

THE POSET OF ELEMENTARY ABELIAN SUBGROUPS
OF RANK AT LEAST 2

by Serge BOUC and Jacques THÉVENAZ

to Guido Mislin

If p is a prime number, the poset of all nontrivial elementary abelian p -subgroups of a finite group plays an important role in both group theory and representation theory. It was studied by Quillen [Qu], who proved among many other things that it is homotopy equivalent to the poset of all nontrivial p -subgroups. In the case of a p -group P , one might believe that this poset has no interest since it is contractible (because the poset of all p -subgroups has a maximal element, namely P). However, it turns out that the subposet $\mathcal{A}(P)_{\geq 2}$ consisting of elementary abelian subgroups of rank at least 2 plays a key role in some recent work about endo-trivial and endo-permutation modules (see [CT], [BT], [Bo], [Th]). Actually it appeared much before in problems related to the classification of finite simple groups (see Sections 1 and 10 of [GLS]).

The purpose of this note is to show the following result and then state an open question related to $\mathcal{A}(P)_{\geq 2}$.

THEOREM 1.1. *Let P be a finite p -group. The poset $\mathcal{A}(P)_{\geq 2}$ of elementary abelian subgroups of P of rank at least 2 has the homotopy type of a wedge of spheres (of possibly different dimensions).*

If X is a poset, we write $\Delta(X)$ for the simplicial complex associated with X , whose set of n -simplices consists of all chains $x_0 < x_1 < \dots < x_n$ in X . If X is empty, we view $\Delta(X)$ as a sphere of dimension -1 . Its suspension consists of 2 points and is therefore a sphere of dimension 0.

Proof. For the reader's convenience, we split the proof in a series of steps.

1. The proof proceeds by induction on $|P|$ and starts with the case where P has rank 1 (i.e. P is either cyclic or generalized quaternion), so $\mathcal{A}(P)_{\geq 2}$ is empty, hence a sphere of dimension -1 . We assume from now on that the rank of P is at least 2, that is, P is neither cyclic nor quaternion.

2. If the centre $Z(P)$ is not cyclic, there is a central elementary abelian subgroup U of rank 2 and $\mathcal{A}(P)_{\geq 2}$ is contractible on U via the conical contraction $Q \leq QU \geq U$ (see 1.5 in [Qu]). Thus we obtain a point, that is, a wedge indexed by the empty set. Therefore we assume from now on that

the centre $Z(P)$ is cyclic and we write Z for the unique subgroup of order p in $Z(P)$.

3. Let $\mathcal{A}(P)_{>Z}$ be the poset of all elementary abelian subgroups strictly containing Z . There is a homotopy equivalence $\mathcal{A}(P)_{\geq 2} \longrightarrow \mathcal{A}(P)_{>Z}$ given by $Q \mapsto QZ$. We therefore work with $\mathcal{A}(P)_{>Z}$.

4. If $p = 2$ and P is either dihedral or semidihedral, there is no elementary abelian subgroup of rank ≥ 3 . So $\mathcal{A}(P)_{>Z}$ consists of isolated points and is therefore a wedge of spheres of dimension 0. Thus we can assume that P is not cyclic, quaternion, dihedral, or semidihedral, and it follows that P has an elementary abelian subgroup E_0 of rank 2 which is normal in P (see Lemma 10.11 in [GLS]). We set $M = C_P(E_0)$. Note that M is a normal subgroup of index p (because P/M embeds in $\text{Aut}(E_0) \cong \text{GL}_2(\mathbb{F}_p)$ whose order is divisible by p , but not by p^2).

5. Let $\{E_1, \dots, E_n\}$ be the subset of $\mathcal{A}(P)_{>Z}$ consisting of all elementary abelian subgroups of rank 2 not contained in M . In other words, in the lattice of subgroups of P containing Z , the subgroups E_1, \dots, E_n are the complements of M which are elementary abelian (i.e. not cyclic of order p^2). Any such subgroup can be written $E_i = Z \times S_i$ where S_i is a complement of M in P . Any $F \in \mathcal{A}(P)_{>Z}$ is either in M or contains some E_i , in which case $F \in \mathcal{A}(C_P(E_i))_{>Z}$. Thus if we define

$$\mathcal{A}_i = \mathcal{A}(M)_{>Z} \cup \mathcal{A}(P)_{\geq E_i},$$

we have

$$\mathcal{A}(P)_{>Z} = \bigcup_{i=1}^n \mathcal{A}_i \quad \text{and} \quad \Delta(\mathcal{A}(P)_{>Z}) = \bigcup_{i=1}^n \Delta(\mathcal{A}_i).$$

6. For any subset I of $\{1, \dots, n\}$ of cardinality ≥ 2 , the intersection $\bigcap_{i \in I} \mathcal{A}_i$ is contractible. Indeed if $Q \in \bigcap_{i \in I} \mathcal{A}_i$ but $Q \not\leq M$, then Q contains E_i and E_j where $i, j \in I$ (and $i \neq j$) and therefore Q has rank ≥ 3 . Thus $Q \cap M > Z$ and we have the contraction $Q \mapsto Q \cap M$ of $\bigcap_{i \in I} \mathcal{A}_i$ onto the poset $\mathcal{A}(M)_{>Z}$, which is contractible by step 2 (the centre of M contains E_0). It follows (see for instance Lemma 2.8 in [KT]) that $\Delta(\mathcal{A}(P)_{>Z})$ has the homotopy type

$$\Delta(\mathcal{A}(P)_{>Z}) \simeq \vee_{i=1}^n \Delta(\mathcal{A}_i).$$

7. Define a poset $\mathcal{B}_i = \{\bullet\} \cup \mathcal{A}(P)_{\geq E_i}$ where $\bullet < Q$ for every $Q \in \mathcal{A}(P)_{>E_i}$ but there is no order relation between \bullet and E_i . There is an order preserving map $f : \mathcal{A}_i \rightarrow \mathcal{B}_i$ which is the identity on $\mathcal{A}(P)_{\geq E_i}$ and such that $f(X) = \bullet$ for every $X \in \mathcal{A}(M)_{>Z}$. Since $\Delta(\mathcal{A}(M)_{>Z})$ is a contractible subcomplex of $\Delta(\mathcal{A}_i)$, the map f is a homotopy equivalence (see Lemma 2.2 in [BW]).

Alternatively, f is a homotopy equivalence by Quillen's fibre theorem (see Prop. 1.6 in [Qu] or Theorem 2.2 in [Wa]), because $f^{-1}((\mathcal{B}_i)_{\geq Y})$ is easily seen to be contractible for every $Y \in \mathcal{B}_i$ (if $Y = \bullet$, use the contraction of step 6).

8. Clearly both $\Delta(\{\bullet\} \cup \mathcal{A}(P)_{>E_i})$ and $\Delta(\mathcal{A}(P)_{\geq E_i})$ are cones on $\Delta(\mathcal{A}(P)_{>E_i})$ and their union is $\Delta(\mathcal{B}_i)$. Hence $\Delta(\mathcal{B}_i)$ is the suspension $\Sigma\Delta(\mathcal{A}(P)_{>E_i})$, and therefore $\Delta(\mathcal{A}_i)$ has the homotopy type of $\Sigma\Delta(\mathcal{A}(P)_{>E_i})$ (by the previous step).

9. There is a poset isomorphism

$$\mathcal{A}(P)_{>E_i} \longrightarrow \mathcal{A}(C_M(E_i))_{>Z}, \quad F \mapsto (F \cap M),$$

with inverse given by $G \mapsto GE_i$.

10. From steps 6, 7, 8, and 9, we obtain that

$$\Delta(\mathcal{A}(P)_{>Z}) \simeq \bigvee_{i=1}^n \Sigma\Delta(\mathcal{A}(P)_{>E_i}) \simeq \bigvee_{i=1}^n \Sigma\Delta(\mathcal{A}(C_M(E_i))_{>Z}).$$

Note that $\mathcal{A}(C_M(E_i))_{>Z}$ has the same homotopy type as $\mathcal{A}(C_M(E_i))_{\geq 2}$ by step 3. By induction, every $\Delta(\mathcal{A}(C_M(E_i))_{\geq 2})$ has the homotopy type of a wedge of spheres. Suspending and taking the wedge $\bigvee_{i=1}^n$, it follows that $\Delta(\mathcal{A}(P)_{>Z})$ has the homotopy type of a wedge of spheres.

REMARK 1.2. (a) We have actually used a well-known technique to split the homotopy type as a wedge of suspensions, according to the set of all complements of a given element in a lattice. This result first appears in [BW] (and in a weak form in [KT]).

(b) The theorem is almost the same as a theorem of Conlon [Co] and the proof follows also the same pattern (see Prop. 2.3 in [Co]). The only difference lies in the definition of the poset: Conlon considers the poset of abelian subgroups Q containing Z such that Q/Z is elementary abelian, so he allows for elements of order p^2 in Q , whereas we only deal with elementary abelian subgroups.

(c) According to the last sentence of Fumagalli's article [Fu], the theorem above also follows from the techniques developed in his paper. However, he only deals with odd primes and his methods are not as direct as the ones developed here.

Computer computations with 3-groups of order $\leq 3^6$ show that all spheres which appear in these examples have the same dimension. However, for 2-groups of order $\leq 2^9$, there are examples (see below) where one gets spheres

of 2 different dimensions (actually 2 consecutive dimensions). This raises the following question :

QUESTION 1.3. *Which dimensions of spheres occur in the above result ? Are they all equal if p is odd ? Does one get only 2 consecutive dimensions when $p = 2$?*

The evidence we have does not seem to be sufficient for stating a conjecture, but there is clearly an open problem which deserves some attention.

EXAMPLE 1.4. Here is an example with $p = 2$. The group P has order 64 and is number 258 among groups of order 64 in the computer package GAP. It has a centre of order 2 and its Frattini subgroup, equal to its derived group, is cyclic of order 4. It is defined below by generators and relations. It turns out that $\mathcal{A}(P)_{\geq 2}$ has the homotopy type of a wedge of two spheres of dimension 0 and four spheres of dimension 1.

Generators : $f_1, f_2, f_3, f_4, f_5, f_6$.

Relations : $f_1^2 = f_6$, $f_2^2 = 1$, $f_3^2 = 1$, $f_4^2 = 1$, $f_5^2 = f_6$, $f_6^2 = 1$,
 $[f_1, f_2] = f_5 f_6$, $[f_1, f_3] = 1$, $[f_1, f_4] = 1$, $[f_1, f_5] = f_6$, $[f_1, f_6] = 1$,
 $[f_2, f_3] = 1$, $[f_2, f_4] = 1$, $[f_2, f_5] = f_6$, $[f_2, f_6] = 1$, $[f_3, f_4] = f_6$,
 $[f_3, f_5] = 1$, $[f_3, f_6] = 1$, $[f_4, f_5] = 1$, $[f_4, f_6] = 1$, $[f_5, f_6] = 1$.

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Serge Bouc, LAMFA - CNRS, Université de Picardie -
Jules Verne, 33, rue St Leu, F-80039 Amiens, France
e-mail: serge.bouc@u-picardie.fr

Jacques Thévenaz, Institut de géométrie, algèbre et
topologie,
Bâtiment BCH, École Polytechnique Fédérale de Lausanne,
CH-1015 Lausanne, Switzerland
e-mail: Jacques.Thevenaz@epfl.ch