.*

(a) The subfunctor $\langle \gamma_T \rangle$ of $F_{T^{op}}$ generated by γ_T is isomorphic to $\mathbb{S}_{E,R}$.
(b) In other words, for any finite set X, the module $\mathbb{S}_{E,R}(X)$ is isomorphic to the k-submodule of $F_{T^{op}}(X)$ generated by the elements $S\star\gamma_T$, for $S \subseteq X \times E$.

Proof : We first show that it suffices to prove the result in the case when T is the lattice $I_\downarrow(E,R)$. For any other lattice T' with the same poset (E,R) of irreducible elements, the inclusion $E \subseteq T'$ extends to a unique surjective map

$$f : T = I_\downarrow(E,R) \longrightarrow T'$$

in the category $\mathcal{L}$ which induces the identity on E (see Lemma 3.5). Then $\langle \gamma_{T'} \rangle$ is isomorphic to $\langle \gamma_T \rangle$ by Lemma 9.4, so we now assume that $T = I_\downarrow(E,R)$, which is a distributive lattice.

We now apply Proposition 5.7 to the element $\gamma_T \in F_{T^{op}}(E)$, using the fact that $\gamma_T \in R \star F_{T^{op}}(E)$, because $R \star \gamma_T = \gamma_T$ by Lemma 9.3. We deduce that $\gamma_T \in F_{T^{op}}(E)$ determines a unique morphism

$$\xi : F_{I_\downarrow(E,R^{op})} \to F_{T^{op}}$$

such that $\xi_E(j) = \gamma_T$, where $j : E \to I_\downarrow(E,R^{op})$ is the inclusion map.

Then for any finite set X and any function $\psi : X \to I_\downarrow(E,R^{op})$, we can compute the map $\xi_X : F_{I_\downarrow(E,R^{op})}(X) \to F_{T^{op}}(X)$ as follows :

$$\xi_X(\psi) = \xi_X(\Gamma_\psi j) = \Gamma_\psi \star \xi_E(j) = \Gamma_\psi \star \gamma_T \ ,$$

using the equality $\psi = \Gamma_\psi j$ of Lemma 5.6. In particular, the image of ξ is the subfunctor $\langle \gamma_T \rangle$ generated by γ_T.

By Theorem 6.5 and the fact that the lattice $I_\downarrow(E,R^{op})$ has (E,R^{op}) as its full subset of irreducible elements, there is a surjective morphism

$$\Theta_{I_\downarrow(E,R^{op})} : F_{I_\downarrow(E,R^{op})} \longrightarrow \mathbb{S}_{E,R}$$

and its kernel is described in Theorem 7.1. We want to prove that, for any finite set X, the kernel of the surjection

$$\xi_X : F_{I_\downarrow(E,R^{op})}(X) \longrightarrow \langle \gamma_T \rangle(X)$$

is equal to the kernel of the surjection

$$\Theta_{I_\downarrow(E,R^{op}),X} : F_{I_\downarrow(E,R^{op})}(X) \longrightarrow \mathbb{S}_{E,R}(X) \ ,$$

from which the isomorphism $\langle \gamma_T \rangle \cong \mathbb{S}_{E,R}$ will follow.

The kernel of the surjection ξ_X is the set of all linear combinations

$$u=\sum_{\psi:X\to I_\downarrow(E,R^{op})}\lambda_\psi\psi\ ,$$

where $\lambda_\psi\in k$, such that

$$\sum_{\psi:X\to I_\downarrow(E,R^{op})}\lambda_\psi\Gamma_\psi\star\gamma_T=0\ .$$

Equivalently, using the nondegeneracy of the bilinear form $(-,-)_X$ of (8.4), proved in Theorem 8.9,

$$\forall\,\varphi:X\to T,\ \ (\varphi,\sum_\psi\lambda_\psi\Gamma_\psi\star\gamma_T)_X=0\ .$$

By Equation 8.6, this is in turn equivalent to

$$\forall\,\varphi:X\to T,\ \sum_\psi\lambda_\psi(\Gamma_\psi^{op}\varphi,\gamma_T)_E=0\ .$$

Now $\gamma_T=\iota^\star$ by Lemma 9.3 and we use $\iota^\star$ instead. By Equation 8.10, we have

$$(\Gamma_\psi^{op}\varphi,\gamma_T)_E=(\Gamma_\psi^{op}\varphi,\iota^\star)_E=\delta_{\Gamma_\psi^{op}\varphi,\iota}$$

and therefore we obtain the condition

$$\forall\,\varphi:X\to T,\qquad\sum_{\substack{\psi:X\to I_\downarrow(E,R^{op})\\ \Gamma_\psi^{op}\varphi=\iota}}\lambda_\psi=0\ .$$

Since we have assumed that $T=I_\downarrow(E,R)$ and since $I_\downarrow(E,R^{op})=I^\uparrow(E,R)$, we have maps $\varphi:X\to I_\downarrow(E,R)$ and $\psi:X\to I^\uparrow(E,R)$ and we know from Theorem 7.3 that the property $\Gamma_\psi^{op}\varphi=\iota$ is equivalent to $\Gamma_\psi^{op}\Gamma_\varphi=R^{op}$.

It follows that $u=\sum\limits_{\psi:X\to I^\uparrow(E,R)}\lambda_\psi\psi$ is in the kernel of ξ_X if and only if

$$\forall\,\varphi:X\to I_\downarrow(E,R),\qquad\sum_{\substack{\psi:X\to I_\downarrow(E,R^{op})\\ \Gamma_\psi^{op}\Gamma_\varphi=R^{op}}}\lambda_\psi=0\ .$$

But the condition $\Gamma_\psi^{op}\Gamma_\varphi=R^{op}$ is in turn is equivalent to $\Gamma_\varphi^{op}\Gamma_\psi=R$, by passing to the opposite. Moreover $I_\downarrow(E,R)=I^\uparrow(E,R^{op})$ and $I^\uparrow(E,R)=I_\downarrow(E,R^{op})$. Therefore $u=\sum\limits_{\psi:X\to I_\downarrow(E,R^{op})}\lambda_\psi\psi$ is in the kernel of ξ_X if and only if

$$\forall\varphi:X\to I^\uparrow(E,R^{op}),\qquad\sum_{\substack{\psi:X\to I_\downarrow(E,R^{op})\\ \Gamma_\varphi^{op}\Gamma_\psi=R}}\lambda_\psi=0\ .$$

By Theorem 7.1, this is equivalent to requiring that $u\in\operatorname{Ker}\Theta_{I_\downarrow(E,R^{op}),X}$. It follows that $\operatorname{Ker}\xi_X=\operatorname{Ker}\Theta_{I_\downarrow(E,R^{op}),X}$. Consequently, the images of ξ_X and $\Theta_{I_\downarrow(E,R^{op}),X}$ are isomorphic, that is, $\langle\gamma_T\rangle\cong\mathbb{S}_{E,R}$. This completes the proof of Theorem 9.5. $\square$

Since we now know that the subfunctor $\langle\gamma_T\rangle$ of $F_{T^{op}}$ is isomorphic to $\mathbb{S}_{E,R}$, we use again duality to obtain more.

9.6. Theorem. *Let T be a finite lattice and let (E,R) be the full subposet of its irreducible elements. We consider orthogonal k-submodules with respect to the family of k-bilinear forms $(-,-)_X$ defined in (8.4).*

(a) $\langle\gamma_T\rangle^\perp = \operatorname{Ker}\Theta_T$, *where* $\Theta_T : F_T \to \mathbb{S}_{E,R^{op}}$ *is the morphism of Theorem 6.5.*
(b) $F_T/\langle\gamma_T\rangle^\perp \cong \mathbb{S}_{E,R^{op}}$.
(c) $\langle\gamma_T\rangle^{\perp\perp} \cong \mathbb{S}^\natural_{E,R^{op}}$.
(d) *There is a canonical injective morphism* $\alpha_{E,R} : \mathbb{S}_{E,R} \longrightarrow \mathbb{S}^\natural_{E,R^{op}}$.

Proof : (a) Let $\sum\limits_{\varphi:X\to T} \lambda_\varphi\varphi \in F_T(X)$, where X is a finite set. Then

$$\begin{aligned}
&\sum_\varphi \lambda_\varphi\varphi \in \langle\gamma_T\rangle(X)^\perp \iff \big(\sum_\varphi \lambda_\varphi\varphi, Q\star\gamma_T\big)_X = 0 \quad \forall Q\in\mathcal{C}(X,E)\\
&\iff \big(\sum_\varphi \lambda_\varphi\varphi, Q\star\gamma_T\big)_X = 0 \quad \forall Q\in\mathcal{C}(X,E)R \qquad (\text{because } R\star\gamma_T=\gamma_T)\\
&\iff \sum_\varphi \lambda_\varphi\big(Q^{op}\varphi,\gamma_T\big)_E = 0 \quad \forall Q\in\mathcal{C}(X,E)R \qquad (\text{by } 8.6)\\
&\iff \sum_{\substack{\varphi\\ Q^{op}\varphi=\iota}} \lambda_\varphi = 0 \quad \forall Q\in\mathcal{C}(X,E)R \qquad (\text{by 8.10 and Lemma 9.3})\\
&\iff \sum_{\substack{\varphi\\ \Gamma_\psi^{op}\varphi=\iota}} \lambda_\varphi = 0 \quad \forall \psi: X\to I^\uparrow(E,R) \qquad (\text{by Proposition 4.5})\\
&\iff \sum_{\substack{\varphi\\ \Gamma_\psi^{op}\Gamma_\varphi=R^{op}}} \lambda_\varphi = 0 \quad \forall \psi: X\to I^\uparrow(E,R) \qquad (\text{by Theorem 7.3})\\
&\iff \sum_{\varphi:X\to T} \lambda_\varphi\varphi \in \operatorname{Ker}\Theta_{T,X} \quad (\text{by Theorem 7.1})
\end{aligned}$$

Therefore $\langle\gamma_T\rangle(X)^\perp = \operatorname{Ker}\Theta_{T,X}$.

(b) This follows immediately from (a) and Theorem 6.5.

(c) This follows immediately from (b) and duality.

(d) There is an obvious inclusion $\langle\gamma_T\rangle \subseteq \langle\gamma_T\rangle^{\perp\perp}$. Now we have $\langle\gamma_T\rangle \cong \mathbb{S}_{E,R}$ by Theorem 9.5 and $\langle\gamma_T\rangle^{\perp\perp} \cong \mathbb{S}^\natural_{E,R^{op}}$ by (c). Thus we obtain a canonical injective morphism $\mathbb{S}_{E,R} \longrightarrow \mathbb{S}^\natural_{E,R^{op}}$. □

9.7. Remark. We will prove in [BT3] that $\alpha_{E,R} : \mathbb{S}_{E,R} \longrightarrow \mathbb{S}^\natural_{E,R^{op}}$ is actually an isomorphism. This is easy to prove if k is a field, because the inclusion $\langle\gamma_T\rangle \subseteq \langle\gamma_T\rangle^{\perp\perp}$ must be an equality since the pairing (8.4) is nondegenerate, by Theorem 8.9.

We end this section with a description of the dual of a simple functor. We assume that k is a field and we let $S_{E,R,V}$ be the simple correspondence functor (over k) parametrized by (E,R,V). Part (d) of Theorem 9.6 suggests that the index R must become R^{op} after applying duality. We now show that this is indeed the case.

9.8. Theorem. *Let k be a field. The dual $S^\natural_{E,R,V}$ of the simple functor $S_{E,R,V}$ is isomorphic to $S_{E,R^{op},V^\natural}$, where $V^\natural$ denotes the ordinary dual of the $k\operatorname{Aut}(E,R)$-module V.*

Proof : For simplicity, write $\mathcal{R}_E = k\mathcal{C}(E,E)$ for the algebra of all relations on E and recall from Section 2 that $\mathcal{P}_E$ is a quotient algebra of $\mathcal{R}_E$. The evaluation $S_{E,R,V}(E)$ is the $\mathcal{R}_E$-module

$$T_{R,V} = \mathcal{P}_E f_R \otimes_{k\operatorname{Aut}(E,R)} V \ .$$

(which is actually a simple $\mathcal{R}_E$-module by Proposition 2.5).

Clearly the dual $S^\natural_{E,R,V}$ is again a simple functor and its minimal set is E again. Moreover, by evaluation at E, we find that

$$S^\natural_{E,R,V}(E) \cong S_{E,R,V}(E)^\natural = T^\natural_{R,V} \ .$$

Here the action of a relation $Q \in \mathcal{R}_E$ on a $\mathcal{R}_E$-module $W^\natural$ is defined by

$$(Q\cdot\alpha)(w) = \alpha(Q^{op}\cdot w) \ , \ \forall \alpha \in W^\natural \ , \ \forall w \in W \ .$$

We are going to define a nondegenerate pairing

$$\langle -,-\rangle : T_{R,V} \times T_{R^{op},V^\natural} \longrightarrow k \ ,$$

satisfying $\langle Q\cdot x\,,\,y\rangle = \langle x\,,\,Q^{op}\cdot y\rangle$ for all $x \in T_{R,V}$, $y \in T_{R^{op},V^\natural}$, and $Q \in \mathcal{R}_E$. This will induce an isomorphism of $\mathcal{R}_E$-modules

$$T^\natural_{R,V} \cong T_{R^{op},V^\natural} \ .$$

But a simple functor with minimal set E is completely determined by its evaluation at E, because it is generated by this minimal nonzero evaluation (by simplicity). Since $S^\natural_{E,R,V}$ and $S_{E,R^{op},V^\natural}$ have both an evaluation at E isomorphic to $T_{R^{op},V^\natural}$, it follows that

$$S^\natural_{E,R,V} \cong S_{E,R^{op},V^\natural} \ ,$$

as required.

Now we construct the required pairing. By Proposition 2.4, $\mathcal{P}_E f_R$ has a k-basis $\{\Delta_\sigma f_R \mid \sigma \in \Sigma_E\}$, where Σ_E is the group of all permutations of E. Moreover, it is a free right $k\,\mathrm{Aut}(E,R)$-module and it follows that we can write

$$T_{R,V} = \mathcal{P}_E f_R \otimes_{k\,\mathrm{Aut}(E,R)} V = \bigoplus_{\sigma\in[\Sigma_E/\,\mathrm{Aut}(E,R)]} \Delta_\sigma f_R \otimes V \ ,$$

where $[\Sigma_E/\,\mathrm{Aut}(E,R)]$ denotes a set of representatives of the left cosets of $\mathrm{Aut}(E,R)$ in Σ_E. Noticing that $\mathrm{Aut}(E,R^{op}) = \mathrm{Aut}(E,R)$, we have a similar decomposition

$$T_{R^{op},V^\natural} = \mathcal{P}_E f_{R^{op}} \otimes_{k\,\mathrm{Aut}(E,R)} V^\natural = \bigoplus_{\varepsilon\in[\Sigma_E/\,\mathrm{Aut}(E,R)]} \Delta_\varepsilon f_{R^{op}} \otimes V^\natural \ .$$

We define the pairing

$$\langle -,-\rangle : T_{R,V} \times T_{R^{op},V^\natural} \longrightarrow k \ , \qquad \langle \Delta_\sigma f_R \otimes v\,,\,\Delta_\varepsilon f_{R^{op}} \otimes \alpha\rangle := \delta_{\sigma,\varepsilon}\,\alpha(v) \ ,$$

where $\sigma,\varepsilon \in [\Sigma_E/\,\mathrm{Aut}(E,R)]$, $v \in V$, $\alpha \in V^\natural$.

By choosing dual bases of V and $V^\natural$, we easily find dual bases of $T_{R,V}$ and $T_{R^{op},V^\natural}$, and it follows that this pairing is nondegenerate. We are left with the proof of the required property of this pairing, namely

$$\langle Q\cdot x\,,\,y\rangle = \langle x\,,\,Q^{op}\cdot y\rangle \qquad \text{for all } \ x \in T_{R,V},\ y \in T_{R^{op},V^\natural},\ Q \in \mathcal{C}(E,E) \ .$$

By Proposition 2.4, the action of Q on $\mathcal{P}_E f_R$ is given by:

$$Q\cdot\Delta_\sigma f_R = \begin{cases} \Delta_{\tau\sigma} f_R & \text{if } \exists \tau \in \Sigma_E \text{ such that } \Delta_E \subseteq \Delta_{\tau^{-1}} Q \subseteq {}^\sigma R, \\ 0 & \text{otherwise .} \end{cases}$$

It follows that

$$\langle Q\cdot\Delta_\sigma f_R \otimes v\,,\,\Delta_\varepsilon f_{R^{op}} \otimes \alpha\rangle = \begin{cases} \alpha(v) & \text{if } \ \varepsilon = \tau\sigma \ \text{ and } \ \Delta_E \subseteq \Delta_{\tau^{-1}} Q \subseteq {}^\sigma R \ , \\ 0 & \text{otherwise .} \end{cases}$$

On the other hand

$$\langle \Delta_\sigma f_R \otimes v\,,\,Q^{op}\cdot\Delta_\varepsilon f_{R^{op}} \otimes \alpha\rangle = \begin{cases} \alpha(v) & \text{if } \ \rho\varepsilon = \sigma \ \text{ and } \ \Delta_E \subseteq \Delta_{\rho^{-1}} Q^{op} \subseteq {}^\varepsilon R^{op} \ , \\ 0 & \text{otherwise .} \end{cases}$$

We put $\rho = \tau^{-1}$ and we assume that $\varepsilon = \tau\sigma$ (that is, $\rho\varepsilon = \sigma$). We then obtain equivalent conditions:

$$\begin{array}{rcll} \Delta_E \subseteq \Delta_\tau Q^{op} \subseteq {}^\varepsilon R^{op} & \iff & \Delta_E \subseteq Q\Delta_{\tau^{-1}} \subseteq {}^\varepsilon R & \text{(taking the opposite)} \\ & \iff & \Delta_E \subseteq Q\Delta_{\tau^{-1}} \subseteq {}^{\tau\sigma}R & \text{(because } \varepsilon = \tau\sigma) \\ & \iff & \Delta_E \subseteq \Delta_{\tau^{-1}}Q \subseteq {}^\sigma R & \text{(conjugating by } \Delta_{\tau^{-1}}) \end{array}$$

Therefore, we obtain the required equality

$$\langle Q \cdot \Delta_\sigma f_R \otimes v \, , \, \Delta_\varepsilon f_{R^{op}} \otimes \alpha \rangle = \langle \Delta_\sigma f_R \otimes v \, , \, Q^{op} \cdot \Delta_\varepsilon f_{R^{op}} \otimes \alpha \rangle \; ,$$

from which it follows that we have an isomorphism of $\mathcal{R}_E$-modules $T^\natural_{R,V} \cong T_{R^{op},V^\natural}$. This completes the proof. ∎

10. Embeddings and idempotents corresponding to total orders

In this section, we construct morphisms of lattices in the category $k\mathcal{L}$ between a finite lattice T and a totally ordered lattice. This will be used in Section 11 to obtain detailed information about correspondence functors associated to a total order.

For a (non negative) integer $n \in \mathbb{N}$, we denote by $\underline{n}$ the set $\{0, 1, \ldots, n\}$, linearly ordered by $0 < 1 < \ldots < n$. Then $\underline{n}$ is a distributive lattice, with least element 0 and greatest element n. Moreover $x \vee y = \sup(x, y)$ and $x \wedge y = \inf(x, y)$, for any $x, y \in \underline{n}$. We denote by $[n] = \{1, \ldots, n\}$ the set of irreducible elements of $\underline{n}$, viewed as a full subposet of $\underline{n}$.

Let $\pi : T \to \underline{n}$ be a surjective morphism of finite lattices. For every $h \in \underline{n}$, let

$$b_h = \sup\big(\pi^{-1}(h)\big) = \bigvee_{\pi(t)=h} t = \bigvee_{\pi(t)\leq h} t \; .$$

In other words, $b_h = \pi^{op}(h)$ with the notation of Lemma 8.1. Then we have $\hat{0} \leq b_0 < b_1 < \ldots < b_{n-1} < b_n = \hat{1}$. If $h \in \mathrm{Irr}(\underline{n}) = [n] = \{1, \ldots, n\}$, the whole interval $]b_{h-1}, b_h]_T$ is mapped to h under π, while b_{h-1} is mapped to $h-1$. The morphism π determines uniquely the totally ordered n-tuple $B = (b_0, b_1, \ldots, b_{n-1})$.

Conversely, if $B = (b_0, b_1, \ldots, b_{n-1})$ is a strictly increasing n-tuple in $T - \{\hat{1}\}$ and if $b_n = \hat{1}$ (so that $\hat{0} \leq b_0 < b_1 < \ldots < b_{n-1} < b_n = \hat{1}$), then B determines a unique surjective morphism $\pi^B : T \to \underline{n}$ by setting $\pi^B(t) = h$ if $t \leq b_h$ and $t \not\leq b_{h-1}$ (hence in particular $\pi^B(\hat{0}) = 0$).

For every $h \in [n]$, choose $a_h \in [b_{h-1}, b_h]_T$, and define the n-tuple

$$A = (a_1, a_2, \ldots, a_n) \; .$$

Set also $a_0 = \hat{0}$. Notice that A is totally ordered : $\hat{0} = a_0 \leq a_1 \leq \ldots \leq a_n \leq \hat{1}$, with possible equalities. Define

$$j_A^B : \underline{n} \longrightarrow T \, , \qquad h \mapsto a_h \; .$$

It is easy to see that j_A^B is a morphism of lattices (because if $e < f$ in $\underline{n}$, then $e \leq f - 1$, hence $a_e \leq b_e \leq b_{f-1} \leq a_f$). Note that j_A is a section of π^B if $a_h \in]b_{h-1}, b_h]_T$ for every $h \in [n]$, but not if $a_h = b_{h-1}$ for some h.

Write

$$\mu(B, A) = \prod_{h\in[n]} \mu_T(b_{h-1}, a_h) \; ,$$

where $\mu_T(-,-)$ denotes the Möbius function of the lattice T. For simplicity, we write $\mu = \mu_T$ throughout this section and the next. Allowing the n-tuple A to vary (i.e. a_h varies in $[b_{h-1}, b_h]_T$ for each h), define

$$j^B = (-1)^n \sum_A \mu(B,A) j^B_A \in k\mathcal{L}(\underline{n}, T) ,$$

or in other words

$$j^B = (-1)^n \prod_{h=1}^{n} \Big(\sum_{a_h \in [b_{h-1}, b_h]_T} \mu(b_{h-1}, a_h) \Big) j^B_{(a_1, a_2, \dots, a_n)} .$$

10.1. Proposition. *Let T be a finite lattice and let B be a strictly increasing n-tuple in $T - \{\widehat{1}\}$. Let $j^B : \underline{n} \to T$ be the corresponding morphism, constructed as above.*

(a) For any finite set X and any function $\varphi : X \to \underline{n}$ such that $[n] \not\subseteq \varphi(X)$, we have $j^B \varphi = 0$.
(b) j^B induces $F_{j^B} : F_{\underline{n}} \to F_T$ vanishing on $H_{\underline{n}}$, hence induces in turn a morphism

$$\overline{F}_{j^B} : F_{\underline{n}}/H_{\underline{n}} \longrightarrow F_T .$$

Proof : (a) We have

$$j^B\varphi = (-1)^n \sum_A \mu(B,A) j^B_A \varphi = \sum_{\psi : X \to T} (-1)^n \Big(\sum_{\substack{A \\ j^B_A \varphi = \psi}} \mu(B,A) \Big) \psi .$$

Let $g \in [n]$ be such that $g \notin \varphi(X)$. Then we can modify each n-tuple A by changing freely the image $j^B_A(g) = a_g \in [b_{g-1}, b_g]_T$ without changing the equality $j^B_A \varphi = \psi$. This is because $\varphi(x) \neq g$ for all $x \in X$. We set $A' = (a_1, \dots, \widehat{a}_g, \dots, a_n)$ and $B' = (b_0, \dots, \widehat{b}_{g-1}, \dots, b_{n-1})$ (where $\widehat{\ }$ denotes omission) and we let

$$j^{B'}_{A'} : \underline{n} - \{g\} \longrightarrow T , \qquad h \mapsto a_h .$$

We obtain

$$\sum_{\substack{A \\ j^B_A \varphi = \psi}} \mu(B,A) = \sum_{\substack{A' \\ j^{B'}_{A'} \varphi = \psi}} \mu(B', A') \sum_{a_g \in [b_{g-1}, b_g]_T} \mu(b_{g-1}, a_g) .$$

But the inner sum on the right is zero by definition of the Möbius function (and because $b_{g-1} < b_g$). Therefore the coefficient of every ψ is zero, hence $j^B \varphi = 0$.

(b) This follows from (a). □

For any subset Y of $[n]$, define

$$\rho_Y : \underline{n} \longrightarrow \underline{n} , \qquad \rho_Y(h) = \begin{cases} 0 & \text{if } h = 0 , \\ h & \text{if } h \in Y , \\ h-1 & \text{if } h \notin Y . \end{cases}$$

10.2. Proposition. *Let T be a finite lattice and let B be a strictly increasing n-tuple in $T-\{\hat{1}\}$. Let $\pi^B : T \to \underline{n}$ be the corresponding surjective morphism and $j^B : \underline{n} \to T$, constructed as above. Let $p : F_{\underline{n}} \to F_{\underline{n}}/H_{\underline{n}}$ be the canonical map.*

(a) The composite of $\overline{F}_{j^B} : F_{\underline{n}}/H_{\underline{n}} \to F_T$ and $p\,F_{\pi^B} : F_T \to F_{\underline{n}}/H_{\underline{n}}$ is the identity morphism of $F_{\underline{n}}/H_{\underline{n}}$.

(b) $\overline{F}_{j^B} : F_{\underline{n}}/H_{\underline{n}} \to F_T$ is injective and embeds $F_{\underline{n}}/H_{\underline{n}}$ as a direct summand of F_T.

(c) $F_{j^B}F_{\pi^B}$ is an idempotent endomorphism of F_T whose image is isomorphic to $F_{\underline{n}}/H_{\underline{n}}$.

(d) $j^B\pi^B$ is an idempotent endomorphism of T.

(e) $\pi^B j^B = (-1)^n \sum\limits_{\emptyset\subseteq Y\subseteq[n]} (-1)^{|Y|}\rho_Y$.

Proof : (a) It is clear that

$$\pi^B j^B_A(h) = \begin{cases} 0 & \text{if } h=0\ , \\ h & \text{if } j^B_A(h) \in\,]b_{h-1}, b_h]_T\ , \\ h-1 & \text{if } j^B_A(h) = b_{h-1}\ . \end{cases}$$

Therefore

$$\pi^B j^B = (-1)^n \sum_{\emptyset\subseteq Y\subseteq[n]} \Big(\sum_{\substack{A \\ \pi^B j^B_A=\rho_Y}} \mu(B,A)\Big)\rho_Y\ .$$

If $h\in Y$, then $j^B_A(h)$ runs freely over $]b_{h-1}, b_h]_T$. If $h\notin Y$, then $j^B_A(h)=b_{h-1}$ is fixed. It follows that

$$\sum_{\substack{A \\ \pi^B j^B_A=\rho_Y}} \mu(B,A) = \prod_{h\in Y}\Big(\sum_{a_h\in\,]b_{h-1},b_h]_T} \mu(b_{h-1},a_h)\Big) = \prod_{h\in Y}(-1) = (-1)^{|Y|}\ ,$$

using the fact that

$$0 = \sum_{a_h\in\,[b_{h-1},b_h]_T} \mu(b_{h-1},a_h) = 1 + \sum_{a_h\in\,]b_{h-1},b_h]_T} \mu(b_{h-1},a_h)\ .$$

This shows that

$$\pi^B j^B = (-1)^n \sum_{\emptyset\subseteq Y\subseteq[n]} (-1)^{|Y|}\rho_Y\ .$$

Suppose now that Y is a proper subset of $[n]$ and let $g\in[n]$ be maximal such that $g\notin Y$. If $h>g$, then $\rho_Y(h)=h>g$, while if $h\leq g$, then $\rho_Y(h)\leq g-1$. Therefore $g\notin\rho_Y(\underline{n})$. This shows that $[n]\not\subseteq\rho_Y(\underline{n})$, for any proper subset Y of $[n]$.

Now for any function $\varphi : X\to\underline{n}$,

$$F_{\pi^B}F_{j^B}(\varphi) = \pi^B j^B\varphi = (-1)^n \sum_{\emptyset\subseteq Y\subseteq[n]} (-1)^{|Y|}\rho_Y\varphi = \varphi + \sum_{Y\neq[n]} (-1)^{n+|Y|}\rho_Y\varphi\ .$$

But $[n]\not\subseteq\rho_Y(\underline{n})$ if $Y\neq[n]$, hence $[n]\not\subseteq\rho_Y\varphi(X)$. In other words, $\rho_Y\varphi\in H_{\underline{n}}(X)$, so that $F_{\pi^B}F_{j^B}(\varphi)=\varphi \pmod{H_{\underline{n}}(X)}$. Composing with the canonical map $p : F_{\underline{n}}(X)\to F_{\underline{n}}(X)/H_{\underline{n}}(X)$ and writing $p(\varphi)=\overline{\varphi}$, we obtain

$$pF_{\pi^B}\overline{F}_{j^B}(\overline{\varphi}) = \overline{\varphi}\ ,$$

as was to be shown.

(b) This follows immediately from (a).

(c) This follows immediately from (a) and the obvious equality $\overline{F}_{j^B}p\,F_{\pi^B} = F_{j^B}F_{\pi^B}$.

(d) Since the functor $F_? : k\mathcal{L} \to \mathcal{F}_k$ is fully faithful by Theorem 4.8, $j^B\pi^B$ must be an idempotent because its image $F_{j^B}F_{\pi^B}$ is an idempotent. Alternatively, it is not difficult to compute directly

$$j^B\pi^B j^B = j^B(-1)^n \sum_{\emptyset\subseteq Y\subseteq[n]} (-1)^{|Y|}\rho_Y = j^B \operatorname{id}_{\underline{n}} + (-1)^n \sum_{Y\neq[n]} (-1)^{|Y|} j^B\rho_Y = j^B \ ,$$

because $[n] \not\subseteq \rho_Y(\underline{n})$ if $Y \neq [n]$, hence $j^B\rho_Y = 0$ by Proposition 10.1.

(e) The required equality has already been proved above. □

Our aim is to show that the idempotents $j^B\pi^B$ are orthogonal. In order to understand the product of two idempotents $j^C\pi^C$ and $j^B\pi^B$ we need to have more information about $\pi^C j^B$. This is the purpose of the next two propositions.

10.3. Proposition. *Let T be a finite lattice and let*

$$\begin{array}{lll} B = (b_0, b_1, \ldots, b_{n-1}) & \textit{with} & \widehat{0} \leq b_0 < b_1 < \ldots < b_{n-1} < b_n = \widehat{1} \ , \\ C = (c_0, c_1, \ldots, c_{m-1}) & \textit{with} & \widehat{0} \leq c_0 < c_1 < \ldots < c_{m-1} < c_m = \widehat{1} \ . \end{array}$$

If $\pi^C j^B \neq 0$, then the restriction of π^C to the subset $\{b_0, b_1, \ldots, b_{n-1}, b_n\}$ is injective. In particular, $n \leq m$.

Proof : We have

$$\pi^C j^B = (-1)^n \sum_A \mu(B,A)\pi^C j^B_A = (-1)^n \sum_{\psi:\underline{n}\to\underline{m}} \Big(\sum_{\substack{A \\ \pi^C j^B_A = \psi}} \mu(B,A) \Big)\psi \ .$$

Now fix some morphism $\psi : \underline{n} \to \underline{m}$ and, for every $h \in [n]$, define

$$U_h = \{a \in [b_{h-1}, b_h]_T \mid \pi^C(a) = \psi(h)\} \subseteq [b_{h-1}, b_h]_T \ .$$

Then the condition $\pi^C j^B_A = \psi$ is equivalent to $j^B_A(h) \in U_h$ for every $h \in [n]$, that is, $a_h \in U_h$ for every $h \in [n]$. In particular $U_h \neq \emptyset$ in that case. Since all elements of U_h have the same image under π^C, so has their join and therefore U_h has a supremum

$$u_h = \sup(U_h) \in U_h \ .$$

Define

$$V_h = \{a \in [b_{h-1}, b_h]_T \mid \pi^C(a) \leq \psi(h) - 1\} \ .$$

Then we clearly have $[b_{h-1}, u_h]_T = V_h \sqcup U_h$ because any $a \leq u_h$ satisfies either $\pi^C(a) \leq \psi(h) - 1$ or $\pi^C(a) = \psi(h)$. If V_h is nonempty (i.e. if $b_{h-1} \notin U_h$), then V_h has a supremum $v_h = \bigvee_{t\in V_h} t$ and $V_h = [b_{h-1}, v_h]_T$.

Now, in the expression above for $\pi^C j^B$, the coefficient of ψ is $(-1)^n z_\psi$, where

$$z_\psi = \sum_{\pi^C j^B_A = \psi} \mu(B,A) = \Big(\sum_{a_1\in U_1} \mu(b_0,a_1)\Big)\Big(\sum_{a_2\in U_2} \mu(b_1,a_2)\Big) \ldots \Big(\sum_{a_n\in U_n} \mu(b_{n-1},a_n)\Big)$$

and we examine one inner sum $\sum_{a\in U_h} \mu(b_{h-1}, a)$. We have already observed that $U_h \neq \emptyset$. Now there are two cases :

(a) If $V_h = \emptyset$, then $U_h = [b_{h-1}, u_h]_T$ and $\sum_{a\in U_h} \mu(b_{h-1}, a) = 0$, unless $b_{h-1} = u_h$.

(b) If $V_h \neq \emptyset$, then we claim that $\sum_{a\in U_h} \mu(b_{h-1}, a) = 0$, unless $b_{h-1} = v_h$.

To prove the claim, suppose that $b_{h-1} < v_h$ (hence $b_{h-1} < u_h$ because $v_h < u_h$). Then we have $[b_{h-1}, u_h]_T = [b_{h-1}, v_h]_T \sqcup U_h$ and therefore

$$\sum_{a\in U_h} \mu(b_{h-1}, a) = \sum_{a\in[b_{h-1},u_h]_T} \mu(b_{h-1}, a) - \sum_{a\in[b_{h-1},v_h]_T} \mu(b_{h-1}, a) = 0 \ ,$$

because $b_{h-1} < u_h$ and $b_{h-1} < v_h$.

It follows that the sum $\sum_{a\in U_h}\mu(b_{h-1},a)$ is nonzero in only two cases :

(a) $U_h=\{b_{h-1}\}$. Then $\sum_{a\in U_h}\mu(b_{h-1},a)=\mu(b_{h-1},b_{h-1})=1$.
(b) $V_h=\{b_{h-1}\}$. Then $U_h=]b_{h-1},u_h]_T$ and $\sum_{a\in U_h}\mu(b_{h-1},a)=-1$ because

$$\begin{aligned}0=\sum_{a\in[b_{h-1},u_h]_T}\mu(b_{h-1},a) &= \mu(b_{h-1},b_{h-1})+\sum_{a\in]b_{h-1},u_h]_T}\mu(b_{h-1},a)\\ &= 1+\sum_{a\in U_h}\mu(b_{h-1},a)\,.\end{aligned}$$

If the coefficient z_ψ is nonzero, then this sum must be nonzero for every $h\in[n]$ and we obtain $z_\psi=(-1)^p$ where p is the number of times the integer h satisfies case (b) (and then the coefficient of ψ in the expression for $\pi^C j^B$ is $(-1)^{n+p}$).

Now suppose that the restriction of π^C to $\{b_0,b_1,\ldots,b_{n-1},b_n\}$ is not injective. We want to prove that the coefficient z_ψ is zero (for our fixed morphism $\psi:\underline{n}\to\underline{m}$). Since π^C is order-preserving and not injective on the subset above, there exists $h\in[n]$ such that $\pi^C(b_{h-1})=\pi^C(b_h)$. We know that $z_\psi=0$ if U_h is empty. Thus we can assume that U_h contains an element a. Since $b_{h-1}\leq a\leq b_h$, we have $\pi^C(b_{h-1})\leq\pi^C(a)\leq\pi^C(b_h)$, hence equality $\pi^C(b_{h-1})=\pi^C(a)=\pi^C(b_h)$. This means that both b_{h-1} and b_h belong to U_h (because $\pi^C(a)=\psi(h)$ by definition of U_h). Now in both cases (a) and (b) above, the set U_h does not contain simultaneously b_{h-1} and b_h (it is either $\{b_{h-1}\}$ or $]b_{h-1},u_h]_T$). This shows that the sum $\sum_{a\in U_h}\mu(b_{h-1},a)$ cannot be nonzero. Therefore $z_\psi=0$.

This argument holds for every morphism $\psi:\underline{n}\to\underline{m}$ and so every coefficient z_ψ is zero. It follows that $\pi^C j^B=0$, as was to be proved. □

10.4. Proposition. *Let T be a finite lattice and let*

$$\begin{array}{lll}B=(b_0,b_1,\ldots,b_{n-1}) & \text{with} & \hat 0\leq b_0<b_1<\ldots<b_{n-1}<b_n=\hat 1\,,\\ C=(c_0,c_1,\ldots,c_{m-1}) & \text{with} & \hat 0\leq c_0<c_1<\ldots<c_{m-1}<c_m=\hat 1\,,\\ D=(d_0,d_1,\ldots,d_{m-1}) & \text{with} & \hat 0\leq d_0<d_1<\ldots<d_{m-1}<d_m=\hat 1\,.\end{array}$$

Then

$$j^D\pi^C j^B=\begin{cases}j^D & \text{if } C=B\,,\\ 0 & \text{if } C\neq B\,.\end{cases}$$

Proof : We assume that $j^D\pi^C j^B\neq 0$, and in particular $\pi^C j^B\neq 0$. Write first

$$\pi^C j^B=\sum_A \mu(B,A)\pi^C j^B_A$$

where $A=(a_1,a_2,\ldots,a_n)$ and $a_h\in[b_{h-1},b_h]$ for every $h\in[n]$. Let $\psi:\underline{n}\to\underline{m}$ be a map appearing with a nonzero coefficient in the expression of $\pi^C j^B\neq 0$ as a linear combination of morphisms in the category $\mathcal{L}$. Let A be such that $\pi^C j^B_A=\psi$. Since $j^D\pi^C j^B\neq 0$, we can also assume that ψ is such that $j^D\psi\neq 0$. Proposition 10.1 implies that the function $\psi:\underline{n}\to\underline{m}$ must satisfy $[m]\subseteq\psi(\underline{n})$. Since ψ is a morphism of lattices, we also have $\psi(0)=0$. Therefore $\psi:\underline{n}\to\underline{m}$ must be surjective. In particular $n\geq m$.

By Proposition 10.3, $\pi^C j^B\neq 0$ implies that $n\leq m$. Therefore $n=m$. Since the map $\pi^C j^B_A:\underline{n}\to\underline{n}$ is order-preserving and surjective, it must be the identity map. This shows that whenever A is such that $j^D\pi^C j^B_A\neq 0$, then $\pi^C j^B_A=\mathrm{id}_{\underline{n}}$. Therefore, the functions ψ which appear with a nonzero coefficient in the expression of $\pi^C j^B$ are $\mathrm{id}_{\underline{n}}$ and maps in the kernel of j^D.

Now Proposition 10.3 also asserts that the restriction of π^C

$$\pi^C:\{b_0,b_1,\ldots,b_{n-1},b_n\}\longrightarrow\underline{m}$$

is injective. Since $n = m$, this must be a bijection, and since π^C is order-preserving, we must have $\pi^C(b_h) = h$ for every $h \in \underline{n}$. As in the proof of Proposition 10.3, associated with the map $\mathrm{id}_{\underline{n}} : \underline{n} \to \underline{n}$, there is a corresponding subset

$$U_h = \{a \in [b_{h-1}, b_h]_T \mid \pi^C(a) = \mathrm{id}_{\underline{n}}(h)\} = \{a \in [b_{h-1}, b_h]_T \mid \pi^C(a) = h\} .$$

Thus we have $b_h \in U_h$ for every $h \in \underline{n}$.

Since $\mathrm{id}_{\underline{n}}$ appears with a nonzero coefficient $(-1)^n z_{\mathrm{id}}$ in the expression of $\pi^C j^B$, the proof of Proposition 10.3 shows that $\sum_{a\in U_h} \mu(b_{h-1}, a)$ is nonzero. Therefore we must be in one of the two cases (a) or (b) of that proof. But the first case cannot hold because $b_h \notin U_h = \{b_{h-1}\}$. So we are in case (b) and we have $b_h \in U_h =]b_{h-1}, u_h]_T \subseteq]b_{h-1}, b_h]_T$. This forces $u_h = b_h$ and $U_h =]b_{h-1}, b_h]_T$.

The equality $U_h =]b_{h-1}, b_h]_T$ means that π^C coincides with π^B on this interval. This holds for every $h \in [n]$. We now show, by descending induction, that $c_h = b_h$ for every $h \in [n]$. First $c_n = \widehat{1} = b_n$. Now since $\pi^C(b_h) = h$, we have $b_h \leq c_h$, but $b_h \not\leq c_{h-1}$, for every $h \in \underline{n}$. Assume that $c_h = b_h$ for every $h \geq i$ and suppose for contradiction that $b_{i-1} < c_{i-1}$. Then $b_{i-1} < c_{i-1} < c_i = b_i$, hence $c_{i-1} \in]b_{i-1}, b_i]$, which implies that $\pi^B(c_{i-1}) = i$. But π^B coincides with π^C on the interval, so $\pi^C(c_{i-1}) = i$, which contradicts the definition of π^C.

We have now proved that $n = m$ and $C = B$ whenever $j^D\pi^C j^B \neq 0$. Moreover, in that case, we have seen that $\pi^B j^B$ is the sum of a multiple of $\mathrm{id}_{\underline{n}}$ and morphisms in the kernel of j^D. The proof of Proposition 10.3 shows that the coefficient of $\mathrm{id}_{\underline{n}}$ is equal to $(-1)^{n+p}$, where p is the number of times the integer h satisfies case (b). But we have noticed that case (a) cannot occur, so $p = n$ and the coefficient is 1. Thus $\pi^B j^B = \mathrm{id}_{\underline{n}} \pmod{\mathrm{Ker}(j^D)}$ and it follows that $j^D\pi^B j^B = j^D$.

Alternatively, the equality $\pi^B j^B = \mathrm{id}_{\underline{n}} \pmod{\mathrm{Ker}(j^D)}$ also follows from Proposition 10.1 and Proposition 10.2, because $\pi^B j^B = (-1)^n \sum_{\emptyset\subseteq Y\subseteq[n]} (-1)^{|Y|}\rho_Y$ and $j^D\rho_Y = 0$ whenever $Y \neq [n]$ since $[n] \not\subseteq \rho_Y(\underline{n})$. □

Given a finite lattice T, let $\mathcal{P}_{T,n}$ be the set of all strictly increasing n-tuples $(d_0, d_1, \ldots, d_{n-1})$ in $T - \{\widehat{1}\}$, that is, such that $\widehat{0} \leq d_0 < d_1 < \ldots < d_{n-1} < d_n = \widehat{1}$. The set $\mathcal{P}_{T,n}$ corresponds bijectively to the set of all surjective morphisms $T \to \underline{n}$. If $D, C \in \mathcal{P}_{T,n}$, we define

$$f_{D,C} = j^D\pi^C : T \longrightarrow T .$$

In particular, $f_{B,B} = j^B\pi^B$ is the idempotent of Proposition 10.1.

10.5. Corollary. *Let T be a finite lattice.*

(a) Let $A, B \in \mathcal{P}_{T,n}$ and $D, C \in \mathcal{P}_{T,m}$, where $n, m \geq 0$ are two integers. Then

$$f_{D,C}f_{B,A} = \begin{cases} f_{D,A} & \text{if } C = B \text{ (hence } n = m), \\ 0 & \text{if } C \neq B . \end{cases}$$

(b) When $n \geq 0$ varies and $B \in \mathcal{P}_{T,n}$ varies, the idempotents $f_{B,B}$ are pairwise orthogonal.

Proof : (a) By Proposition 10.4, we have

$$f_{D,C}f_{B,A} = j^D\pi^C j^B\pi^A = j^D\pi^A$$

if $C = B$, and zero otherwise.

(b) This follows from (a). □

Let $M_{|\mathcal{P}_{T,n}|}(k)$ denote the matrix algebra of size $|\mathcal{P}_{T,n}|$, with rows and columns indexed by the set $\mathcal{P}_{T,n}$. If $D, C \in \mathcal{P}_{T,n}$, we let $m_{D,C}$ denote the matrix with coefficient 1 in position (D, C) and 0 elsewhere.

10.6. Theorem. *Let T be a finite lattice, let $\mathcal{P}_{T,n}$ denote the set of all strictly increasing n-tuples in $T - \{\widehat{1}\}$, and let N be the maximal length of a strictly increasing sequence in $T - \{\widehat{1}\}$.*

(a) The map

$$\mathcal{I}_T : \bigoplus_{n=0}^{N} M_{|\mathcal{P}_{T,n}|}(k) \longrightarrow \mathrm{End}_{k\mathcal{L}}(T)\,, \qquad m_{D,C} \mapsto f_{D,C}$$

is an algebra homomorphism (without unit elements).

(b) $\mathcal{I}_T$ is injective.

(c) The image of $\mathcal{I}_T$ is equal to the subalgebra $\mathcal{E}_T$ (without unit element) of $\mathrm{End}_{k\mathcal{L}}(T)$ *having a k-basis consisting of all maps whose image is totally ordered (and which are morphisms in the category $\mathcal{L}$).*

Proof : (a) If $|C| \neq |B|$, then $m_{D,C}$ and $m_{B,A}$ are not in the same block, so their product is 0, while the product $f_{D,C} f_{B,A}$ is also zero. If $|C| = |B|$, then the relations of Corollary 10.5 are the standard relations within a matrix algebra of size $|\mathcal{P}_{T,n}|$.

(b) Since the elements $m_{D,C}$ form a k-basis of $\bigoplus_{n=0}^{N} M_{|\mathcal{P}_{T,n}|}(k)$, it suffices to prove that their images $f_{D,C}$ are k-linearly independent. Suppose that

$$\sum_{n=0}^{N} \sum_{D,C \in \mathcal{P}_{T,n}} \lambda_{D,C} f_{D,C} = 0\;,$$

where $\lambda_{D,C} \in k$. Multiply on the left by the idempotent $f_{A,A}$ and on the right by the idempotent $f_{B,B}$. By Corollary 10.5, we obtain $\lambda_{A,B} f_{A,B} = 0$. Since $f_{A,B}$ is a linear combination of distinct maps $T \to T$, one of them appearing with coefficient ± 1, we must have $\lambda_{A,B} = 0$.

(c) It is clear that $\mathcal{E}_T$ is a subalgebra. Moreover, every map $j_A^D \pi^C$ has an image which is totally ordered, by construction. Therefore $f_{D,C} = j^D \pi^C \in \mathcal{E}_T$ and hence $\mathrm{Im}(\mathcal{I}_T) \subseteq \mathcal{E}_T$.

Now $\mathcal{E}_T$ has a k-basis consisting of all maps $\varphi_{U,V} : T \to T$ described as follows. First $U = (u_0, u_1, \ldots, u_{n-1}) \in \mathcal{P}_{T,n}$, while $V \in \mathcal{Y}_{T,n}$, where $\mathcal{Y}_{T,n}$ denotes the set of all strictly increasing n-tuples $V = (v_1, v_2, \ldots, v_n)$ in $T - \{\widehat{0}\}$ (that is $\widehat{0} = v_0 < v_1 < v_2 < \ldots < v_n \leq \widehat{1}$). Define $\lambda^V : \underline{n} \to T$ by $\lambda^V(i) = v_i$ for every $i \in \underline{n}$ and then set $\varphi_{U,V} = \lambda^V \pi^U$. Then

$$\mathcal{B}_n = \{\varphi_{U,V} \mid U \in \mathcal{P}_{T,n}\,,\, V \in \mathcal{Y}_{T,n}\}$$

is a k-basis of the submodule $\mathcal{E}_{T,n}$ generated by all endomorphisms whose image is isomorphic to $\underline{n}$, while $\mathcal{B} = \bigcup_{n=0}^{N} \mathcal{B}_n$ is a k-basis of $\mathcal{E}_T = \bigoplus_{n=0}^{N} \mathcal{E}_{T,n}$. We have seen that $\mathcal{B}' = \bigcup_{n=0}^{N} \{f_{D,C} \mid D, C \in \mathcal{P}_{T,n}\}$ is a k-basis of $\mathrm{Im}(\mathcal{I}_T)$. It is an easy exercise to show that there is a bijection between $\mathcal{Y}_{T,n}$ and $\mathcal{P}_{T,n}$. Therefore $\mathcal{B}$ and $\mathcal{B}'$ have the same cardinality. In other words $\mathrm{Im}(\mathcal{I}_T)$ and $\mathcal{E}_T$ are free k-modules of the same rank.

Now we allow the base ring k to vary and we write an index k to emphasize the dependence on k. Thus we have an injective algebra homomorphism

$$\mathcal{I}_{T,k} : \bigoplus_{n=0}^{N} M_{|\mathcal{P}_{T,n}|}(k) \longrightarrow \mathcal{E}_{T,k} \subseteq \mathrm{End}_{k\mathcal{L}}(T)$$

and we let $\mathcal{Q}_k := \mathcal{E}_{T,k}/\operatorname{Im}(\mathcal{I}_{T,k})$, so that we have a short exact sequence

$$0 \longrightarrow \operatorname{Im}(\mathcal{I}_{T,k}) \xrightarrow{j_k} \mathcal{E}_{T,k} \xrightarrow{p_k} \mathcal{Q}_k \longrightarrow 0\ ,$$

where j_k is the inclusion map and p_k the canonical surjection. In the case of the ring of integers $\mathbb{Z}$, we see that $\mathcal{Q}_\mathbb{Z}$ is a finite abelian group, because $\operatorname{Im}(\mathcal{I}_{T,\mathbb{Z}})$ and $\mathcal{E}_{T,\mathbb{Z}}$ are free $\mathbb{Z}$-modules of the same rank. Tensoring with the prime field $\mathbb{F}_p$, where p is a prime, we obtain

$$0 \longrightarrow \operatorname{Tor}(\mathbb{F}_p, \mathcal{Q}_\mathbb{Z}) \longrightarrow \mathbb{F}_p \otimes \operatorname{Im}(\mathcal{I}_{T,\mathbb{Z}}) \xrightarrow{1\otimes j_\mathbb{Z}} \mathbb{F}_p \otimes \mathcal{E}_{T,\mathbb{Z}} \xrightarrow{1\otimes p_\mathbb{Z}} \mathbb{F}_p \otimes \mathcal{Q}_\mathbb{Z} \longrightarrow 0\ .$$

Using the canonical bases of $\operatorname{Im}(\mathcal{I}_{T,k})$ and $\mathcal{E}_{T,k}$ respectively, we see that $k \otimes \operatorname{Im}(\mathcal{I}_{T,\mathbb{Z}}) \cong \operatorname{Im}(\mathcal{I}_{T,k})$, and similarly $k \otimes \mathcal{E}_{T,\mathbb{Z}} \cong \mathcal{E}_{T,k}$. Moreover the inclusion map $1\otimes j_k$ corresponds, under these isomorphisms, to the inclusion map j_k. In particular, we obtain

$$0 \longrightarrow \operatorname{Tor}(\mathbb{F}_p, \mathcal{Q}_\mathbb{Z}) \longrightarrow \operatorname{Im}(\mathcal{I}_{\mathbb{F}_p}) \xrightarrow{j_{\mathbb{F}_p}} \mathcal{E}_{\mathbb{F}_p} \xrightarrow{1\otimes p_\mathbb{Z}} \mathbb{F}_p \otimes \mathcal{Q}_\mathbb{Z} \longrightarrow 0\ .$$

By (b), we know that $j_{\mathbb{F}_p}$ is injective, hence $\operatorname{Tor}(\mathbb{F}_p, \mathcal{Q}_\mathbb{Z}) = \{0\}$. But since this holds for every prime number p and $\mathcal{Q}_\mathbb{Z}$ is finite, we must have $\mathcal{Q}_\mathbb{Z} = \{0\}$. It follows that the inclusion map $j_\mathbb{Z} : \operatorname{Im}(\mathcal{I}_{T,\mathbb{Z}}) \to \mathcal{E}_{T,\mathbb{Z}}$ is an isomorphism. Tensoring with k, it follows that the inclusion map $j_k : \operatorname{Im}(\mathcal{I}_{T,k}) \to \mathcal{E}_{T,k}$ is an isomorphism as well. In other words, $\operatorname{Im}(\mathcal{I}_{T,k}) = \mathcal{E}_{T,k}$, as was to be shown. □

10.7. Remark. Let $\mathcal{B}$ be the canonical basis of $\mathcal{E}_T$ described in the proof. The change of basis from $\mathcal{B}$ to the basis $\{f_{D,C}\}$ of $\operatorname{Im}(\mathcal{I}_T)$ is not obvious. By construction, every map $j_A^D\pi^C$ belongs to $\mathcal{B}$, but beware of the fact that if C and D are n-tuples, then $j_A^D\pi^C$ may be a composite $T \to \underline{m} \to T$ for some $m < n$, because the n-tuple $A = (a_1, a_2, \ldots, a_n)$ is increasing but not necessarily strictly increasing, hence may consist of m distinct elements for some $m < n$.

The image under $\mathcal{I}_T$ of the identity element of $\bigoplus_{n=0}^N M_{|\mathcal{P}_{T,n}|}(k)$ is an idempotent e_T of $\operatorname{End}_{k\mathcal{L}}(T)$ and e_T is an identity element of $\mathcal{E}_T$. We now prove that the we actually get central idempotents.

10.8. Theorem. *For every finite lattice T, let $\mathcal{E}_T = \operatorname{Im}(\mathcal{I}_T)$ be the subalgebra of $\operatorname{End}_{k\mathcal{L}}(T)$ appearing in Theorem 10.6, and let e_T be the identity element of $\mathcal{E}_T$.*

(a) $e_T = \displaystyle\sum_{n=0}^{N} \sum_{B\in\mathcal{P}_{T,n}} f_{B,B}$.

(b) For any finite lattice T' and any morphism $\theta \in \operatorname{Hom}_{k\mathcal{L}}(T, T')$, we have $\theta e_T = e_{T'}\theta$. In other words, the family of idempotents e_T, for $T \in \mathcal{L}$, is a natural transformation of the identity functor $\mathrm{id}_{k\mathcal{L}}$.

(c) e_T is a central idempotent of $\operatorname{End}_{k\mathcal{L}}(T)$.

(d) The subalgebra $\mathcal{E}_T$ is a direct product factor of $\operatorname{End}_{k\mathcal{L}}(T)$, that is, there exists a subalgebra $\mathcal{D}$ such that $\operatorname{End}_{k\mathcal{L}}(T) = \mathcal{E}_T \times \mathcal{D}$ (where $\mathcal{E}_T$ is identified with $\mathcal{E}_T \times \{0\}$ and $\mathcal{D}$ with $\{0\} \times \mathcal{D}$, as usual).

Proof : (a) The identity element of $\displaystyle\bigoplus_{n=0}^{N} M_{|\mathcal{P}_{T,n}|}(k)$ is equal to $\displaystyle\sum_{n=0}^{N} \sum_{B\in\mathcal{P}_{T,n}} m_{B,B}$. Taking its image under $\mathcal{I}_T$ yields the required formula.

(b) We have seen in the proof of Theorem 10.6 that every element of the canonical basis of $\mathcal{E}_T$ has the form $\lambda^V\pi^U$, where $U = (u_0, u_1, \ldots, u_{n-1}) \in \mathcal{P}_{T,n}$ is a strictly increasing n-tuple in $T - \{\hat{1}\}$, while $V = (v_1, v_2, \ldots, v_n)$ belongs to the set $\mathcal{Y}_{T,n}$ of all strictly increasing n-tuples in $T - \{\widehat{0}\}$. We now compute their opposite, as in Lemma 8.1.

From the surjective morphism $\pi^U : T \to \underline{n}$, we obtain $(\pi^U)^{op} : \underline{n}^{op} \to T^{op}$ defined as follows : for every $h \in \underline{n}$,

$$(\pi^U)^{op}(h) = \bigvee_{\pi^U(t) \leq h} t = \sup\big((\pi^U)^{-1}(h)\big) = u_h \ ,$$

in view of the way U is associated to π^U (see the beginning of Section 10). But $u_h = \lambda^{U^{op}}(h)$, hence $(\pi^U)^{op} = \lambda^{U^{op}}$.

Applying this to the case $U = V^{op}$, where $V \in \mathcal{Y}_{T,n}$, we obtain $(\pi^{V^{op}})^{op} = \lambda^{(V^{op})^{op}} = \lambda^V$. Taking opposites yields $(\lambda^V)^{op} = (\pi^{V^{op}})$.

It follows that the opposite of the canonical basis element $\lambda^V \pi^U$ of $\mathcal{E}_T$ is the canonical basis element $\lambda^{U^{op}} \pi^{V^{op}}$ of $\mathcal{E}_{T^{op}}$. Therefore, the opposite of the identity element e_T of $\mathcal{E}_T$ must belong to $\mathcal{E}_{T^{op}}$. Moreover, it must be the identity element of $\mathcal{E}_{T^{op}}$, because taking opposites behaves well with respect to composition, by Lemma 8.1. Therefore $(e_T)^{op} = e_{T^{op}}$.

Now if $\theta : T \to T'$ is a morphism in $\mathcal{L}$, then the image of a totally ordered subset of T is a totally ordered subset of T'. It follows that composition with θ maps $\mathcal{E}_T$ to linear combination of maps with a totally ordered image, hence invariant under the idempotent element $e_{T'}$. In other words, we have

$$\theta\, e_T = e_{T'}\, \theta\, e_T \ .$$

Applying this equation to T'^{op}, T^{op}, and the morphism $\theta^{op} : T'^{op} \to T^{op}$, we obtain $\theta^{op}\, e_{T'^{op}} = e_{T^{op}}\, \theta^{op}\, e_{T'^{op}}$. Passing to opposites and using the above equality $(e_T)^{op} = e_{T^{op}}$, we get

$$e_{T'}\, \theta\, e_T = e_{T'}\, \theta \ .$$

The two displayed equations then yield $\theta e_T = e_{T'} \theta$. This holds as well if θ is replaced by a linear combination of morphisms, as was to be shown.

(c) This is a special case of (b).

(d) This follows immediately from (c). ∎

11. The case of a total order

In this section, we consider the case of a totally ordered lattice $\underline{n}$, where $n \in \mathbb{N}$. We determine completely the ring of endomorphisms of $\underline{n}$ in the category $k\mathcal{L}$ and we deduce a direct sum decomposition of $F_{\underline{n}}$. For the lattice $\underline{n}$, the set $\mathcal{P}_{\underline{n},r}$ of all strictly increasing r-tuples $(d_0, d_1, \ldots, d_{r-1})$ in $\underline{n} - \{n\}$ is just the set of all subsets of $\{0, 1, \ldots, n-1\}$ of size r, because any such subset is totally ordered. Throughout this section, we use subsets of size r instead of strictly increasing r-tuples. In particular $|\mathcal{P}_{\underline{n},r}| = \binom{n}{r}$.

11.1. Theorem. *Let $n \in \mathbb{N}$.*

(a) The homomorphism of k-algebras of Theorem 10.6

$$\mathcal{I}_{\underline{n}} : \bigoplus_{m=0}^{n} M_{|\mathcal{P}_{\underline{n},r}|}(k) \longrightarrow \mathrm{End}_{k\mathcal{L}}(\underline{n}) \,, \qquad m_{D,C} \mapsto f_{D,C} \ ,$$

is an isomorphism.

(b) In particular, if k is a field, then $\mathrm{End}_{k\mathcal{L}}(\underline{n})$ is semi-simple.

Proof : First note that (b) follows from (a), because any matrix algebra $M_q(k)$ is simple, so that the direct sum is semi-simple. Since any map $\varphi : \underline{n} \to \underline{n}$ has an image which is totally ordered, the subalgebra $\mathcal{E}_T$ of $\mathrm{End}_{k\mathcal{L}}(\underline{n})$ appearing in Theorem 10.6 is the whole of $\mathrm{End}_{k\mathcal{L}}(\underline{n})$. By Theorem 10.6, the morphism $\mathcal{I}_{\underline{n}}$ is surjective and injective, hence an isomorphism. □

11.2. Remark. Since $\mathcal{E}_T$ is the whole of $\mathrm{End}_{k\mathcal{L}}(\underline{n})$, the canonical basis $\mathcal{B}$ of $\mathcal{E}_T$ is a basis of $\mathrm{End}_{k\mathcal{L}}(\underline{n})$, which also has another basis consisting of the morphisms $f_{D,C}$. As mentioned in Remark 10.7, the change of basis is not straightforward, but it can be made more explicit in the case of a totally ordered lattice considered here.

Every element of the canonical basis of $\mathcal{E}_T$ has the form $\lambda^V\pi^U$, where $U = (u_0, u_1, \ldots, u_{n-1}) \in \mathcal{P}_{T,n}$ is a strictly increasing n-tuple in $T - \{\widehat{1}\}$, while $V = (v_1, v_2, \ldots, v_n) \in \mathcal{Y}_{T,n}$ is a strictly increasing n-tuple in $T - \{\widehat{0}\}$. On the other hand, the morphisms $f_{D,C}$ are parametrized by pairs $D, C \in \mathcal{P}_{T,n}$. But we have an obvious bijection from $\mathcal{P}_{T,n}$ to $\mathcal{Y}_{T,n}$, mapping an n-tuple $(u_0, u_1, \ldots, u_{n-1}) \in \mathcal{P}_{T,n}$ to $(u_0+1, u_1+1, \ldots, u_{n-1}+1) \in \mathcal{Y}_{T,n}$. Thus we can parametrize both bases by the same set $\mathcal{P}_{T,n} \times \mathcal{P}_{T,n}$. Then it is not hard to see that the matrix of the change of basis is unitriangular. Actually, this provides another proof of the fact that the map $\mathcal{I}_{\underline{n}}$ is an isomorphism.

11.3. Remark. Theorem 11.1 is similar to the result proved in [FHH] about the planar rook algebra. Over the field $\mathbb{C}$ of complex numbers, this algebra is actually isomorphic to $\mathrm{End}_{\mathbb{C}\mathcal{L}}(\underline{n})$. However, the planar rook monoid is not isomorphic to the monoid of endomorphisms of $\underline{n}$ in $\mathcal{L}$, because it turns out that they do not have the same number of idempotents, even when $n = 2$. Only the corresponding monoid algebras become isomorphic (over $\mathbb{C}$).

Now we consider the central idempotents of $\mathrm{End}_{k\mathcal{L}}(\underline{n})$ corresponding to the above decomposition into matrix algebras.

11.4. Notation. *For an integer m with $0 \le m \le n$, set*

$$\beta_{n,m} := \sum_{B \in \mathcal{P}_{\underline{n},m}} f_{B,B} \ .$$

In particular, for $m = n$ and $B = \{0, 1, \ldots, n-1\}$, we have $\pi^B = \mathrm{id}_{\underline{n}}$ and we define

$$\varepsilon_n := \beta_{n,n} = f_{B,B} = j^B\pi^B = j^B = (-1)^n \sum_{\emptyset \subseteq Y \subseteq [n]} (-1)^{|Y|}\rho_Y \ ,$$

where, as before, $\rho_Y \in \mathrm{End}_{\mathcal{L}}(\underline{n})$ is defined by $\rho_Y(h) = h$ if $h \in Y$ and $\rho_Y(h) = h-1$ otherwise.

11.5. Proposition. *The elements $\beta_{n,m}$, for $0 \le m \le n$, are orthogonal central idempotents of $\mathrm{End}_{k\mathcal{L}}(\underline{n})$, and their sum is equal to the identity. In particular, the central idempotent ε_n satisfies*

$$\varepsilon_n \,\mathrm{End}_{k\mathcal{L}}(\underline{n}) = k\varepsilon_n \ .$$

Proof : For $B \in \mathcal{P}_{\underline{n},m}$, the inverse image of $f_{B,B}$ under the algebra isomorphism $\mathcal{I}$ of Theorem 11.1 is the matrix $m_{B,B}$ of the component $M_{|\mathcal{P}_{\underline{n},m}|}(k)$ indexed by m. Summing over all $B \in \mathcal{P}_{\underline{n},m}$, it follows that the inverse image of $\beta_{n,m}$ under $\mathcal{I}$ is the identity element of $M_{|\mathcal{P}_{\underline{n},m}|}(k)$. The first statement follows.

In the case $m = n$, the set $\mathcal{P}_{\underline{n},m}$ consists of the singleton $B = \{0, 1, \ldots, n-1\}$ and the corresponding matrix algebra has size 1. We see that the inverse image of ε_n under $\mathcal{I}$ is the identity element $m_{B,B}$ of the component $M_1(k) = k$. Clearly $\varepsilon_n \operatorname{End}_{k\mathcal{L}}(\underline{n}) \cong M_1(k)$ is a free k-module of rank 1, hence $\varepsilon_n \operatorname{End}_{k\mathcal{L}}(\underline{n}) = k\varepsilon_n$. □

We want to use the functor $F_? : k\mathcal{L} \to \mathcal{F}_k$ of Section 4 to deduce information on the correspondence functor $F_{\underline{n}}$. By Theorem 4.12, we already know that $F_{\underline{n}}$ is projective, because the total order $\underline{n}$ is a distributive lattice. If $n = 0$ (in which case $[n] = \emptyset$) and if $n = 1$ (in which case $[n] = \{1\}$), we recover the cases already considered in Section 5 of [BT2]. Our purpose is to treat now the general case.

We apply the functor $F_? : k\mathcal{L} \to \mathcal{F}_k$ to the map $j^B \in \operatorname{Hom}_{k\mathcal{L}}(\underline{m}, \underline{n})$ defined in Section 10, where $B \in \mathcal{P}_{\underline{n},m}$. By Proposition 10.1 we obtain a morphism

$$F_{j^B} : F_{\underline{m}} \longrightarrow F_{\underline{n}}$$

which vanishes on $H_{\underline{m}}$. By Proposition 10.2, this induces an injective morphism

$$\overline{F}_{j^B} : F_{\underline{m}}/H_{\underline{m}} \longrightarrow F_{\underline{n}}$$

which embeds $F_{\underline{m}}/H_{\underline{m}}$ as a direct summand of $F_{\underline{n}}$, corresponding to the idempotent $f_{B,B} = j^B\pi^B$. In particular, for $m = n$, we have $f_{B,B} = j^B = \varepsilon_n$ and we obtain an idempotent endomorphism F_{ε_n} of $F_{\underline{n}}$ with kernel $H_{\underline{n}}$.

11.6. Theorem. *Let $n \in \mathbb{N}$ and let $\mathbb{S}_n = F_{\underline{n}}/H_{\underline{n}}$. There are isomorphisms of correspondence functors*

$$\begin{aligned} F_{\varepsilon_n} F_{\underline{n}} &\cong \mathbb{S}_n \ , \\ F_{\beta_{n,m}} F_{\underline{n}} &\cong \mathbb{S}_m^{\oplus\binom{n}{m}} \ , \ \ \textit{for each} \ \ 0 \le m \le n \ , \\ F_{\underline{n}} &\cong \bigoplus_{\substack{0\le m\le n \\ B\in\mathcal{P}_{\underline{n},m}}} \mathbb{S}_{|B|} \ . \end{aligned}$$

Proof : By Theorem 4.8, the functor $F_?$ induces an isomorphism of k-algebras

$$\operatorname{End}_{k\mathcal{L}}(\underline{n}) \cong \operatorname{End}_{\mathcal{F}_k}(F_{\underline{n}}) \ .$$

Now the idempotents $f_{B,B}$ of $\operatorname{End}_{k\mathcal{L}}(\underline{n})$, for $B \in \mathcal{P}_{\underline{n},m}$ and $0 \le m \le n$, are orthogonal and their sum is equal to the identity, by Theorem 11.1. It follows that the endomorphisms $F_{f_{B,B}}$ of $F_{\underline{n}}$ are orthogonal idempotents, and their sum is the identity. Hence we obtain a decomposition of correspondence functors

$$F_{\underline{n}} = \bigoplus_{\substack{0\le m\le n \\ B\in\mathcal{P}_{\underline{n},m}}} F_{f_{B,B}}\big(F_{\underline{n}}\big) \ .$$

By surjectivity of $\pi^B : \underline{n} \to \underline{m}$, the image of $F_{f_{B,B}} = F_{j^B}F_{\pi^B} : F_{\underline{n}} \to F_{\underline{n}}$ is equal to the image of $F_{j^B} : F_{\underline{m}} \to F_{\underline{n}}$. Therefore $F_{f_{B,B}}\big(F_{\underline{n}}\big) = F_{j^B}\big(F_{\underline{m}}\big)$. By Proposition 10.2, the image $F_{j^B}\big(F_{\underline{m}}\big)$ is isomorphic to $\mathbb{S}_m = F_{\underline{m}}/H_{\underline{m}}$ and it follows that

$$F_{f_{B,B}}\big(F_{\underline{n}}\big) \cong \mathbb{S}_m \ .$$

Taking $m = n$ and $f_{B,B} = j^B = \epsilon_n$, we obtain the first isomorphism $F_{\varepsilon_n}F_{\underline{n}} \cong \mathbb{S}_n$. Summing over all $B \in \mathcal{P}_{\underline{n},m}$ for a fixed m, we obtain the second isomorphism, because $|\mathcal{P}_{\underline{n},m}| = \binom{n}{m}$. Finally, summing over all $0 \le m \le n$ and all $B \in \mathcal{P}_{\underline{n},m}$, we obtain the third isomorphism. □

11.7. Corollary. *Let $m,n\in\mathbb{N}$. Then*

$$\mathrm{Hom}_{\mathcal{F}_k}(\mathbb{S}_n,\mathbb{S}_m)=\left\{\begin{array}{ll}0 & \text{if } n\neq m\ ,\\ k\cdot\mathrm{id}_{\mathbb{S}_n} & \text{if } n=m\ .\end{array}\right.$$

Proof : Since $\mathbb{S}_n\cong F_{\varepsilon_n}F_{\underline{n}}$, the case $n=m$ follows from Proposition 11.5. Now for integers $l,m\in\{0,\ldots,n\}$, we use the central idempotents $\beta_{n,l}$ of Proposition 11.5 and we obtain

$$\mathrm{Hom}_{\mathcal{F}_k}(F_{\beta_{n,l}}F_{\underline{n}},F_{\beta_{n,m}}F_{\underline{n}})\cong\mathrm{Hom}_{\mathcal{F}_k}(\mathbb{S}_l,\mathbb{S}_m)^{\oplus\binom{n}{l}\binom{n}{m}}\ .$$

Since $F_{\beta_{n,l}}$ and $F_{\beta_{n,m}}$ are central idempotents of $\mathrm{End}_{\mathcal{F}_k}(F_{\underline{n}})$, and since they are orthogonal if $l\neq m$, it follows that $\mathrm{Hom}_{\mathcal{F}_k}(F_{\beta_{n,l}}F_{\underline{n}},F_{\beta_{n,m}}F_{\underline{n}})=0$ if $l\neq m$, hence $\mathrm{Hom}_{\mathcal{F}_k}(\mathbb{S}_l,\mathbb{S}_m)=0$. □

Now we prove that the functor $\mathbb{S}_n$ is actually isomorphic to a fundamental functor and we compute the ranks of all its evaluations.

11.8. Theorem. *Let $\mathbb{S}_n=F_{\underline{n}}/H_{\underline{n}}$.*

(a) $\mathbb{S}_n$ is isomorphic to the fundamental functor $\mathbb{S}_{[n],\mathrm{tot}}$, where tot *denotes the total order on $[n]$.*

(b) For any finite set X, the k-module $\mathbb{S}_n(X)$ is free of rank

$$\mathrm{rank}\big(\mathbb{S}_n(X)\big)=\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}(i+1)^{|X|}\ .$$

Proof : (a) We are going to use the results of Section 8 applied to the lattice $T=\underline{n}^{op}$. The set of its irreducible elements is

$$E=\{0,1,\ldots,n{-}1\}\ ,$$

with a total order tot^{op} being the opposite of the usual order. Now we have

$$F_{T^{op}}=F_{(\underline{n}^{op})^{op}}=F_{\underline{n}}$$

and its evaluation at E contains an element

$$\gamma_T=\gamma_{\underline{n}^{op}}=\sum_{A\subseteq E}(-1)^{|A|}\eta_A^\circ\ .$$

Recall from Notation 9.2 that $\eta_A^\circ:E\to T^{op}=\underline{n}$ denotes the same map as $\eta:E\to T=\underline{n}^{op}$ and that η is defined by

$$\forall e\in E,\ \ \eta_A(e)=\left\{\begin{array}{ll}r(e) & \text{if } e\in A\ ,\\ e & \text{if } e\notin A\ ,\end{array}\right.\qquad\text{that is,}\qquad \eta_A(e)=\left\{\begin{array}{ll}e+1 & \text{if } e\in A\ ,\\ e & \text{if } e\notin A\ ,\end{array}\right.$$

because $r(e)=e+1$ in the lattice $\underline{n}^{op}$.

Now we define $\omega:E\to\underline{n}$ by $\omega(e)=e+1$. Then $\omega\in F_{\underline{n}}(E)$ and when we apply the idempotent F_{ε_n} we claim that we obtain

(11.9) $$F_{\varepsilon_n}(\omega)=(-1)^n\gamma_{\underline{n}^{op}}\ .$$

The definition of ε_n yields

$$F_{\varepsilon_n}(\omega)=(-1)^n\sum_{Y\subseteq[n]}(-1)^{|C|}\rho_Y\,\omega$$

and the definition of ρ_Y gives

$$(\rho_Y\,\omega)(e)=\rho_Y(e+1)=\left\{\begin{array}{ll}e+1 & \text{if } e+1\in Y\ ,\\ e & \text{if } e+1\notin Y\ .\end{array}\right.$$

Setting $Y = A + 1$ for each $A \subseteq E$, we see that $\rho_Y\, \omega = \eta_A^\circ$ and it follows that

$$\begin{aligned} F_{\varepsilon_n}(\omega) &= (-1)^n \sum_{Y \subseteq [n]} (-1)^{|Y|} \rho_Y\, \omega \\ &= (-1)^n \sum_{A \subseteq E} (-1)^{|A|} \eta_A^\circ \\ &= (-1)^n \gamma_{\underline{n}^{op}} \,. \end{aligned}$$

This proves Claim 11.9 above.

Now $F_{\underline{n}}$ is generated by $\omega \in F_{\underline{n}}(E)$, because it is generated by $\iota \in F_{\underline{n}}([n])$ (where $\iota : [n] \to \underline{n}$ is the inclusion), hence also by any injection from a set E of cardinality n to $\underline{n}$ (by composing with a bijection between E and $[n]$). Since F_{ε_n} is an idempotent endomorphism of the correspondence functor $F_{\underline{n}}$, we see that $F_{\varepsilon_n} F_{\underline{n}}$ is generated by $F_{\varepsilon_n}(\omega)$. In other words, in view of Claim 11.9 above, $F_{\varepsilon_n} F_{\underline{n}}$ is generated by $\gamma_{\underline{n}^{op}} \in F_{\underline{n}}(E)$. Now Theorem 9.5 asserts that the subfunctor of $F_{\underline{n}}$ generated by $\gamma_T = \gamma_{\underline{n}^{op}}$ is isomorphic to $\mathbb{S}_{E,\mathrm{tot}^{op}}$. But $(E, \mathrm{tot}^{op}) \cong ([n], \mathrm{tot})$ via the map $e \mapsto n - e$. Therefore, using the isomorphism of Theorem 11.6, we obtain

$$\mathbb{S}_n \cong F_{\varepsilon_n} F_{\underline{n}} = \langle \gamma_{\underline{n}^{op}} \rangle \cong \mathbb{S}_{E,\mathrm{tot}^{op}} \cong \mathbb{S}_{[n],\mathrm{tot}} \,.$$

(b) The canonical k-basis of the k-module $\mathbb{S}_n(X) = F_{\underline{n}}(X)/H_{\underline{n}}(X)$ is the set $Z_n(X)$ of all maps $\varphi : X \to \underline{n}$ such that $[n] \subseteq \varphi(X) \subseteq \underline{n}$. Therefore $\mathbb{S}_n(X)$ is free of rank $|Z_n(X)|$. The number of maps in $Z_n(X)$ has been computed in Lemma 9.1 of [BT2] and the formula is actually well-known. The formula shows that this rank is equal to

$$|Z_n(X)| = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n+1-i)^{|X|} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (j+1)^{|X|}$$

as required. □

11.10. Remark. We shall see in [BT3] that a similar formula holds for the rank of the evaluation of any fundamental functor, but the proof in the general case is much more elaborate. Also, Corollary 11.7 holds more generally for fundamental functors and the general case will be proved in [BT3].

When k is a field, we get even more.

11.11. Corollary. *Let k be a field.*

(a) The functor $\mathbb{S}_n$ is simple, isomorphic to $S_{[n],\mathrm{tot},k}$, where k is the trivial module for the trivial group $\mathrm{Aut}([n], \mathrm{tot}) = \{\mathrm{id}\}$.

(b) $\mathbb{S}_n$ is simple, projective, and injective in $\mathcal{F}_k$.

Proof : (a) It is clear that $\mathrm{Aut}([n], \mathrm{tot})$ is the trivial group, with a single simple module k. Recall from Section 2 that the fundamental functor $\mathbb{S}_{E,R}$ and the simple functor $S_{E,R,V}$ are given by

$$\mathbb{S}_{E,R} = L_{E,\mathcal{P}_E f_R}/J_{E,\mathcal{P}_E f_R} \,, \qquad S_{E,R,V} = L_{E,T_{R,V}}/J_{E,T_{R,V}} \,,$$

where $\mathcal{P}_E f_R$ is the fundamental module corresponding to (E, R) and $T_{R,V} = \mathcal{P}_E f_R \otimes_{k\,\mathrm{Aut}(E,R)} V$. Here $E = [n]$, $R = \mathrm{tot}$, $k\,\mathrm{Aut}(E, R) = k$, and $V = k$, hence

$$T_{\mathrm{tot},k} = \mathcal{P}_{[n]} f_{\mathrm{tot}} \otimes_k k \cong \mathcal{P}_{[n]} f_{\mathrm{tot}} \,.$$

Therefore $\mathbb{S}_{[n],\mathrm{tot}} \cong S_{[n],\mathrm{tot},k}$.

(b) Since $\underline{n}$ is a distributive lattice, $F_{\underline{n}}$ is projective and injective by Corollary 8.11. Therefore so is its direct summand $\mathbb{S}_n$. It follows that $\mathbb{S}_n$ is simple, projective, and injective. □

Our last purpose in this section is to find, for any finite lattice T, all the direct summands of F_T isomorphic to a functor $\mathbb{S}_n$ corresponding to a total order. Recall that e_T denotes the central idempotent of $\mathrm{End}_{k\mathcal{L}}(T)$ which is an identity element for the subalgebra $\mathcal{E}_T$ (see Theorem 10.8).

11.12. Theorem. *Let T be a finite lattice and let N be the maximal length of a strictly increasing sequence in T. For every finite set X, let $F_T^{\mathrm{tot}}(X)$ be the k-submodule of $F_T(X)$ generated by all the maps $\varphi : X \to T$ such that $\varphi(X)$ is a totally ordered subset of T.*

(a) F_T^{tot} is a subfunctor of F_T, equal to $F_{e_T}(F_T)$.
(b) F_T^{tot} is a direct summand of F_T, isomorphic to

$$F_T^{\mathrm{tot}} \cong \bigoplus_{\substack{0\le m\le N\\ B\in\mathcal{P}_{T,m}}} \mathbb{S}_{|B|} \ .$$

(c) The image of any morphism $F_{\underline{n}} \to F_T$ is contained in F_T^{tot}. In particular, any subfunctor of F_T isomorphic to a functor $\mathbb{S}_n$ is contained in F_T^{tot}.
(d) $\mathrm{Hom}_{\mathcal{F}_k}\big(F_T^{\mathrm{tot}}, F_{\mathrm{id}-e_T}(F_T)\big) = \{0\}$ and $\mathrm{Hom}_{\mathcal{F}_k}\big(F_{\mathrm{id}-e_T}(F_T), F_T^{\mathrm{tot}}\big) = \{0\}$.
(e) The splitting of the surjection $F_T \to F_T^{\mathrm{tot}}$ is natural in T.

Proof : (a) Let $\varphi \in F_T^{\mathrm{tot}}(X)$. Writing $\mathrm{Im}(\varphi) = \{t_1, t_2, \ldots, t_n\}$ in increasing order, we can write $\varphi = j\psi$, where $\psi : X \to \underline{n}$ is the map defined by $\psi(x) = i$ if $\varphi(x) = t_i$, and where $j : \underline{n} \to T$ is the map defined by $j(i) = t_i$ for $1 \le i \le n$ and $j(0) = \widehat{0}$. Clearly j is a morphism in the category $\mathcal{L}$. By Theorem 10.8, we have $e_T j = j e_{\underline{n}} = j$, because $e_{\underline{n}}$ is the identity element of $\mathrm{End}_{k\mathcal{L}}(\underline{n})$ by Theorem 11.1. Therefore

$$\varphi = j\psi = e_T j\psi = e_T\varphi = F_{e_T}(\varphi) \ ,$$

proving that $\varphi \in F_{e_T}(F_T(X))$.

Conversely, if $\varphi \in F_{e_T}(F_T(X))$, then we can write $\varphi = F_{e_T}(\psi) = e_T\psi$ for some map $\psi : X \to T$. Since e_T is, by construction, a linear combination of maps with a totally ordered image, so is $e_T\psi$, proving that $\varphi \in F_T^{\mathrm{tot}}(X)$.

This shows that $F_T^{\mathrm{tot}} = F_{e_T}(F_T)$ and the latter is a subfunctor of F_T.

(b) As in the proof of Theorem 11.6, we apply the fully faithful functor $k\mathcal{L} \to \mathcal{F}_k$ defined by $T \mapsto F_T$. There is direct sum decomposition of functors

$$F_T = F_{e_T}(F_T) \oplus F_{\mathrm{id}-e_T}(F_T) = F_T^{\mathrm{tot}} \oplus F_{\mathrm{id}-e_T}(F_T) \ .$$

The idempotent e_T is the sum of the orthogonal idempotents $f_{B,B}$ of $\mathrm{End}_{k\mathcal{L}}(T)$, for $B \in \mathcal{P}_{T,m}$ and $0 \le m \le N$. It follows that the endomorphisms $F_{f_{B,B}}$ of F_T are orthogonal idempotents with sum F_{e_T}. Hence we obtain a direct sum decomposition of correspondence functors

$$F_T^{\mathrm{tot}} = F_{e_T}(F_T) = \bigoplus_{\substack{0\le m\le n\\ B\in\mathcal{P}_{T,m}}} F_{f_{B,B}}(F_T) \ .$$

By Proposition 10.2, the image of $F_{f_{B,B}}$ is isomorphic to $F_{\underline{m}}/H_{\underline{m}} = \mathbb{S}_m$, where $m = |B|$, proving the result.

(c) Let $\alpha : F_{\underline{n}} \to F_T$ be a morphism of correspondence functors. Since the functor $T \mapsto F_T$ is full, α is the image of a morphism $\underline{n} \to T$ in $k\mathcal{L}$, which is in turn a linear combination of order-preserving maps $f : \underline{n} \to T$. For such a map f and for any function $\varphi : X \to \underline{n}$, the image of $f\varphi$ is a totally ordered subset of T. It follows that the image of the map $F_f : F_{\underline{n}}(X) \to F_T(X)$ is contained in $F_T^{\mathrm{tot}}(X)$.

The special case follows from the fact that $\mathbb{S}_n$ is a quotient of $F_{\underline{n}}$, by Theorem 11.6.

(d) The first statement is a consequence of (b) and (c), while the second one follows from a dual argument. Details are left to the reader.

(e) By Theorem 10.8, the family of idempotents e_T, for $T \in \mathcal{L}$, is a natural transformation of the identity functor $\mathrm{id}_{k\mathcal{L}}$. Therefore the family of idempotents F_{e_T}, for $T \in \mathcal{L}$, is a natural transformation of the identity functor $\mathrm{id}_{\mathcal{F}_k}$. □

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