A NOTE ON THE λ-STRUCTURE ON THE BURNSIDE RING

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Abstract. Let G be a finite group and let S be a G-set. The Burnside ring of G has a natural structure of a λ-ring, $\{\lambda^n\}_{n\in\mathbb{N}}$. However, a priori $\lambda^n(S)$, where S is a G-set, can only be computed recursively, by first computing $\lambda^1(S),\dots,\lambda^{n-1}(S)$. In this paper we establish an explicit formula, expressing $\lambda^n(S)$ as a linear combination of classes of G-sets.

1. Introduction

We use $\mathcal{B}(G)$ to denote the Burnside ring of the finite group G. Recall that, as an abelian group, $\mathcal{B}(G)$ is free on $\{[S]\}_{S\in R}$, where R is a set of representatives of the isomorphism classes of transitive G-sets, and that its rank equals the number of conjugacy classes of subgroups of G. When f is a function on $\mathcal{B}(G)$ and S is a G-set, we write $f(S)$ for $f([S])$.

There is a λ-structure on $\mathcal{B}(G)$, $\{\lambda^n\}_{n\in\mathbb{N}}$, defined as the opposite structure[1] of $\{\sigma^n\}_{n\in\mathbb{N}}$, where $\sigma^n(S)$ is the class of the nth symmetric power of S. It should be considered the natural λ-structure on $\mathcal{B}(G)$, one reason for this being that there is a canonical homomorphism to the ring of rational representations of G, $\mathrm{h}\colon \mathcal{B}(G)\to \mathrm{R}_{\mathbb{Q}}(G)$, defined by $\mathrm{h}(S)=\big[\mathbb{Q}[S]\big]$, and the given λ-structure on $\mathcal{B}(G)$ makes h into a λ-homomorphism. (Note however that this λ-structure is non-special.)

The implicit nature of the definition of the λ-structure on $\mathcal{B}(G)$ makes it hard to compute with. The main result of this paper is a closed formula for $\lambda^n(S)$, where S is any G-set. To state it, we use the following notation: let $\mu=(\mu_1,\dots,\mu_l)\vdash n$, i.e., μ is a partition of n. We use $\ell(\mu):=l$ to denote the length of μ, and if $\mu=(1^{\alpha_1},2^{\alpha_2},\dots)$, we define the tuple $\alpha(\mu):=(\alpha_1,\dots,\alpha_{l'})$, and write $\binom{\ell(\mu)}{\alpha(\mu)}$ for $\frac{l!}{\alpha_1!\cdots\alpha_{l'}!}$. Using this notation we can express $\lambda^n(S)$, for any G-set S, as a linear combination of classes of G-sets:

Theorem 1.1. *Let n be a positive integer and let $\mu=(\mu_1,\dots,\mu_l)\vdash n$. For S a G-set, let $\mathcal{P}_\mu(S)$ be the G-set consisting of $\ell(\mu)$-tuples of pairwise disjoint subsets of S, where the first one has cardinality μ_1, and so on. Then*

$$\lambda^n(S)=(-1)^n\sum_{\mu\vdash n}(-1)^{\ell(\mu)}\binom{\ell(\mu)}{\alpha(\mu)}[\mathcal{P}_\mu(S)]\in\mathcal{B}(G). \tag{1.2}$$

In particular, $\lambda^n(S)=0$ when $n>|S|$.

This result was first stated and proved by the second author in an earlier version of this paper (preprint [Rök07b]), showing that $\lambda^n(S)$ lies in a subring of $\mathcal{B}(G)$ on which h is injective, and then that the image of (1.2) in $\mathrm{R}_{\mathbb{Q}}(G)$ is satisfied. The proof given in the present version is more intrinsic, using only the structure of Burnside rings. It relies on the construction of a ring of formal power series with coefficients in Burnside rings, and an exponential map on this ring, developed in

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[1] Let $\{\sigma^n\}_{n\in\mathbb{N}}$ be a λ-structure on the ring R and define $\sigma_t(x):=\sum_{i\geq 0}\sigma^i(x)t^i\in R[[t]]$. The λ-structure opposite to $\{\sigma^n\}$ is defined by $\sigma_t(x)\cdot\lambda_{-t}(x)=1\in R[[t]]$, where $\lambda_t(x):=\sum_{i\geq 0}\lambda^i(x)t^i$.

[Bou92]. Using this framework, the proof reduces to some explicit combinatorial computations. In Section 2 we give a survey of the relevant constructions and results from [Bou92]. Then in Section 3 we use these results to obtain a formula for $\lambda^n(S)$, which we then show to be the requested one using some combinatorial arguments.

Theorem 1.1 originates in the paper [Rök07a], in which the second author computes the classes of certain tori in the Grothendieck ring of varieties, in terms of the λ-structure on that ring. This formula is suggested by the corresponding class of the cohomology of the torus, and its proof uses a map from the Burnside ring of the absolute Galois group of the base field, where formula (1.2) can be applied. Actually, it was these computations that led the second author conjecture Theorem 1.1.

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2. Background Material

An introduction to λ-rings, representation rings and the Burnside ring is given in [Knu73]. The standard reference for λ-rings is [AT69]. We now give a quick review of some definitions and results:

2.1. **Posets.** A G-poset P is a G-set with a partial ordering compatible with the G-action, in the sense that if $s \leq t \in P$ then $gs \leq gt$ for all $g \in G$. A G-map of G-posets is a map $f\colon P \to Q$ of posets such that $gf(s) = f(gs)$ for $s \in P$ and $g \in G$. If also $f'\colon P \to Q$, then $f \leq f'$ if this holds pointwise. In connection with this, when S is a G-set and we use it as a G-poset this means that we view S as a G-poset using its discrete ordering.

Let P be a G-poset. We recall the definition of the Lefschetz invariant of P: for every $i \in \mathbb{N}$, $\mathrm{Sd}_i P$ is the G-set of chains $x_0 < \cdots < x_i$ in P of length $i+1$. The *Lefschetz invariant* of the G-poset P, Λ_P, is the alternating sum $\sum_{i\geq 0}(-1)^i[\mathrm{Sd}_i P] \in \mathcal{B}(G)$. The *reduced Lefschetz invariant* of P is $\widetilde{\Lambda}_P := \Lambda_P - 1$.

We also need the notion of homotopic posets. We say that the G-posets P and Q are *simplicially homotopic*, or just *homotopic*[2], if there are G-maps $f\colon P \to Q$ and $g\colon Q \to P$ such that $gf \leq \mathrm{Id}_P$ or $gf \geq \mathrm{Id}_P$, and similarly for fg. if P and Q are homotopic as G-posets then $\widetilde{\Lambda}_P = \widetilde{\Lambda}_Q$ (see e.g. Proposition 4.2.5 in [Bou00]). In particular, if P has a largest or smallest element then $\widetilde{\Lambda}_P = 0$.

2.2. **Results from** [Bou92]**.** In this subsection we give a review of the definitions and results from [Bou92] that we use to prove Theorem 1.1. We use G_n to denote the wreath product of G with Σ_n, $G_n := G\wr\Sigma_n$ (by definition, $G_0 = 1$). One defines the ring $\mathbb{B}(G)$ in the folling way: as a group it is the direct product of the Burnside rings $\mathcal{B}(G \wr \Sigma_n)$, indexed over all $n \in \mathbb{N}$. We represent the elements of this group as a power series, $\sum_{i\geq 0} x_i t^i$ where $x_i \in \mathcal{B}(G \wr \Sigma_i)$. This is a ring in a natural way, see [Bou92] for the construction of the multiplication.

Let $\tilde{g} = ((g_1, \ldots, g_n), \sigma)$, where $\sigma \in \Sigma_n$ and $g_i \in G$, be an element of G_n. When S is a G-set we view S^n as a G_n-set by $\tilde{g}(s_1, \ldots, s_n) = (g_1 s_{\sigma^{-1}1}, \ldots, g_n s_{\sigma^{-1}n})$. Moreover, let $\underline{S}$ be the poset defined by adding a smallest element 0 to S, and define the G_n-poset S^{*n} as the set of maps $\{1, \ldots, n\} \to \underline{S}$ which are not constant equal to zero, where the partial ordering is defined pointwise, and with the G_n-action defined in the same way as on S^n, with G acting trivially on the minimal element 0.

[2]Note however that two non-homotopic posets may admit homotopic realizations.

Next one defines maps $u_i\colon \mathcal{B}(G) \to \mathcal{B}(G_i)$ by $x \mapsto \Lambda_{P^i}$, where P is a G-poset such that $\Lambda_P = x$. Let $\mathbb{I}(G)$ be the ideal of $\mathbb{B}(G)$ consisting of those series with zero as constant coefficient. The u_i are then used to define an exponential map $\mathcal{E}\mathrm{xp}\colon \mathbb{I}(G) \to \mathbb{B}(G)$ having the property that if $f, g \in \mathbb{I}(G)$ then $\mathcal{E}\mathrm{xp}(f+g) = \mathcal{E}\mathrm{xp}(f)\,\mathcal{E}\mathrm{xp}(g)$. In the case we are interested in, when $f = xt$ for $x \in \mathcal{B}(G)$, we have by definition $\mathcal{E}\mathrm{xp}(xt) = \sum_{i\geq 0} u_i(x)t^i$. (We omit the construction in the general case, see [Bou92].) In particular, when H is a subgroup of G we have $\mathcal{E}\mathrm{xp}([G/H]t) = \sum_{i\geq 0}[G_i/H_i]t^i$. Moreover, since, for any G-set S, $S = \Lambda_S = \widetilde{\Lambda}_{S_+}$, where $S_+ := S\dot{\cup}\{\bullet\}$, we have $\mathcal{E}\mathrm{xp}(-[S]t) = \sum_{i\geq 0} u_i(-\widetilde{\Lambda}_{S_+})t^i$. By Lemme 4 in [Bou92] it follows that

$$\mathcal{E}\mathrm{xp}(-[S]t) = -\sum_{i\geq 0} \widetilde{\Lambda}_{(S_+)^{*i}}\, t^i. \tag{2.1}$$

For every $i \in \mathbb{N}$ we have a map $\mathrm{m}_i\colon \mathcal{B}(G_i) \to \mathcal{B}(G)$, induced by taking the G_i-set S to the G-set $\Sigma_i\backslash S$. Together the m_i give a homomorphism of rings $\mathrm{m}\colon \mathbb{B}(G) \to \mathcal{B}(G)[[t]]$.

3. Proof of Theorem 1.1

The property that allows us to use the above theory on our problem is the following:

Lemma 3.1. *For any $x \in \mathcal{B}(G)$ we have $\mathrm{m}\big(\mathcal{E}\mathrm{xp}(xt)\big) = \sigma_t(x)$ and $\mathrm{m}\big(\mathcal{E}\mathrm{xp}(-xt)\big) = \lambda_{-t}(x)$.*

Proof. Let S^n denote the nth symmetric power. When $x = [G/H]$, we have to show that $\Sigma_n\backslash(G_n/H_n) \simeq \mathrm{S}^n(G/H)$ as G-sets, for every positive integer n: firstly, the map

$$(g_1, \dots, g_n, \sigma) \mapsto (\overline{g_1}, \dots, \overline{g_n})\colon G_n \to (G/H)^n$$

factors through G_n/H_n, for if $(g_1, \dots, g_n, \sigma) \in G_n$ then, for any $(h_1, \dots, h_n, \tau) \in H_n$, the element

$$(g_1, \dots, g_n, \sigma)(h_1, \dots, h_n, \tau) = (g_1 h_{\sigma 1}, \dots, g_n h_{\sigma n}, \sigma\tau) \in G_n$$

maps to $(\overline{g_1 h_{\sigma 1}}, \dots, \overline{g_n h_{\sigma n}}) = (\overline{g_1}, \dots, \overline{g_n}) \in (G/H)^n$, which is also the image of $(g_1, \dots, g_n, \sigma)$. Denote the resulting map $\phi\colon G_n/H_n \to (G/H)^n$. If we give $(G/H)^n$ the G_n-set structure $(g_1, \dots, g_n, \sigma)\cdot(\overline{f_1}, \dots, \overline{f_n}) = (\overline{g_1 f_{\sigma 1}}, \dots, \overline{g_n f_{\sigma n}})$, then ϕ is G_n-equivariant. Moreover it is surjective. Since both G_n/H_n and $(G/H)^n$ have $|G|^n/|H|^n$ elements, it follows that ϕ is an isomorphism of G_n-sets. Consequently it induces an isomorphism of G-sets $\Sigma_n\backslash(G_n/H_n) \to \mathrm{S}^n(G/H)$.

For arbitrary x the result now follows from the properties of m and $\mathcal{E}\mathrm{xp}$. For suppose that it holds for $x, y \in \mathcal{B}(G)$. Firstly $\mathrm{m}(\mathcal{E}\mathrm{xp}(x+y)) = \mathrm{m}(\mathcal{E}\mathrm{xp}(x))\,\mathrm{m}(\mathcal{E}\mathrm{xp}(y)) = \sigma_t(x)\sigma_t(y) = \sigma_t(x+y)$. Moreover $1 = \mathrm{m}(\mathcal{E}\mathrm{xp}(x))\,\mathrm{m}(\mathcal{E}\mathrm{xp}(-x)) = \sigma_t(x)\,\mathrm{m}(\mathcal{E}\mathrm{xp}(-x))$, hence $\mathrm{m}(\mathcal{E}\mathrm{xp}(-x)) = \sigma_t(-x)$. Since every element of $\mathcal{B}(G)$ is a linear combination of elements $[G/H]$ we are done.

The second assertion follows immediately, since $\sigma_t(x)\lambda_{-t}(x) = 1$, so $\lambda_{-t}(x) = \sigma_t(-x)$. ☐

Using this lemma together with (2.1) shows that, when S is a G-set, $\lambda_{-t}(S) = -\,\mathrm{m}\big(\sum_{n\geq 0} \widetilde{\Lambda}_{(S_+)^{*i}}\, t^n\big)$, hence that

$$\lambda^n(S) = (-1)^{n-1}\,\mathrm{m}_n\big(\widetilde{\Lambda}_{(S_+)^{*i}}\big). \tag{3.2}$$

Thus we have in some sense achieved our goal; we have expressed $\lambda^n(S)$ in a non-recursive way, without using $\lambda^i(S)$ for $i < n$. However, we want to be more concrete, and the major step is the following proposition, which allows us to express $\lambda^n(S)$ without using $\mathcal{B}(G_n)$.

Proposition 3.3. *For S a G-set, let $\Omega_{\leq n}(S)$ be the G-poset of nonempty subsets of S of cardinality $\leq n$. For any $n \in \mathbb{N}$,*

$$\mathrm{m}_n\big(\widetilde{\Lambda}_{S^{*n}}\big) = \widetilde{\Lambda}_{\Omega_{\leq n}(S)} \,.$$

Proof. Given the G-set S and a positive integer n we define the G-poset S_n,

$$S_n := \{\alpha\colon S \to \mathbb{N} : 1 \leq \sum_{s\in S} \alpha(s) \leq n\}$$

with the ordering given by $\alpha \leq \alpha'$ if $\alpha(s) \leq \alpha'(s)$ for every $s \in S$, and the G-action $(g\alpha)(s) := \alpha(g^{-1}s)$. Note that S_n is G-homotopic to $\Omega_{\leq n}(S)$, for we have maps $\theta\colon S_n \to \Omega_{\leq n}(S)$, given by $\alpha \mapsto \alpha^{-1}(\mathbb{N} \smallsetminus \{0\})$, and $\theta'\colon \Omega_{\leq n}(S) \to S_n$ sending $A \subseteq S$ to its characteristic function. The composition $\theta\theta'$ is the identity and $\theta'\theta \leq \mathrm{Id}_{S_n}$. Hence $\widetilde{\Lambda}_{S_n} = \widetilde{\Lambda}_{\Omega_{\leq n}(S)}$, so it suffices to show that $\mathrm{m}_n\big(\widetilde{\Lambda}_{S^{*n}}\big) = \widetilde{\Lambda}_{S_n}$. We will do this by proving that, for every i,

$$\Sigma_n \backslash \mathrm{Sd}_i(S^{*n}) \simeq \mathrm{Sd}_i(S_n)$$

as G-sets.

We proceed to constructing this isomorphism: first, we have a map $\phi\colon S^{*n} \to S_n$, defined by $\phi(f)(s) = |f^{-1}(s)|$ for $s \in S$. One checks that this is a well-defined map of G-posets (where we view S^{*n} as a G-poset via restriction). The map ϕ is surjective, for given $\alpha \in S_n$ one may construct an element f in its preimage in the following way: for $s \in S$, choose $E_s \subseteq \{1, \ldots, n\}$ such that $|E_s| = \alpha(s)$ (possibly, $E_s = \emptyset$). Since $\sum_{s\in S} \alpha(s) \leq n$ we may do this such that the E_s are mutually disjoint. We now define $f \in S^{*n}$ by $f(i) = 0$ if $i \notin \cup_{s\in S} E_s$ and $f(i) = s$ if $i \in E_s$. It then follows that $\phi(f)(s) = |f^{-1}(s)| = |E_s| = \alpha(s)$ for all $s \in S$, i.e., $\phi(f) = \alpha$.

Next one shows that ϕ induces, for every i, a map of G-sets $\Phi\colon \mathrm{Sd}_i(S^{*n}) \to \mathrm{Sd}_i(S_n)$ defined by

$$\Phi(f_0 < \cdots < f_i) := \big(\phi(f_0) < \cdots < \phi(f_i)\big).$$

Since we already know that ϕ is a map of G-posets it suffices to show that Φ does not map chains to shorter chains, i.e., that if $f < f'$ then $\varphi(f) < \varphi(f')$. This follows since there exists an $i_0 \in \{1, \ldots, n\}$ such that $f(i_0) < f'(i_0)$, i.e., $f(i_0) \notin S$ whereas $f'(i_0) = s_0 \in S$, hence $f^{-1}(s_0)$ is strictly contained in $f'^{-1}(s_0)$, i.e., $\phi(f)(s_0) < \phi(f')(s_0)$.

The map Φ is surjective, for ϕ is and from the construction it follows that we may choose elements in the preimages such that the chain property is not destroyed.

Finally, for $c = (f_0 < \cdots < f_i)$ and $c = (f'_0 < \cdots < f'_i)$ in $\mathrm{Sd}_i(S^{*n})$ we have $\Phi(c) = \Phi(c')$ if and only if there exists a $\sigma \in \Sigma_n$ such that $\sigma(c) = c'$. For suppose that $\Phi(c) = \Phi(c')$. Then, for every $0 \leq j \leq i$, $\phi(f_j) = \phi(f'_j)$, i.e., for every $s \in S$ we have $|f_j^{-1}(s)| = |f_j'^{-1}(s)|$. Since $f_0^{-1}(s) \subseteq \cdots \subseteq f_i^{-1}(s)$ and $f_0'^{-1}(s) \subseteq \cdots \subseteq f_i'^{-1}(s)$ this means that we may chose a bijection $\sigma_s\colon f_i^{-1}(s) \to f_i'^{-1}(s)$ such that $\sigma_s\big(f_j^{-1}(s)\big) = f_j'^{-1}(s)$ for every $0 \leq j \leq i$. Since the sets $f_j^{-1}(s)$, for $s \in S$, are mutually disjoint there exists a $\sigma \in \Sigma_n$ which, viewed as an automorphism of $\{1, \ldots, n\}$, restrict to σ_s on $f_i^{-1}(s)$ for every $s \in S$. Then, for any $1 \leq j \leq i$ and for any $m \in \{1, \ldots, n\}$ and $s \in S$ we have that

$$f_j(m) = s \iff m \in f_j^{-1}(s) \iff \sigma m \in f'^{-1}(s) \iff f'_j(\sigma m) = s,$$

and also that $f_j(m) = 0 \iff f'_j(\sigma m) = 0$. Hence $\sigma f_j = f'_j$ for $0 \leq j \leq i$, i.e., $\sigma c = c'$.

It follows that Φ induces an isomorphism of G-sets $\Sigma_n \backslash \mathrm{Sd}_i(S^{*n}) \to \mathrm{Sd}_i(S_n)$. $\square$

Therefore, from (3.2),

$$\lambda^n(S) = (-1)^{n-1}\,\widetilde{\Lambda}_{\Omega_{\leq n}(S)}\,.$$

Theorem 1.1 therefore follows from the following computation:

Lemma 3.4. *Let S be a G-set and let $S_+ := S \cup \{\bullet\}$. In $\mathcal{B}(G)$ we then have the equality*

$$\widetilde{\Lambda}_{\Omega_{\leq n}(S_+)} = -\sum_{\mu\vdash n}(-1)^{\ell(\mu)}\binom{\ell(\mu)}{\alpha(\mu)}[\mathcal{P}_\mu(S)].$$

Proof. The inclusion $S \to S_+$ induces an inclusion $i\colon \Omega_{\leq n}(S) \to \Omega_{\leq n}(S_+)$. By Proposition 4.2.7 of [Bou00] we have

$$\widetilde{\Lambda}_{\Omega_{\leq n}(S_+)} = \widetilde{\Lambda}_{\Omega_{\leq n}(S)} + \sum_{A\in[G\backslash\,\Omega_{\leq n}(S_+)]} \mathrm{ind}_{G_A}^{G}(\widetilde{\Lambda}_{i^A}\,\widetilde{\Lambda}_{]A,.[}),$$

where $i^A = \{B \in \Omega_{\leq n}(S) : B = i(B) \subseteq A\}$. However, when $A \neq \{\bullet\}$ the set i^A has a largest element (namely $A \smallsetminus \{\bullet\}$), hence $\widetilde{\Lambda}_{i^A} = 0$. Therefore the sum after the summation sign has only one non-zero element, namely the one with index $\{\bullet\}$, which equals $-\widetilde{\Lambda}_{]\bullet,.[}$ (where $]\bullet,.[$ is the set of elements of $\Omega_{\leq n}(S_+)$ containing $\bullet$). Since $]\bullet,.[$ is homotopic (more precisely isomorphic) to $\Omega_{\leq n-1}(S)$, it follows that

$$\widetilde{\Lambda}_{\Omega_{\leq n}(S_+)} = \widetilde{\Lambda}_{\Omega_{\leq n}(S)} - \widetilde{\Lambda}_{\Omega_{\leq n-1}(S)}\,.$$

It is easy to see that this last expression is the desired one: let S be a G-set and define, for any tuple of positive integers $\alpha = (\alpha_0, \ldots, \alpha_i)$, the G-set $\mathcal{P}_\alpha(S)$ similarly as when α is a partition of an integer. Then the map sending the sequence $(S_0 \subset \cdots \subset S_i) \mapsto (S_0, S_1 \smallsetminus S_0, \ldots, S_i \smallsetminus S_{i-1})$ is an isomorphism of G-sets $\mathrm{Sd}_i(\Omega_{\leq n}(S)) \to \bigcup_{\substack{\alpha=(\alpha_0,\ldots,\alpha_i):\\ \sum\alpha_j\leq n\\ \alpha_j>0}} \mathcal{P}_\alpha(S)$, hence

$$[\mathrm{Sd}_i(\Omega_{\leq n}(S))] = \sum_{\substack{\alpha=(\alpha_0,\ldots,\alpha_i):\\ \sum\alpha_j\leq n\\ \alpha_j>0}} [\mathcal{P}_\alpha(S)].$$

We therefore have

$$[\mathrm{Sd}_i(\Omega_{\leq n}(S))] - [\mathrm{Sd}_i(\Omega_{\leq n-1}(S))] = \sum_{\substack{\alpha=(\alpha_0,\ldots,\alpha_i):\\ \sum\alpha_j= n\\ \alpha_j>0}} [\mathcal{P}_\alpha(S)] = \sum_{\substack{\mu\vdash n\\ \ell(\mu)=i+1}} \binom{\ell(\mu)}{\alpha(\mu)}[\mathcal{P}_\mu(S)],$$

and consequently

$$\begin{aligned}\widetilde{\Lambda}_{\Omega_{\leq n}(S)} - \widetilde{\Lambda}_{\Omega_{\leq n-1}(S)} &= \sum_{i\geq 0}(-1)^i \sum_{\substack{\mu\vdash n\\ \ell(\mu)=i+1}} \binom{\ell(\mu)}{\alpha(\mu)}[\mathcal{P}_\mu(S)]\\ &= -\sum_{\mu\vdash n}(-1)^{\ell(\mu)}\binom{\ell(\mu)}{\alpha(\mu)}[\mathcal{P}_\mu(S)].\end{aligned}$$ ☐

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