# The representation theory of finite sets and correspondences

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Abstract. We investigate correspondence functors, namely the functors from the category of finite sets and correspondences to the category of $k$-modules, where $k$ is a commutative ring. They have various specific properties which do not hold for other types of functors. In particular, if $k$ is a field and if $F$ is a correspondence functor, then $F$ is finitely generated if and only if the dimension of $F(X)$ grows exponentially in terms of the cardinality of the finite set $X$. In such a case, $F$ has finite length. Also, if $k$ is noetherian, then any subfunctor of a finitely generated functor is finitely generated. When $k$ is a field, we give a description of all the simple functors and we determine the dimension of their evaluations at any finite set.

A main tool is the construction of a functor associated to any finite lattice $T$. We prove for instance that this functor is projective if and only if the lattice $T$ is distributive. Moreover, it has quotients which play a crucial role in the analysis of simple functors. The special case of total orders yields some more specific results. Several other properties are also discussed, such as projectivity, duality, and symmetry. In an appendix, all the lattices associated to a given poset are described.

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# Contents

# 1. Introduction

Representations of categories have been used by many authors in different contexts. The purpose of the present memoir is to develop the theory in the case of the category of finite sets, when morphisms are all correspondences between finite sets. For representing the category of finite sets, there are several possible choices. Pirashvili [**Pi**] treats the case of pointed sets and maps, while Church, Ellenberg and Farb [**CEF**] consider the case where the morphisms are all injective maps. Putman and Sam [**PS**] use all $k$-linear splittable injections between finite-rank free $k$-modules (where $k$ is a commutative ring). Here, we move away from such choices by using all correspondences as morphisms. The cited papers are concerned with applications to cohomological stability, while we develop our theory without any specific application in mind. The main motivation is provided by the fact that finite sets are basic objects in mathematics. Moreover, the theory turns out to have many quite surprising results, which justify the development presented here.

Let $\mathcal{C}$ be the category of finite sets and correspondences. A correspondence functor over a commutative ring $k$ is a functor from $\mathcal{C}$ to the category $k$-Mod of all $k$-modules. As much as possible, we develop the theory for an arbitrary commutative ring $k$. Let us start however with the case when $k$ is a field. If $F$ is a correspondence functor over a field $k$, then we prove that $F$ is finitely generated if and only if the dimension of $F(X)$ grows exponentially in terms of the cardinality of the finite set $X$ (Theorem 8.4). In such a case, we also obtain that $F$ has finite length (Theorem 8.6). This result was obtained independently by Gitlin [**Gi**], using a criterion proved by Wiltshire-Gordon [**WG**]. Moreover, for finitely generated correspondence functors, we show that the Krull-Remak-Schmidt theorem holds (Proposition 6.6) and that projective functors coincide with injective functors (Theorem 9.6).

Suppose that $k$ is a field. By well-known results about representations of categories, simple correspondence functors can be easily classified. In our case, they are parametrized by triples $(E,R,V)$, where $E$ is a finite set, $R$ is a partial order relation on $E$, and $V$ is a simple $k\,\mathrm{Aut}(E,R)$-module (Theorem 3.12). This is the first indication of the importance of posets in our work. However, if $S_{E,R,V}$ is a simple functor parametrized by $(E,R,V)$, then it is quite hard to describe the evaluation $S_{E,R,V}(X)$ at a finite set $X$. We achieve this in Section 17 by giving a closed formula for its dimension (Theorem 17.19).

Simple functors have precursors depending only on a poset $(E,R)$. There is a correspondence functor $\mathbb{S}_{E,R}$, which we call a *fundamental correspondence functor* (Definition 4.7), from which we recover in a straightforward fashion every simple correspondence functor $S_{E,R,V}$ by means of a suitable tensor product with $V$. Actually, $\mathbb{S}_{E,R}$ can be defined over any commutative ring $k$ and a large part of our work is concerned with proving properties of the fundamental correspondence functors $\mathbb{S}_{E,R}$. In particular, we prove in Section 17 that the evaluation $\mathbb{S}_{E,R}(X)$ at a finite set $X$ is a free $k$-module, by describing a $k$-basis of $\mathbb{S}_{E,R}(X)$ and giving a closed formula for its rank (Corollary 17.17). This is the key which is used, when $k$ is a field, to obtain the formula for the dimension of $S_{E,R,V}(X)$, for any simple $k\,\mathrm{Aut}(E,R)$-module $V$.

A natural question when dealing with a commutative ring $k$ is to obtain specific results when $k$ is noetherian. We follow this track in Section 10 and show for instance that any subfunctor of a finitely generated correspondence functor is again finitely generated (Corollary 10.5). Also, we obtain stabilization results for Hom and Ext between correspondence functors evaluated at large enough finite sets (Theorem 10.10).

In Section 19, we also introduce a natural tensor structure on the category of correspondence functors, and show that there is an adjoint internal hom construction.

A main tool in this work is the construction of a correspondence functor $F_T$ associated to a finite lattice $T$ (Section 11). This is the second indication of the importance of posets and lattices in our work. Part 2 of this memoir describes the interplay between lattices and functors. For instance, one of our first results asserts that the functor $F_T$ is projective if and only if the lattice $T$ is distributive (Theorem 11.11).

The fundamental functors can be analyzed by using lattices: if $(E,R)$ is the subposet of irreducible elements in a lattice $T$, then the functor $F_T$ has a fundamental functor as a quotient, which turns out to be $\mathbb{S}_{E,R^{op}}$ where $R^{op}$ denotes the opposite order relation (Theorem 13.1). We show that there is a duality between $F_T$ and $F_{T^{op}}$ over any commutative ring $k$ (Theorem 14.9) and the fundamental functor $\mathbb{S}_{E,R}$ also appears as a subfunctor of $F_{T^{op}}$ (Theorem 14.16). The special case of a totally ordered lattice $T$ yields some specific results (Section 15), connected to the fact that $F_T$ and its associated fundamental functor are projective functors.

A surprising byproduct of our work is concerned with the theory of finite lattices. In any lattice $T$, we define some operations $r^\infty$ and $s^\infty$, as well as some special elements which we call *bulbs* (Section 16). Also, a canonical forest (disjoint union of trees) can be constructed in $T$, from which some idempotents can be defined in $F_T(T)$ (Theorem 17.7 and Theorem 17.9). All this plays a crucial role in the description of a $k$-basis of $\mathbb{S}_{E,R}(X)$ in Section 17.

Some specific results about lattices have been postponed to two appendices (Section 21 and Section 22) because they are not used directly in our analysis of correspondence functors.

There are a few basic results in Section 2 which have been imported from elsewhere, but this memoir is almost self-contained. The main exception is the algebra $\mathcal{E}_E$ of essential relations on a finite set $E$. This algebra has been analyzed in [**BT**] and all its simple modules have been classified there. This uses the *fundamental module* $\mathcal{P}_E f_R$ and there is an explicit description of the action of relations on $\mathcal{P}_E f_R$. All the necessary background on this algebra $\mathcal{E}_E$ of essential relations is recalled in Section 3. Note that the fundamental module $\mathcal{P}_E f_R$ is a main ingredient for the definition of the fundamental functor $\mathbb{S}_{E,R}$.

# Part 1

# CORRESPONDENCE FUNCTORS

## 2. The representation theory of categories

Before introducing the category $\mathcal{C}$ of finite sets and correspondences, we first recall some standard facts from the representation theory of categories. Let $\mathcal{D}$ be a category and let $X$ and $Y$ be two objects of $\mathcal{D}$. We adopt a slightly unusual notation by writing $\mathcal{D}(Y,X)$ for the set of all morphisms from $X$ to $Y$. We reverse the order of $X$ and $Y$ in view of having later a left action of morphisms behaving nicely under composition.

We assume that $\mathcal{D}$ is small (or more generally that a skeleton of $\mathcal{D}$ is small). This allows us to talk about the *set* of natural transformations between two functors starting from $\mathcal{D}$.

Throughout this paper, $k$ denotes a commutative ring. It will sometimes be noetherian and sometimes a field, but we shall always emphasize when we make additional assumptions.

**2.1. Definition.** *The* $k$-linearization *of a category $\mathcal{D}$, where $k$ is any commutative ring, is defined as follows :*

- *The objects of $k\mathcal{D}$ are the objects of $\mathcal{D}$.*
- *For any two objects $X$ and $Y$, the set of morphisms from $X$ to $Y$ is the free $k$-module $k\mathcal{D}(Y,X)$ with basis $\mathcal{D}(Y,X)$.*
- *The composition of morphisms in $k\mathcal{D}$ is the $k$-bilinear extension*

$$k\mathcal{D}(Z,Y) \times k\mathcal{D}(Y,X) \longrightarrow k\mathcal{D}(Z,X)$$

*of the composition in $\mathcal{D}$.*

**2.2. Definition.** *Let $\mathcal{D}$ be a category and $k$ a commutative ring. A* $k$-representation *of the category $\mathcal{D}$ is a $k$-linear functor from $k\mathcal{D}$ to the category* $k$-Mod *of $k$-modules.*

We could have defined a $k$-representation of $\mathcal{D}$ as a functor from $\mathcal{D}$ to $k$-Mod, but it is convenient to linearize first the category $\mathcal{D}$ (just as for group representations, where one can first introduce the group algebra).

If $F : k\mathcal{D} \to k$-Mod is a $k$-representation of $\mathcal{D}$ and if $X$ is an object of $\mathcal{D}$, then $F(X)$ will be called the *evaluation* of $F$ at $X$. Morphisms in $k\mathcal{D}$ act on the left on the evaluations of $F$ by setting, for every $m \in F(X)$ and for every morphism $\alpha \in k\mathcal{D}(Y,X)$,

$$\alpha \cdot m \ := F(\alpha)(m) \in F(Y) \ .$$

We often use a dot for this action of morphisms on evaluation of functors. With our choice of notation, if $\beta \in k\mathcal{D}(Z,Y)$, then

$$(\beta\alpha) \cdot m = \beta \cdot (\alpha \cdot m) \ .$$

The category $\mathcal{F}(k\mathcal{D}, k\text{-Mod})$ of all $k$-representations of $\mathcal{D}$ is an abelian category. (We need to restrict to a small skeleton of $\mathcal{D}$ in order to have *sets* of natural transformations, which are morphisms in $\mathcal{F}(k\mathcal{D}, k\text{-Mod})$, but we will avoid this technical discussion). A sequence of functors

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

is exact if and only if, for every object $X$, the evaluation sequence

$$0 \longrightarrow F_1(X) \longrightarrow F_2(X) \longrightarrow F_3(X) \longrightarrow 0$$

is exact. Also, a $k$-representation of $\mathcal{D}$ is called *simple* if it is nonzero and has no proper nonzero subfunctor.

For any object $X$ of $\mathcal{D}$, consider the representable functor $k\mathcal{D}(-,X)$ (which is a projective functor). Its evaluation at an object $Y$ is the $k$-module $k\mathcal{D}(Y,X)$, which has a natural structure of a $(k\mathcal{D}(Y,Y),k\mathcal{D}(X,X))$-bimodule by composition.

For any $k\mathcal{D}(X,X)$-module $W$, we define, as in [**Bo1**], the functor

$$L_{X,W} = k\mathcal{D}(-,X) \otimes_{k\mathcal{D}(X,X)} W \ ,$$

which satisfies the following adjunction property.

**2.3. Lemma.** *Let $\mathcal{F} = \mathcal{F}(k\mathcal{D}, k\text{-Mod})$ be the category of all $k$-representations of $\mathcal{D}$ and let $X$ be an object of $\mathcal{D}$.*

*(a) The functor*

$$k\mathcal{D}(X,X)\!-\!\operatorname{Mod} \longrightarrow \mathcal{F} \ , \quad W \mapsto L_{X,W}$$

*is left adjoint of the evaluation functor*

$$\mathcal{F} \longrightarrow k\mathcal{D}(X,X)\!-\!\operatorname{Mod} \ , \quad F \mapsto F(X) \ .$$

*In other words, for any $k$-representation $F : k\mathcal{D} \to k\text{-Mod}$ and any $k\mathcal{D}(X,X)$-module $W$, there is a natural isomorphism*

$$\operatorname{Hom}_{\mathcal{F}}(L_{X,W}, F) \cong \operatorname{Hom}_{k\mathcal{D}(X,X)}(W, F(X)) \ .$$

*Moreover $L_{X,W}(X) \cong W$ as $k\mathcal{D}(X,X)$-modules.*

*(b) This functor $k\mathcal{D}(X,X)\!-\!\operatorname{Mod} \longrightarrow \mathcal{F}$ is right exact. It maps projective modules to projective functors, and indecomposable projective modules to indecomposable projective functors.*

**Proof :** Part (a) is straightforward and is proved in Section 2 of [**Bo1**]. Part (b) follows because this functor is left adjoint of an exact functor. ∎

Our next result is a slight extension of the first lemma of [**Bo1**].

**2.4. Lemma.** *Let $X$ be an object of $\mathcal{D}$ and let $W$ be a $k\mathcal{D}(X,X)$-module. For any object $Y$ of $\mathcal{D}$, let*

$$J_{X,W}(Y) = \Big\{ \sum_i \phi_i \otimes w_i \in L_{X,W}(Y) \mid \forall \psi \in k\mathcal{D}(X,Y), \sum_i (\psi\phi_i)\cdot w_i = 0 \Big\} \ .$$

*(a) $J_{X,W}$ is the unique subfunctor of $L_{X,W}$ which is maximal with respect to the condition that it vanishes at $X$.*

*(b) If $W$ is a simple module, then $J_{X,W}$ is the unique maximal subfunctor of $L_{X,W}$ and $L_{X,W}/J_{X,W}$ is a simple functor.*

**Proof :** The proof is sketched in Lemma 2.3 of [**BST**] in the special case of biset functors for finite groups, but it extends without change to representations of an arbitrary category $\mathcal{D}$. ∎

**2.5. Notation.** *Let $X$ be an object of $\mathcal{D}$ and let $W$ be a $k\mathcal{D}(X,X)$-module. We define*

$$S_{X,W} = L_{X,W}/J_{X,W} \ .$$

*If $W$ is a simple $k\mathcal{D}(X,X)$-module, then $S_{X,W}$ is a simple functor.*

We emphasize that $L_{X,W}$ and $S_{X,W}$ are defined for any $k\mathcal{D}(X,X)$-module $W$ and any commutative ring $k$. Note that we always have $J_{X,W}(X) = \{0\}$ because if $a = \sum_i \phi_i \otimes w_i \in J_{X,W}(X)$, then $a = \mathrm{id}_X \otimes (\sum_i \phi_i \cdot w_i) = 0$.

Therefore, we have isomorphisms of $k\mathcal{D}(X,X)$-modules

$$L_{X,W}(X) \cong S_{X,W}(X) \cong W \ .$$

**2.6. Proposition.** *Let $S$ be a simple $k$-representation of $\mathcal{D}$ and let $Y$ be an object of $\mathcal{D}$ such that $S(Y) \neq 0$.*

(a) *$S$ is generated by $S(Y)$, that is, $S(X) = k\mathcal{D}(X,Y)S(Y)$ for all objects $X$. More precisely, if $0 \neq u \in S(Y)$, then $S(X) = k\mathcal{D}(X,Y) \cdot u$.*
(b) *$S(Y)$ is a simple $k\mathcal{D}(Y,Y)$-module.*
(c) *$S \cong S_{Y,S(Y)}$.*

**Proof :** (a) Given $0 \neq u \in S(Y)$, let $S'(X) = k\mathcal{D}(X,Y) \cdot u$ for all objects $X$. This clearly defines a nonzero subfunctor $S'$ of $S$, so $S' = S$ by simplicity of $S$.

(b) This follows from (a).

(c) By the adjunction of Lemma 2.3, the identity id : $S(Y) \to S(Y)$ corresponds to a non-zero morphism $\theta : L_{Y,S(Y)} \to S$. Since $S$ is simple, $\theta$ must be surjective. But $S_{Y,S(Y)}$ is the unique simple quotient of $L_{Y,S(Y)}$, by Lemma 2.4 and Notation 2.5, so $S \cong S_{Y,S(Y)}$. □

It should be noted that $S$ has many realizations $S \cong S_{Y,W}$ as above, where $W = S(Y) \neq 0$. However, if there is a notion of unique minimal object, then one can parametrize simple functors $S$ by setting $S \cong S_{Y,W}$, where $Y$ is the unique minimal object such that $S(Y) \neq 0$ (see Theorem 3.7 for the case of correspondence functors).

Our next proposition is Proposition 3.5 in [**BST**] in the case of biset functors, but it holds more generally and we just recall the proof of [**BST**].

**2.7. Proposition.** *Let $S$ be a simple $k$-representation of $\mathcal{D}$ and let $Y$ be an object of $\mathcal{D}$ such that $S(Y) \neq 0$. Let $F$ be any $k$-representation of $\mathcal{D}$. Then the following are equivalent:*

(a) *$S$ is isomorphic to a subquotient of $F$.*
(b) *The simple $k\mathcal{D}(Y,Y)$-module $S(Y)$ is isomorphic to a subquotient of the $k\mathcal{D}(Y,Y)$-module $F(Y)$.*

**Proof :** It is clear that (a) implies (b). Suppose that (b) holds and let $W_1$, $W_2$ be submodules of $F(Y)$ such that $W_2 \subset W_1$ and $W_1/W_2 \cong S(Y)$. For $i \in \{1,2\}$, let $F_i$ be the subfunctor of $F$ generated by $W_i$. Explicitly, for any object $X$ of $\mathcal{D}$, $F_i(X) = k\mathcal{D}(X,Y) \cdot W_i \subseteq F(X)$. Then $F_i(Y) = W_i$ and $(F_1/F_2)(Y) = W_1/W_2 \cong S(Y)$. The isomorphism $S(Y) \to (F_1/F_2)(Y)$ induces, by the adjunction of Lemma 2.3, a nonzero morphism $\theta : L_{Y,S(Y)} \to F_1/F_2$. Since $S(Y)$ is simple, $L_{Y,S(Y)}$ has a unique maximal subfunctor $J_{Y,S(Y)}$, by Lemma 2.4, and $L_{Y,S(Y)}/J_{Y,S(Y)} \cong S_{Y,S(Y)} \cong S$, by Proposition 2.6. Let $F_1' = \theta(L_{Y,S(Y)})$ and $F_2' = \theta(J_{Y,S(Y)})$. Since $\theta \neq 0$, we obtain

$$F_1'/F_2' \cong L_{Y,S(Y)}/J_{Y,S(Y)} \cong S_{Y,S(Y)} \cong S\,,$$

showing that $S$ is isomorphic to a subquotient of $F$. □

## 3. Correspondence functors

Leaving the general case, we now prepare the ground for the category $\mathcal{C}$ we are going to work with.

**3.1. Definition.** *Let $X$ and $Y$ be sets.*

(a) *A* correspondence *from $X$ to $Y$ is a subset of the cartesian product $Y \times X$. Note that we have reversed the order of $X$ and $Y$ for the reasons mentioned at the beginning of Section 2.*

(b) *A correspondence is often called a relation but we use this terminology only when $X = Y$, in which case we say that a subset of $X \times X$ is a* relation on $X$.

(c) *If $\sigma$ is a permutation of $X$, then there is a corresponding relation on $X$ which we write*

$$\Delta_\sigma = \{(\sigma(x), x) \in X \times X \mid x \in X\} .$$

*In particular, when $\sigma = \mathrm{id}$, we also write*

$$\Delta_{\mathrm{id}} = \Delta_X = \{(x, x) \in X \times X \mid x \in X\} .$$

**3.2. Definition.** *Let $\mathcal{C}$ denote the following category :*

- *The objects of $\mathcal{C}$ are the finite sets.*
- *For any two finite sets $X$ and $Y$, the set $\mathcal{C}(Y, X)$ is the set of all correspondences from $X$ to $Y$.*
- *The composition of correspondences is as follows. If $R \subseteq Z \times Y$ and $S \subseteq Y \times X$, then $RS$ is defined by*

$$RS = \{\, (z, x) \in Z \times X \ \mid\ \exists\, y \in Y \ \text{ such that } \ (z, y) \in R \ \text{ and } \ (y, x) \in S \,\} .$$

The identity morphism $\mathrm{id}_X$ is the diagonal subset $\Delta_X \subseteq X \times X$ (in other words the equality relation on $X$).

**3.3. Definition.** *Let $k\mathcal{C}$ be the linearization of the category $\mathcal{C}$, where $k$ is any commutative ring.*

(a) *A* correspondence functor (over $k$) *is a $k$-representation of the category $\mathcal{C}$, that is, a $k$-linear functor from $k\mathcal{C}$ to the category $k$-Mod of $k$-modules.*

(b) *We let $\mathcal{F}_k = \mathcal{F}(k\mathcal{C}, k\text{-Mod})$ be the category of all such correspondence functors (an abelian category).*

In part (b), we need to restrict to a small skeleton of $\mathcal{C}$ in order to have *sets* of natural transformations, which are morphisms in $\mathcal{F}_k$, but we avoid this technical discussion. It is clear that $\mathcal{C}$ has a small skeleton, for instance by taking the full subcategory having one object for each cardinality.

**3.4. Proposition.** *Let $X$ and $Y$ be finite sets. If $R \subseteq Y \times X$, let $R^{op}$ denote the* opposite correspondence*, defined by*

$$R^{op} = \{(x, y) \in X \times Y \mid (y, x) \in R\} \subseteq X \times Y .$$

*Then the assignment $R \mapsto R^{op}$ induces an isomorphism from $\mathcal{C}$ to the opposite category $\mathcal{C}^{op}$, which extends to an isomorphism from $k\mathcal{C}$ to $k\mathcal{C}^{op}$.*

**Proof :** One checks easily that if $X$, $Y$, $Z$ are finite sets, if $R \subseteq Y \times X$ and $S \subseteq Z \times Y$, then $(SR)^{op} = R^{op} S^{op}$. □

We use opposite correspondences to define dual functors. The notion will be used in Section 9 and Section 14.

**3.5. Definition.** *Let $F$ be a correspondence functor over $k$. The* dual $F^\natural$ *of $F$ is the correspondence functor defined on a finite set $X$ by*

$$F^\natural(X) = \mathrm{Hom}_k\big(F(X), k\big) \ .$$

*If $Y$ is a finite set and $R \subseteq Y \times X$, then the map $F^\natural(R) : F^\natural(X) \to F^\natural(Y)$ is defined by*

$$\forall \alpha \in F^\natural(X), \ \ F^\natural(R)(\alpha) = \alpha \circ F(R^{op}) \ .$$

In order to study simple modules or simple functors, it suffices to work over a field $k$, by standard commutative algebra. If we assume that $k$ is a field, then the evaluation at a finite set $X$ of a representable functor or of a simple functor is always a finite-dimensional $k$-vector space.

**3.6. Definition.** *A* minimal set *for a correspondence functor $F$ is a finite set $X$ of minimal cardinality such that $F(X) \neq 0$.*

Clearly, for a nonzero functor, such a minimal set always exists and is unique up to bijection.

Our next task is to describe the parametrization of simple correspondence functors, assuming that $k$ is a field. This uses the finite-dimensional algebra

$$\mathcal{R}_X := k\mathcal{C}(X,X)$$

of all relations on $X$, which was studied in [**BT**]. A relation $R$ on $X$ is called *essential* if it does not factor through a set of cardinality strictly smaller than $|X|$. The $k$-submodule generated by set of inessential relations is a two-sided ideal

$$I_X = \sum_{|Y|<|X|} k\mathcal{C}(X,Y)k\mathcal{C}(Y,X)$$

and the quotient

$$\mathcal{E}_X := k\mathcal{C}(X,X)/I_X$$

is called the *essential algebra*. A large part of its structure has been elucidated in [**BT**].

The following parametrization theorem is similar to Proposition 2 in [**Bo1**] or Theorem 4.3.10 in [**Bo2**]. The context here is different, but the proof is essentially the same.

**3.7. Theorem.** *Assume that $k$ is a field.*

- *(a) Let $S$ be a simple correspondence functor, let $E$ be a minimal set for $S$, and let $W = S(E)$. Then $W$ is a simple module for the essential algebra $\mathcal{E}_E$ (with $I_E$ acting by zero) and $S \cong S_{E,W}$.*
- *(b) Let $E$ be a finite set and let $W$ be a simple module for the essential algebra $\mathcal{E}_E$, viewed as a module for the algebra $\mathcal{R}_E$ by making $I_E$ act by zero on $W$. Then $E$ is a minimal set for $S_{E,W}$. Moreover, $S_{E,W}(E) \cong W$ (as $\mathcal{E}_E$-modules).*
- *(c) The set of isomorphism classes of simple correspondence functors is parametrized by the set of isomorphism classes of pairs $(E,W)$ where $E$ is a finite set and $W$ is a simple $\mathcal{E}_E$-module.*

**Proof :** (a) Since $S(Y) = \{0\}$ if $|Y| < |E|$, we have

$$I_E \cdot S(E) = \sum_{|Y|<|E|} k\mathcal{C}(E,Y)k\mathcal{C}(Y,E) \cdot S(E) \subseteq \sum_{|Y|<|E|} k\mathcal{C}(E,Y) \cdot S(Y) = \{0\} \ ,$$

so $S(E)$ is a module for the essential algebra $\mathcal{E}_E$. Now the identity of $S(E)$ corresponds by adjunction to a nonzero homomorphism $L_{E,W} \to S$, where $W = S(E)$

(see Lemma 2.3). This homomorphism is surjective since $S$ is simple. But $L_{E,W}$ has a unique simple quotient, namely $S_{E,W}$, hence $S \cong S_{E,W}$.

(b) Suppose that $S_{E,W}(Y) \neq \{0\}$. Then $L_{E,W}(Y) \neq J_{E,W}(Y)$, so there exists a correspondence $\phi \in \mathcal{C}(Y,E)$ and $v \in W$ such that $\phi \otimes v \in L_{E,W}(Y) - J_{E,W}(Y)$. By definition of $J_{E,W}$, this means that there exists a correspondence $\psi \in \mathcal{C}(E,Y)$ such that $\psi\phi \cdot v \neq 0$. Since $W$ is a module for the essential algebra $\mathcal{E}_E = k\mathcal{C}(E,E)/I_E$, we have $\psi\phi \notin I_E$. But $\psi\phi$ factorizes through $Y$, so we must have $|Y| \geq |E|$. Thus $E$ is a minimal set for $S_{E,W}$. The isomorphism $S_{E,W}(E) \cong W$ is a general fact mentioned before.

(c) This follows from (a) and (b). □

Theorem 3.7 reduces the classification of simple correspondence functors to the question of classifying all simple modules for the essential algebra $\mathcal{E}_E$. Fortunately, this has been achieved in [**BT**]. The simple $\mathcal{E}_E$-modules are actually modules for a quotient $\mathcal{P}_E = \mathcal{E}_E/N$ where $N$ is a nilpotent ideal defined in [**BT**]. We call $\mathcal{P}_E$ the *algebra of permuted orders*, because it has a $k$-basis consisting of all relations on $E$ of the form $\Delta_\sigma R$, where $\sigma$ runs through the symmetric group $\Sigma_E$ of all permutations of $E$, and $R$ is an order on $E$. By an order, we always mean a partial order relation. We let $\mathcal{O}$ be the set of all orders on $E$ and $\mathrm{Aut}(E,R)$ the stabilizer of the order $R$ in the symmetric group $\Sigma_E$. For the description of simple $\mathcal{E}_E$-modules, we need the following new basis of $\mathcal{P}_E$ (see Theorem 6.2 in [**BT**] for details).

**3.8. Lemma.**

(a) *There is a set $\{f_R \mid R \in \mathcal{O}\}$ of orthogonal idempotents whose sum is 1, such that $\mathcal{P}_E$ has a $k$-basis consisting of all elements of the form $\Delta_\sigma f_R$, where $\sigma \in \Sigma_E$ and $R \in \mathcal{O}$.*

(b) *For any $\sigma \in \Sigma_E$, we have ${}^\sigma f_R = f_{{}^\sigma R}$, where ${}^\sigma x = \Delta_\sigma x \Delta_{\sigma^{-1}}$ for any $x \in \mathcal{P}_E$. In particular, $\Delta_\sigma f_R \Delta_{\sigma^{-1}} = f_R$ if $\sigma \in \mathrm{Aut}(E,R)$.*

(c) *For any order $Q$ on $E$, we have :*

$$Qf_R \neq 0 \iff Qf_R = f_R \iff Q \subseteq R\,.$$

For the description of simple $\mathcal{E}_E$-modules and then simple correspondence functors, we will make use of the left $\mathcal{E}_E$-module $\mathcal{P}_E f_R$. This module is actually defined without assuming that $k$ is a field.

**3.9. Definition.** *Let $E$ be a finite set and $R$ an order on $E$. We call $\mathcal{P}_E f_R$ the* fundamental module *for the algebra $\mathcal{E}_E$, associated with the poset $(E,R)$.*

We now describe its structure.

**3.10. Proposition.** *Let $E$ be a finite set and $R$ an order on $E$.*

(a) *The fundamental module $\mathcal{P}_E f_R$ is a left module for the algebra $\mathcal{P}_E$, hence also a left module for the essential algebra $\mathcal{E}_E$ and for the algebra of relations $\mathcal{R}_E$.*

(b) *$\mathcal{P}_E f_R$ is a $(\mathcal{P}_E, k\,\mathrm{Aut}(E,R))$-bimodule and the right action of $k\,\mathrm{Aut}(E,R)$ is free.*

(c) *$\mathcal{P}_E f_R$ is a free $k$-module with a $k$-basis consisting of the elements $\Delta_\sigma f_R$, where $\sigma$ runs through the group $\Sigma_E$ of all permutations of $E$.*

(d) *The action of the algebra of relations $\mathcal{R}_E$ on the module $\mathcal{P}_E f_R$ is given as follows. For any relation $Q \in \mathcal{C}(E,E)$,*

$$Q \cdot \Delta_\sigma f_R = \begin{cases} \Delta_{\tau\sigma} f_R & \text{if } \exists \tau \in \Sigma_E \text{ such that } \Delta_E \subseteq \Delta_{\tau^{-1}} Q \subseteq {}^\sigma R, \\ 0 & \text{otherwise}\,, \end{cases}$$

*where $\Delta_E$ is the diagonal of $E \times E$, and ${}^\sigma R = \big\{\big(\sigma(e), \sigma(f)\big) \mid (e,f) \in R\big\}$ (recall that $\tau$ is unique in the first case).*

**Proof :** See Corollary 7.3 and Proposition 8.5 in [**BT**]. □

The description of all simple $\mathcal{E}_E$-modules is as follows (see Theorem 8.1 in [**BT**] for details).

**3.11. Theorem.** *Assume that $k$ is a field.*

(a) *Let $R$ be an order on $E$ and let $\mathcal{P}_E f_R$ be the corresponding fundamental module. If $V$ is a simple $k\,\mathrm{Aut}(E,R)$-module, then*
$$T_{R,V} := \mathcal{P}_E f_R \otimes_{k\,\mathrm{Aut}(E,R)} V$$
*is a simple $\mathcal{P}_E$-module (hence also a simple $\mathcal{E}_E$-module).*

(b) *Every simple $\mathcal{E}_E$-module is isomorphic to a module $T_{R,V}$ as in (a).*

(c) *For any permutation $\sigma \in \Sigma_E$, we have $T_{{}^\sigma R,\,{}^\sigma V} \cong T_{R,V}$, where ${}^\sigma R = \Delta_\sigma R \Delta_{\sigma^{-1}}$ is the conjugate order and ${}^\sigma V$ is the conjugate module.*

(d) *The set of isomorphism classes of simple $\mathcal{E}_E$-modules is parametrized by the set of conjugacy classes of pairs $(R,V)$ where $R$ is an order on $E$ and $V$ is a simple $k\,\mathrm{Aut}(E,R)$-module.*

Putting together Theorem 3.7 and Theorem 3.11, we finally obtain the following parametrization, which is essential for our purposes.

**3.12. Theorem.** *Assume that $k$ is a field. The set of isomorphism classes of simple correspondence functors is parametrized by the set of isomorphism classes of triples $(E,R,V)$ where $E$ is a finite set, $R$ is an order on $E$, and $V$ is a simple $k\,\mathrm{Aut}(E,R)$-module.*

**3.13. Notation.** *Let $E$ be a finite set and $R$ an order on $E$.*

(a) *If $V$ is a simple $k\,\mathrm{Aut}(E,R)$-module, we denote by $S_{E,R,V}$ the simple correspondence functor parametrized by the triple $(E,R,V)$.*

(b) *More generally, for any commutative ring $k$ and any $k\,\mathrm{Aut}(E,R)$-module $V$, we define*
$$S_{E,R,V} = \big(L_{E,\mathcal{P}_E f_R}/J_{E,\mathcal{P}_E f_R}\big) \otimes_{k\,\mathrm{Aut}(E,R)} V \;,$$
*by which we mean that $S_{E,R,V}(X) = \big(L_{E,\mathcal{P}_E f_R}/J_{E,\mathcal{P}_E f_R}\big)(X) \otimes_{k\,\mathrm{Aut}(E,R)} V$ for any finite set $X$.*

We end this section with a a basic result concerning the correspondence functors $S_{E,R,V}$, where $k$ is any commutative ring and $V$ is any $k\,\mathrm{Aut}(E,R)$-module.

**3.14. Lemma.** *Let $E$ be a finite set, let $R$ be an order on $E$, and let $V$ be any $k\,\mathrm{Aut}(E,R)$-module.*

(a) *$E$ is a minimal set for $S_{E,R,V}$.*

(b) *$S_{E,R,V}(E) \cong \mathcal{P}_E f_R \otimes_{k\,\mathrm{Aut}(E,R)} V$.*

**Proof :** Both $\mathcal{P}_E f_R$ and $\mathcal{P}_E f_R \otimes_{k\,\mathrm{Aut}(E,R)} V$ are left modules for the essential algebra $\mathcal{E}_E$. Therefore, the argument given in part (b) of Theorem 3.7 shows again that $E$ is a minimal set for $S_{E,R,V}$. Moreover, since $J_{E,\mathcal{P}_E f_R}$ vanishes on evaluation at $E$, we have
$$S_{E,R,V}(E) = L_{E,\mathcal{P}_E f_R}(E) \otimes_{k\,\mathrm{Aut}(E,R)} V = \mathcal{P}_E f_R \otimes_{k\,\mathrm{Aut}(E,R)} V \;,$$
as required. □

When $k$ is a field and $V$ is simple, we obtain $S_{E,R,V}(E) = T_{R,V}$, so we recover the module $T_{R,V}$ of Theorem 3.11.

## 4. Some basic examples and results

In this section we present a few important examples of correspondence functors and prove some of their basic properties.

For any finite set $E$, the representable functor $k\mathcal{C}(-,E)$ (sometimes called Yoneda functor) is the very first example of a correspondence functor. By definition, it is actually isomorphic to the functor $L_{E,k\mathcal{C}(E,E)}$. If $W$ is a $k\mathcal{C}(E,E)$-module generated by a single element $w$ (for instance a simple module), then the functor $L_{E,W}$ is isomorphic to a quotient of $k\mathcal{C}(-,E)$ via the surjective homomorphism

$$k\mathcal{C}(-,E) \longrightarrow L_{E,W} = k\mathcal{C}(-,E) \otimes_{k\mathcal{C}(E,E)} W \, , \qquad \phi \mapsto \phi \otimes w \, .$$

The representable functor $k\mathcal{C}(-,E)$ is projective (by Yoneda's lemma).

The next result is basic. It has several important corollaries which are often used.

**4.1. Lemma.** *Let $E$ and $F$ be finite sets with $|E| \leq |F|$. There exist correspondences $i_* \in \mathcal{C}(F,E)$ and $i^* \in \mathcal{C}(E,F)$ such that $i^*i_* = \mathrm{id}_E$.*

**Proof :** Since $|E| \leq |F|$, there exists an injective map $i : E \hookrightarrow F$. Let $i_* \subseteq (F \times E)$ denote the correspondence

$$i_* = \big\{ \big(i(e), e\big) \mid e \in E \big\} \, ,$$

and $i^* \subseteq (E \times F)$ denote the opposite correspondence

$$i^* = \big\{ \big(e, i(e)\big) \mid e \in E \big\} \, .$$

As $i$ is injective, one checks easily that $i^*i_* = \Delta_E$, that is, $i^*i_* = \mathrm{id}_E$. ∎

In other words, this lemma says that the object $E$ of $\mathcal{C}$ is a direct summand of the object $F$ whenever $|E| \leq |F|$.

**4.2. Corollary.** *Let $E$ and $F$ be finite sets with $|E| \leq |F|$. The representable functor $k\mathcal{C}(-,E)$ is isomorphic to a direct summand of the representable functor $k\mathcal{C}(-,F)$.*

**Proof :** Right multiplication by $i^*$ defines a homomorphism of correspondence functors

$$k\mathcal{C}(-,E) \longrightarrow k\mathcal{C}(-,F) \, ,$$

and right multiplication by $i_*$ defines a homomorphism of correspondence functors

$$k\mathcal{C}(-,F) \longrightarrow k\mathcal{C}(-,E) \, .$$

Their composite is the identity of $k\mathcal{C}(-,E)$, because $i^*i_* = \mathrm{id}_E$. ∎

**4.3. Corollary.** *Let $E$ and $F$ be finite sets with $|E| \leq |F|$. The left $\mathcal{C}(F,F)$-module $\mathcal{C}(F,E)$ is projective.*

**Proof :** By Corollary 4.2, $k\mathcal{C}(F,E)$ is isomorphic to a direct summand of $k\mathcal{C}(F,F)$, which is free. ∎

**4.4. Corollary.** *Let $E$ and $F$ be finite sets with $|E| \leq |F|$. Let $M$ be a correspondence functor. If $M(F) = 0$, then $M(E) = 0$.*

**Proof :** For any $m \in M(E)$, we have $m = i^*i_* \cdot m$. But $i_* \cdot m \in M(F)$, so $i_* \cdot m = 0$. Therefore $m = 0$. ∎

**4.5. Corollary.** *Let $E$ and $F$ be finite sets with $|E| \leq |F|$. For every finite set $X$, composition in the category $k\mathcal{C}$*

$$\mu : k\mathcal{C}(X,F) \otimes_{\mathcal{R}_F} k\mathcal{C}(F,E) \longrightarrow k\mathcal{C}(X,E)$$

*is an isomorphism.*

**Proof :** The inverse of $\mu$ is given by

$$\phi : k\mathcal{C}(X,E) \longrightarrow k\mathcal{C}(X,F) \otimes_{\mathcal{R}_F} k\mathcal{C}(F,E)\,, \qquad \phi(\alpha) = \alpha i^* \otimes i_*\ .$$

Composing with $\mu$, we obtain $\mu\phi(\alpha) = \mu(\alpha i^* \otimes i_*) = \alpha i^* i_* = \alpha$, so $\mu\phi = \mathrm{id}$. On the other hand, if $\beta \in k\mathcal{C}(X,F)$ and $\gamma \in k\mathcal{C}(F,E)$, then $\gamma i^*$ belongs to $\mathcal{R}_F$ and therefore

$$\phi\mu(\beta \otimes \gamma) = \phi(\beta\gamma) = \beta\gamma i^* \otimes i_* = \beta \otimes \gamma i^* i_* = \beta \otimes \gamma\ ,$$

showing that $\phi\mu = \mathrm{id}$. □

Now we move to direct summands of representable functors, given by some idempotent. If $R$ is an idempotent in $k\mathcal{C}(E,E)$, then $k\mathcal{C}(-,E)R$ is a direct summand of $k\mathcal{C}(-,E)$, hence projective again. In particular, if $R$ is a preorder on $E$, that is, a relation which is reflexive and transitive, then $R$ is idempotent (because $R \subseteq R^2$ by reflexivity and $R^2 \subseteq R$ by transitivity). There is an equivalence relation $\sim$ associated with $R$, defined by

$$x \sim y \iff (x,y) \in R \ \text{ and } \ (y,x) \in R\ .$$

Then $R$ induces an order relation $\overline{R}$ on the quotient set $\overline{E} = E/\sim$ such that

$$(x,y) \in R \iff (\overline{x},\overline{y}) \in \overline{R}\ ,$$

where $\overline{x}$ denotes the equivalence class of $x$ under $\sim$.

**4.6. Lemma.** *Let $E$ be a finite set and let $R$ be a preorder on $E$. Let $\overline{R}$ be the corresponding order on the quotient set $\overline{E} = E/\sim$ and write $e \mapsto \overline{e}$ for the quotient map $E \to \overline{E}$. The correspondence functors $k\mathcal{C}(-,E)R$ and $k\mathcal{C}(-,\overline{E})\overline{R}$ are isomorphic via the isomorphism*

$$k\mathcal{C}(-,E)R \longrightarrow k\mathcal{C}(-,\overline{E})\overline{R}\,, \qquad S \mapsto \overline{S}\ ,$$

*where for any correspondence $S \subseteq X \times E$, the correspondence $\overline{S} \subseteq X \times \overline{E}$ is defined by*

$$(x,\overline{e}) \in \overline{S} \iff (x,e) \in S\ .$$

**Proof :** It is straightforward to check that $\overline{S}$ is well-defined. If $S \in \mathcal{C}(X,E)R$, then $S = SR$ and it follows that $\overline{S} = \overline{S}\,\overline{R}$. Surjectivity is easy and injectivity follows from the definition of $\overline{S}$. □

This shows that it is relevant to consider the functors $k\mathcal{C}(-,E)R$ where $R$ is an order on $E$. We will see later that they turn out to be connected with the simple functors parametrized by $R^{op}$.

We now introduce the fundamental functors $\mathbb{S}_{E,R}$ which will play a crucial role in the sequel.

**4.7. Definition.** *If $E$ is a finite set and $R$ is an order on $E$, we denote by $\mathbb{S}_{E,R}$ the correspondence functor*

$$\mathbb{S}_{E,R} = L_{E,\mathcal{P}_E f_R}/J_{E,\mathcal{P}_E f_R} ,$$

*where $\mathcal{P}_E f_R$ is the fundamental module associated with the poset $(E,R)$ (see Proposition 3.10). We call it the* fundamental correspondence functor *associated with the poset $(E,R)$. In other words $\mathbb{S}_{E,R} \cong S_{E,R,k\mathrm{Aut}(E,R)}$ .*

By Lemma 3.14, $E$ is a minimal set for $\mathbb{S}_{E,R}$ and $\mathbb{S}_{E,R}(E) = \mathcal{P}_E f_R$.

**4.8. Remark.** Since $\mathcal{P}_E f_R$ has a bimodule structure (see Proposition 3.10), we can tensor on the right with a $k\,\mathrm{Aut}(E,R)$-module $V$. Then we recover the functor

$$S_{E,R,V} = \mathbb{S}_{E,R} \otimes_{k\,\mathrm{Aut}(E,R)} V .$$

This is because $\mathbb{S}_{E,R}$ is a quotient of the functor $L_{E,\mathcal{P}_E f_R}$ and there is a compatible right $k\,\mathrm{Aut}(E,R)$-module structure on each of its evaluations

$$L_{E,\mathcal{P}_E f_R}(X) = k\mathcal{C}(X,E) \otimes_{k\mathcal{C}(E,E)} \mathcal{P}_E f_R .$$

Whenever $k$ is a field and $V$ is a simple $k\,\mathrm{Aut}(E,R)$-module, then we recover in this way the simple functor $S_{E,R,V}$.

Evaluations of functors are not easy to control. One of our main tasks will be to show that the evaluations of the fundamental correspondence functors $\mathbb{S}_{E,R}$ are free $k$-modules and to find their rank. This will be completely achieved in Corollary 17.17. We start here by giving a first result in this direction. It holds over an arbitrary commutative ring $k$.

**4.9. Lemma.** *Let $(E,R)$ be a finite poset.*

(a) *There is a unique morphism of correspondence functors*

$$\omega_{E,R} : k\mathcal{C}(-,E) \longrightarrow \mathbb{S}_{E,R}$$

*such that $\omega_{E,R}(\Delta_E) = f_R \in \mathcal{P}_E f_R = \mathbb{S}_{E,R}(E)$.*

(b) *The morphism $\omega_{E,R}$ is surjective. For any finite set $X$, the kernel of the map*

$$\omega_{E,R,X} : k\mathcal{C}(X,E) \longrightarrow \mathbb{S}_{E,R}(X)$$

*consists of the set of linear combinations $\sum\limits_{S\subseteq X\times E} \lambda_S S$ with coefficients $\lambda_S$ in $k$, such that*

$$\forall U \subseteq E\times X, \quad \sum_{\Delta\subseteq US\subseteq R} \lambda_S = 0 ,$$

*where $\Delta = \Delta_E$ for simplicity.*

(c) *The morphism $\omega_{E,R}$ factorises as*

$$k\mathcal{C}(-,E) \twoheadrightarrow k\mathcal{C}(-,E)R \xrightarrow{\overline{\omega}_{E,R}} \mathbb{S}_{E,R} ,$$

*where the morphism on the left hand side is right multiplication by $R$. For any finite set $X$, the kernel of $\overline{\omega}_{E,R,X}$ is equal to the set of linear combinations $\sum\limits_{S\in\mathcal{C}(X,E)R} \lambda_S S$ such that*

$$\forall U \in R\mathcal{C}(E,X), \quad \sum_{US=R} \lambda_S = 0 .$$

(d) *Both $\omega_{E,R}$ and $\overline{\omega}_{E,R}$ are homomorphisms of right $k\,\mathrm{Aut}(E,R)$-modules.*

**Proof :** (a) This is just an application of Yoneda's lemma.

(b) The morphism $\omega_{E,R}$ is surjective, because $f_R$ generates $\mathbb{S}_{E,R}$. Let $u = \sum_{S\in\mathcal{C}(X,E)} \lambda_S S$ be an element of $k\mathcal{C}(X,E)$, where $\lambda_S \in k$. Then $\omega_{E,R,X}(u) = 0$ if and only if the image in $\mathbb{S}_{E,R}(X)$ of the element

$$\sum_{S\in\mathcal{C}(X,E)} \lambda_S S \otimes f_R \;\in\; L_{E,\mathcal{P}_E f_R}(X)$$

is equal to 0. In other words, this means that $\sum_{S\in\mathcal{C}(X,E)} \lambda_S S \otimes f_R$ belongs to $J_{E,\mathcal{P}_E f_R}(X)$. By definition, this is equivalent to saying that

$$\textbf{(4.10)} \qquad \forall U \in \mathcal{C}(E,X),\quad \sum_{S\in\mathcal{C}(X,E)} \lambda_S (US)\cdot f_R = 0 \;\text{ in } \mathcal{P}_E f_R\,.$$

Now the action of correspondences on $\mathcal{P}_E f_R$ is described in Proposition 3.10. We obtain $(US)\cdot f_R = 0$, unless there exists $\tau \in \Sigma_E$ such that

$$\Delta \subseteq \Delta_{\tau^{-1}} US \subseteq R\,.$$

In this case $\tau$ is unique, and $(US)\cdot f_R = \Delta_\tau f_R$. Thus Condition 4.10 is equivalent to

$$\forall U \subseteq E\times X,\quad \sum_{\tau\in\Sigma_E} \Big( \sum_{\Delta\subseteq\Delta_{\tau^{-1}}US\subseteq R} \lambda_S \Big) \Delta_\tau f_R = 0 \quad \text{in } \mathcal{P}_E f_R\,.$$

Equivalently,

$$\forall U \subseteq E\times X,\ \forall \tau\in\Sigma_E,\qquad \sum_{\Delta\subseteq\Delta_{\tau^{-1}}US\subseteq R} \lambda_S = 0\,.$$

Changing $U$ to $\Delta_\tau U$, this amounts to requiring that

$$\forall U \subseteq E\times X,\quad \sum_{\Delta\subseteq US\subseteq R} \lambda_S = 0\,.$$

(c) We know that $Rf_R = f_R$, by Lemma 3.8. Then clearly $S - SR \in \operatorname{Ker}\omega_{E,R,X}$, for any finite set $X$ and any $S \in \mathcal{C}(X,E)$. Hence $\omega_{E,R}$ factorizes through $k\mathcal{C}(-,E)R$. Replacing $S$ by $SR$, we can assume that $S = SR$. Then the condition $\Delta \subseteq US \subseteq R$ yields $R \subseteq USR \subseteq R^2 \subseteq R$, that is, $US = R$. The equation $\sum_{US=R} \lambda_S = 0$ for any given $U$ is the same as the equation for $RU$, so we can assume that $U \in R\mathcal{C}(E,X)$. The description of $\operatorname{Ker}\overline{\omega}_{E,R,X}$ follows.

(d) The action of $k\operatorname{Aut}(E,R)$ on both $k\mathcal{C}(X,E)$ and $\mathbb{S}_{E,R}(X)$ is induced by right multiplication of correspondences $S \mapsto S\Delta_\sigma$, for every $S \in \mathcal{C}(X,E)$ and $\sigma \in \operatorname{Aut}(E,R)$. We clearly have

$$\omega_{E,R,X}(S\Delta_\sigma) = S\Delta_\sigma f_R = Sf_R\Delta_\sigma = \omega_{E,R,X}(S)\Delta_\sigma\,.$$

because $\Delta_\sigma$ commutes with $f_R$ whenever $\sigma \in \operatorname{Aut}(E,R)$ (Lemma 3.8). □

We end this section with the description of two small examples.

**4.11. Example.** Let $E = \emptyset$ be the empty set and consider the representable functor $k\mathcal{C}(-,\emptyset)$. Then $k\mathcal{C}(X,\emptyset) = \{\emptyset\}$ is a singleton for any finite set $X$, so $k\mathcal{C}(X,\emptyset) \cong k$. Moreover, any correspondence $S \in \mathcal{C}(Y,X)$ is sent to the identity map from $k\mathcal{C}(X,\emptyset) \cong k$ to $k\mathcal{C}(Y,\emptyset) \cong k$. This functor deserves to be called the *constant functor*. We will denote it by $\underline{k}$.

Assume that $k$ is a field. The algebra $k\mathcal{C}(\emptyset,\emptyset) \cong k$ has a unique simple module $k$. It is then easy to check that $L_{\emptyset,k} = k\mathcal{C}(-,\emptyset)$ and $J_{\emptyset,k} = \{0\}$. Therefore

$$k\mathcal{C}(-,\emptyset) = S_{\emptyset,\emptyset,k}$$

the simple functor indexed by $(\emptyset,\emptyset,k)$. Here the second $\emptyset$ denotes the only relation on the empty set, while $k$ is the only simple module for the group algebra $k\,\mathrm{Aut}(E,R) = k\Sigma_\emptyset \cong k$, where $\Sigma_\emptyset = \{\mathrm{id}\}$ is the symmetric group of the empty set. Note that $S_{\emptyset,\emptyset,k}$ is projective, because it is a representable functor.

**4.12. Example.** Let $E = \{1\}$ be a set with one element and consider the representable functor $\mathcal{C}(-,\{1\})$. Then $\mathcal{C}(X,\{1\})$ is in bijection with the set $\mathcal{B}(X)$ of all subsets of $X$, because $X \times \{1\} \cong X$. It is easy to see that a correspondence $S \in \mathcal{C}(Y,X)$ is sent to the map

$$\mathcal{B}(X) \longrightarrow \mathcal{B}(Y)\,, \qquad A \mapsto S_A\,,$$

where $S_A = \{y \in Y \mid \exists\, a \in A \text{ such that } (y,a) \in S\}$. Thus $k\mathcal{B} \cong k\mathcal{C}(-,\{1\})$ is a correspondence functor such that $k\mathcal{B}(X)$ is a free $k$-module with basis $\mathcal{B}(X)$ and rank $2^{|X|}$ for every finite set $X$.

The functor $k\mathcal{B}$ has a subfunctor isomorphic to the constant functor $S_{\emptyset,\emptyset,k}$, because $\mathcal{B}(X)$ contains the element $\emptyset$ which is mapped to $\emptyset$ by any correspondence. We claim that, if $k$ is a field, the quotient $k\mathcal{B}/S_{\emptyset,\emptyset,k}$ is a simple functor.

Assume that $k$ is a field. The algebra $k\mathcal{C}(\{1\},\{1\})$ has dimension 2, actually isomorphic to $k \times k$ with two primitive idempotents $\emptyset$ and $\{(1,1)\} - \emptyset$. The essential algebra $\mathcal{E}_{\{1\}}$ is a one-dimensional quotient and its unique simple module $W$ is one-dimensional and corresponds to the pair $(R,k)$, where $R$ is the only order relation on $\{1\}$ and $k$ is the only simple module for the group algebra $k\,\mathrm{Aut}(E,R) = k\Sigma_{\{1\}} \cong k$, with $\Sigma_{\{1\}} = \{\mathrm{id}\}$ the symmetric group of $\{1\}$. Thus there is a simple functor $S_{\{1\},W} = S_{\{1\},R,k}$.

The kernel of the quotient map

$$k\mathcal{B} \cong k\mathcal{C}(-,\{1\}) \;\longrightarrow\; k\mathcal{C}(-,\{1\}) \otimes_{k\mathcal{C}(\{1\},\{1\})} W = L_{\{1\},W}$$

is the constant subfunctor $S_{\emptyset,\emptyset,k}$ mentioned above, because $\emptyset \in \mathcal{C}(X,\{1\})$ can be written $\emptyset \cdot \emptyset$, with the second empty set belonging to $\mathcal{C}(\{1\},\{1\})$, thus acting by zero on $W$. Now we know that $L_{\{1\},W}/J_{\{1\},W} = S_{\{1\},W}$ and we are going to show that $J_{\{1\},W} = \{0\}$. It then follows that $L_{\{1\},W} = S_{\{1\},W}$ is simple and isomorphic to $k\mathcal{B}/S_{\emptyset,\emptyset,k}$, proving the claim above.

In order to prove that $J_{\{1\},W} = \{0\}$, we let $u \in L_{\{1\},W}(X)$, which can be written

$$u = \sum_{A \in \mathcal{B}(X)} \lambda_A (A \times \{1\}) \otimes w\,,$$

where $w$ is a generator of $W$ and $\lambda_A \in k$ for all $A$. Since the empty set acts by zero on $w$, the sum actually runs over nonempty subsets $A \in \mathcal{B}(A)$. Then $u \in J_{\{1\},W}(X)$ if and only if, for all $(\{1\} \times B) \in \mathcal{C}(\{1\},X)$, we have

$$\sum_{A \in \mathcal{B}(X)} \lambda_A\, (\{1\} \times B)(A \times \{1\}) \cdot w = 0\,.$$

Since $\emptyset$ acts by zero and $\{1\}$ acts as the identity, we obtain

$$(\{1\} \times B)(A \times \{1\}) \cdot w = \begin{cases} 0 & \text{if } B \cap A = \emptyset\,, \\ w & \text{if } B \cap A \neq \emptyset\,. \end{cases}$$

This yields the condition

$$\sum_{A\cap B\neq\emptyset}\lambda_A=0\,,\qquad \text{for every nonempty}\ \ B\in\mathcal{B}(X)\,.$$

We prove by induction that $\lambda_C=0$ for every nonempty $C\in\mathcal{B}(X)$. Subtracting the condition for $B=X$ and for $B=X-C$, we obtain

$$0=\sum_{A\neq\emptyset}\lambda_A-\sum_{A\not\subseteq C}\lambda_A=\sum_{\emptyset\neq A\subseteq C}\lambda_A\,.$$

If $C=\{c\}$ is a singleton, we obtain $\lambda_{\{c\}}=0$ and this starts the induction. In the general case, we obtain by induction $\lambda_A=0$ for $\emptyset\neq A\neq C$, so we are left with $\lambda_C=0$. Therefore $u=0$ and so $J_{\{1\},W}=\{0\}$.

There is a special feature of this small example, namely that the exact sequence

$$0\longrightarrow S_{\emptyset,\emptyset,k}\longrightarrow k\mathcal{B}\longrightarrow S_{\{1\},R,k}\longrightarrow 0$$

splits. This is because there is a retraction $k\mathcal{B}\to S_{\emptyset,\emptyset,k}$ defined by

$$k\mathcal{B}(X)\longrightarrow k\,,\qquad A\mapsto 1\,,$$

which is easily checked to be a homomorphism of functors. Since $k\mathcal{B}$ is projective (because it is a representable functor), its direct summand $S_{\{1\},R,k}$ is projective.

**4.13. Remark.** In both Example 4.11 and Example 4.12, there is a unique order relation $R$ on $E$, which is a total order. Actually, these examples are special cases of the general situation of a total order, which is studied in Section 15.

## 5. Posets and lattices

Finite posets and lattices play an essential role in this paper. Indeed we already know that pairs $(E,R)$, where $R$ is an order relation on $E$ (that is, a finite poset), appear in the parametrization of simple correspondence functors (Theorem 3.12) and in the definition of the fundamental functor $\mathbb{S}_{E,R}$ (Definition 4.7). Moreover, lattices will play a central role in Part 2 (see Section 11).

This section is an interlude, in which we give some definitions, fix some notation, and prove some basic lemmas, which will be used throughout.

**5.1. Notation and definitions.** *Let $(E,R)$ be a finite poset (or more generally a preorder).*

*(a) We write $\leq_R$ for the order relation, so that $(a,b)\in R$ if and only if $a\leq_R b$. Moreover $a<_R b$ means that $a\leq_R b$ and $a\neq b$.*

*(b) If $a,b\in E$ with $a\leq_R b$, we define intervals*

$$\begin{aligned}
&[a,b]_E=\{x\in E\mid a\leq_R x\leq_R b\}\,, && ]a,b[_E=\{x\in E\mid a<_R x<_R b\}\,,\\
&[a,b[_E=\{x\in E\mid a\leq_R x<_R b\}\,, && ]a,b]_E=\{x\in E\mid A<_R x\leq_R b\}\,,\\
&[a,\cdot[_E=\{x\in E\mid a\leq_R x\}\,, && ]\cdot,b]_E=\{x\in E\mid x\leq_R b\}\,.
\end{aligned}$$

*When the context is clear, we write $[a,b]$ instead of $[a,b]_E$.*

*(c) A subset $A$ of $E$ is a* lower $R$-ideal*, or simply a* lower ideal*, if, whenever $a\in A$ and $x\leq_R a$, we have $x\in A$. Similarly, a subset $A$ of $E$ is an* upper $R$-ideal*, or simply an* upper ideal*, if, whenever $a\in A$ and $a\leq_R x$, we have $x\in A$.*

*(d) A* principal lower ideal*, or simply* principal ideal*, is a subset of the form $]\cdot,a]_E$, where $a\in E$. A* principal upper ideal *is defined similarly.*

**5.2. Notation and definitions.** *Let $T$ be a finite lattice.*

(a) *We write $\leq_T$, or sometimes simply $\leq$, for the order relation, $\vee$ for the join (least upper bound), $\wedge$ for the meet (greatest lower bound), $\hat{0}$ for the least element and $\hat{1}$ for the greatest element.*
(b) *An element $e \in T$ is called* join-irreducible*, or simply* irreducible*, if, whenever $e = \bigvee_{a \in A} a$ for some subset $A$ of $T$, then $e \in A$. In case $A = \emptyset$, the join is $\hat{0}$ and it follows that $\hat{0}$ is not irreducible. If $e \neq \hat{0}$ is irreducible and $e = s \vee t$ with $s, t \in T$, then either $e = s$ or $e = t$. In other words, if $e \neq \hat{0}$, then $e$ is irreducible if and only if $[\hat{0}, e[$ has a unique maximal element.*
(c) *Let $E$ be a subposet of $T$. We say that $E$ is a* full *subposet of $T$ if for all $e, f \in E$ we have :*

$$e \leq_R f \iff e \leq_T f \ .$$

(d) *We write $\mathrm{Irr}(T)$ for the set of irreducible elements in $T$, viewed as a full subposet of $T$.*

**5.3. Notation.** *Let $(E, R)$ be a finite poset (or more generally a preorder).*

(a) *Let $I_\downarrow(E, R)$ denote the set of lower $R$-ideals of $E$. Then $I_\downarrow(E, R)$, ordered by inclusion of subsets, is a lattice : the join operation is union of subsets, and the meet operation is intersection.*
(b) *Similarly, $I^\uparrow(E, R)$ denotes the set of upper $R$-ideals of $E$, which is also a lattice. If $R^{op}$ is the relation opposite to $R$, then clearly $I^\uparrow(E, R) = I_\downarrow(E, R^{op})$.*

**5.4. Lemma.** *Let $R$ be a preorder on a finite set $E$. As in Lemma 4.6, let $\overline{R}$ be the corresponding order on the quotient set $\overline{E} = E/\sim$. The quotient map $E \to \overline{E}$ induces an isomorphism of lattices $I_\downarrow(E, R) \cong I_\downarrow(\overline{E}, \overline{R})$.*

**Proof :** The proof is straightforward and is left to the reader. □

Lemma 5.4, as well as Lemma 4.6, imply that it is enough to consider orders rather than preorders. In the rest of this paper, we shall work with orders, without loss of generality.

**5.5. Lemma.** *Let $(E, R)$ be a finite poset.*

(a) *The irreducible elements in the lattice $I_\downarrow(E, R)$ are the principal ideals $]\cdot, e]_E$, where $e \in E$. Thus the poset $E$ is isomorphic to the poset of all irreducible elements in $I_\downarrow(E, R)$ by mapping $e \in E$ to the principal ideal $]\cdot, e]_E$.*
(b) *$I_\downarrow(E, R)$ is a distributive lattice.*
(c) *If $T$ is a distributive lattice and $(E, R)$ is its subposet of irreducible elements, then $T$ is isomorphic to $I_\downarrow(E, R)$.*

**Proof :** This is not difficult and well-known. For details, see Theorem 3.4.1 and Proposition 3.4.2 in [**St**]. □

**5.6. Convention.** *In the situation of Lemma 5.5, we shall identify $E$ with its image via the map*

$$E \longrightarrow I_{\downarrow}(E,R)\,, \qquad e \mapsto \,]\cdot,e]_E\;.$$

*Thus we view $E$ as a full subposet of $I_{\downarrow}(E,R)$. This abusive convention is a conceptual simplification and has many advantages for the rest of this paper.*

Lattices having a given poset $(E,R)$ as poset of irreducible elements will be described in an appendix (Section 21). They are all quotients of $I_{\downarrow}(E,R)$, and $I_{\downarrow}(E,R)$ is the only one which is distributive, up to isomorphism (by part c) of Lemma 5.5).

Note that if $(E,R)$ is the poset of irreducible elements in a finite lattice $T$, then $T$ is *generated* by $E$ in the sense that any element $x \in T$ is a join of elements of $E$. To see this, define the height of $t \in T$ to be the maximal length of a chain in $[\hat{0},t]_T$. If $x$ is not irreducible and $x \neq \hat{0}$, then $x = t_1 \vee t_2$ with $t_1$ and $t_2$ of smaller height than $x$. By induction on the height, both $t_1$ and $t_2$ are joins of elements of $E$. Therefore $x = t_1 \vee t_2$ is also a join of elements of $E$.

Given a poset $(E,R)$, the map

$$E \longrightarrow I^{\uparrow}(E,R)\,, \qquad e \mapsto \;[e,\cdot[_E$$

is order-reversing, so it is in fact $(E,R^{op})$ which is identified with the poset of irreducible elements in $I^{\uparrow}(E,R)$. Since $I^{\uparrow}(E,R) = I_{\downarrow}(E,R^{op})$, this is actually just Convention 5.6 applied to $R^{op}$.

We now introduce a notation which will play an important role in the rest of the paper.

**5.7. Notation.** *Let $T$ be a finite lattice and let $(E,R)$ be the full subposet of its irreducible elements. For any finite set $X$ and any map $\varphi : X \to T$, we associate the correspondence*

$$\Gamma_\varphi = \{(x,e) \in X \times E \mid e \leq_T \varphi(x)\} \subseteq X \times E\;.$$

*In the special case where $T = I_{\downarrow}(E,R)$ and in view of Convention 5.6, we obtain*

$$\Gamma_\varphi = \{(x,e) \in X \times E \mid e \in \varphi(x)\}\;.$$

**5.8. Lemma.** *Let $T$ be a finite lattice and let $(E,R)$ be the full subposet of its irreducible elements.*

*(a) For any map $\varphi : X \to T$, we have $\Gamma_\varphi R^{op} = \Gamma_\varphi$.*
*(b) If $T = I_{\downarrow}(E,R)$, then a correspondence $S \subseteq X \times E$ has the form $S = \Gamma_\varphi$ for some map $\varphi : X \to I_{\downarrow}(E,R)$ if and only if $SR^{op} = S$.*
*(c) If $T = I^{\uparrow}(E,R)$, then a correspondence $S \subseteq X \times E$ has the form $S = \Gamma_\varphi$ for some map $\varphi : X \to I^{\uparrow}(E,R)$ if and only if $SR = S$.*

**Proof :** (a) Since $\Delta_E \subseteq R^{op}$, we always have $\Gamma_\varphi = \Gamma_\varphi \Delta_E \subseteq \Gamma_\varphi R^{op}$. Conversely, if $(x,f) \in \Gamma_\varphi R^{op}$, then there exists $e \in E$ such that $(x,e) \in \Gamma_\varphi$ and $(e,f) \in R^{op}$, that is, $e \leq_T \varphi(x)$ and $f \leq_R e$. But $f \leq_R e$ if and only if $f \leq_T e$ (because $(E,R)$ is a full subposet of $T$). It follows that $f \leq_T \varphi(x)$, that is, $(x,f) \in \Gamma_\varphi$. Thus $\Gamma_\varphi R^{op} \subseteq \Gamma_\varphi$ and equality follows.

(b) This follows from (c) applied to $R^{op}$, because of the equality $I_{\downarrow}(E,R) = I^{\uparrow}(E,R^{op})$.

(c) One direction follows from (a). For the other direction, let $S \in \mathcal{C}(X,E)$ such that $SR = S$, or equivalently $S \in \mathcal{C}(X,E)R$. Then the set

$$\psi(x) = \big\{e \in E \mid (x,e) \in S\big\}$$

is an upper $R$-ideal in $E$, thus $\psi$ is a function $X \to I^{\uparrow}(E,R)$. Clearly $\Gamma_\psi = S$. □

## 6. Finite generation

In this section, we analyze the property of finite generation for correspondence functors.

**6.1. Definition.** *Let $\{E_i \mid i \in I\}$ be a family of finite sets indexed by a set $I$ and, for every $i \in I$, let $m_i \in M(E_i)$. A correspondence functor $M$ is said to be* generated *by the set $\{m_i \mid i \in I\}$ if for every finite set $X$ and every element $m \in M(X)$, there exists a finite subset $J \subseteq I$ such that*

$$m = \sum_{j\in J} \alpha_j m_j\,, \qquad \text{for some } \alpha_j \in k\mathcal{C}(X,E_j) \quad (j\in J)\ .$$

*In the case where $I$ is finite, then $M$ is said to be* finitely generated*.*

We remark that, in the sum above, each $\alpha_j$ decomposes as a finite $k$-linear combination $\alpha_j = \sum_{S\in\mathcal{C}(X,E_j)} \lambda_S\, S$ where $\lambda_S \in k$. Therefore, every $m \in M(X)$ decomposes further as a (finite) $k$-linear combination

$$m = \sum_{\substack{j\in J\\ S\in\mathcal{C}(X,E_j)}} \lambda_S\, Sm_j\ .$$

**6.2. Example.** If $E$ is a finite set, the representable functor $k\mathcal{C}(-,E)$ is finitely generated. It is actually generated by a single element, namely $\Delta_E \in k\mathcal{C}(E,E)$.

**6.3. Lemma.** *Let $M$ be a finitely generated correspondence functor over $k$. Then, for every finite set $X$, the evaluation $M(X)$ is a finitely generated $k$-module. In particular, if $k$ is a field, then $M(X)$ is finite-dimensional.*

**Proof :** Let $\{m_i \mid i=1,\ldots,n\}$ be a finite set of generators of $M$, with $m_i \in M(E_i)$. Let $B_X = \{Sm_i \mid S\in\mathcal{C}(X,E_i), i=1,\ldots,n\}$. By definition and by the remark above, every element of $M(X)$ is a $k$-linear combination of elements of $B_X$. But $B_X$ is a finite set, so $M(X)$ is finitely generated. If $k$ is a field, this means that $M(X)$ is finite-dimensional. □

It follows that, in order to understand finitely generated correspondence functors, we could assume that all their evaluations are finitely generated $k$-modules. But we do not need this for our next characterizations.

**6.4. Proposition.** *Let $M$ be a correspondence functor over $k$. The following conditions are equivalent :*

*(a)* *$M$ is finitely generated.*

*(b)* *$M$ is isomorphic to a quotient of a functor of the form $\bigoplus_{i=1}^{n} k\mathcal{C}(-,E_i)$ for some finite sets $E_i$ $(i=1,\ldots,n)$.*

*(c)* *$M$ is isomorphic to a quotient of a functor of the form $\bigoplus_{i\in I} kC(-,E)$ for some finite set $E$ and some finite index set $I$.*

*(d)* *There exists a finite set $E$ and a finite subset $B$ of $M(E)$ such that $M$ is generated by $B$.*

**Proof :** (a) $\Rightarrow$ (b). Suppose that $M$ is generated by the set $\{m_i \mid i = 1, \ldots, n\}$, where $m_i \in M(E_i)$. It follows from Yoneda's lemma that there is a morphism

$$\psi_i : k\mathcal{C}(-, E_i) \to M$$

mapping $\Delta_{E_i} \in \mathcal{C}(E_i, E_i)$ to the element $m_i \in M(E_i)$, hence mapping $\beta \in \mathcal{C}(X, E_i)$ to $\beta m_i \in M(X)$. Their sum yields a morphism

$$\psi : \bigoplus_{i=1}^{n} k\mathcal{C}(-, E_i) \to M \ .$$

For any $X$ and any $m \in M(X)$, we have $m = \sum\limits_{i=1}^{n} \alpha_i m_i$ for some $\alpha_i \in k\mathcal{C}(X, E_i)$, hence $m = \psi(\alpha_1, \ldots, \alpha_n)$, proving the surjectivity of $\psi$.

(b) $\Rightarrow$ (c). Suppose that $M$ is isomorphic to a quotient of a functor of the form $\bigoplus\limits_{i=1}^{n} k\mathcal{C}(-, E_i)$. Let $F$ be the largest of the sets $E_i$. By Corollary 4.2, each $k\mathcal{C}(-, E_i)$ is a direct summand of $k\mathcal{C}(-, F)$. Therefore, $M$ is also isomorphic to a quotient of the functor $\bigoplus\limits_{i=1}^{n} k\mathcal{C}(-, F)$.

(c) $\Rightarrow$ (d). By Example 6.2, $k\mathcal{C}(-, E)$ is generated by $\Delta_E \in k\mathcal{C}(E, E)$. Let $b_i \in \bigoplus\limits_{i\in I} k\mathcal{C}(E, E)$ having zero components everywhere, except the $i$-th equal to $\Delta_E$. Since $M$ is a quotient of $\bigoplus\limits_{i\in I} k\mathcal{C}(-, E)$, it is generated by the images of the elements $b_i$. This is a finite set because $I$ is finite by assumption.

(d) $\Rightarrow$ (a). Since $M$ is generated by $B$, it is finitely generated. □

We apply this to the functors $L_{E,V}$ and $S_{E,V}$ defined in Lemma 2.3 and Notation 2.5.

**6.5. Corollary.** *Let $V$ be a finitely generated $\mathcal{R}_E$-module, where $E$ is a finite set and $\mathcal{R}_E$ is the algebra of relations on $E$. Then $L_{E,V}$ and $S_{E,V}$ are finitely generated correspondence functors.*

**Proof :** Let $\{v_i \mid i \in I\}$ be a finite set of generators of $V$ as an $\mathcal{R}_E$-module. There is a morphism $\pi_i : k\mathcal{C}(-, E) \to L_{E,V}$ mapping $\Delta_E$ to $v_i \in L_{E,V}(E) = V$. Therefore, we obtain a surjective morphism

$$\sum_{i\in I} \pi_i : \bigoplus_{i\in I} k\mathcal{C}(-, E) \longrightarrow L_{E,V} \ ,$$

showing that $L_{E,V}$ is finitely generated. Now $S_{E,V}$ is a quotient of $L_{E,V}$, so it is also finitely generated. □

**6.6. Proposition.** *Let $k$ be a noetherian ring.*

*(a) For any finitely generated correspondence functor $M$ over $k$, the algebra $\mathrm{End}_{\mathcal{F}_k}(M)$ is a finitely generated $k$-module.*
*(b) For any two finitely generated correspondence functors $M$ and $N$, the $k$-module $\mathrm{Hom}_{\mathcal{F}_k}(M, N)$ is finitely generated.*
*(c) If $k$ is a field, the Krull-Remak-Schmidt theorem holds for finitely generated correspondence functors over $k$.*

**Proof :** (a) Since $M$ is finitely generated, there exists a finite set $E$ and a surjective morphism $\pi : \bigoplus_{i\in I} k\mathcal{C}(-,E) \to M$ for some finite set $I$ (Proposition 6.4). Denote by $\mathcal{A}$ the subalgebra of $\mathrm{End}_{\mathcal{F}_k}\big(\bigoplus_{i\in I} k\mathcal{C}(-,E)\big)$ consisting of endomorphisms $\varphi$ such that $\varphi(\mathrm{Ker}\,\pi) \subseteq \mathrm{Ker}\,\pi$. The algebra $\mathcal{A}$ is isomorphic to a $k$-submodule of $\mathrm{End}_{\mathcal{F}_k}\big(\bigoplus_{i\in I} k\mathcal{C}(-,E)\big)$, which is isomorphic to a matrix algebra of size $|I|$ over the $k$-algebra $k\mathcal{C}(E,E)$ (because $\mathrm{End}_{\mathcal{F}_k}\big(k\mathcal{C}(-,E)\big) \cong k\mathcal{C}(E,E)$ by Yoneda's lemma). This matrix algebra is free of finite rank as a $k$-module. As $k$ is noetherian, it follows that $\mathcal{A}$ is a finitely generated $k$-module.

Now by definition, any $\varphi \in \mathcal{A}$ induces an endomorphism $\overline{\varphi}$ of $M$ such that $\overline{\varphi}\pi = \pi\varphi$. This yields an algebra homomorphism $\mathcal{A} \to \mathrm{End}_{\mathcal{F}_k}(M)$, which is surjective, since the functor $k\mathcal{C}(-,E)$ is projective. It follows that $\mathrm{End}_{\mathcal{F}_k}(M)$ is also a finitely generated $k$-module.

(b) The functor $M \oplus N$ is finitely generated, hence $V = \mathrm{End}_{\mathcal{F}_k}(M \oplus N)$ is a finitely generated $k$-module, by (a). Since $\mathrm{Hom}_{\mathcal{F}_k}(M,N)$ embeds in $V$, it is also a finitely generated $k$-module.

(c) If moreover $k$ is a field, then $\mathrm{End}_{\mathcal{F}_k}(M)$ is a finite dimensional $k$-vector space, by (a). Finding decompositions of $M$ as a direct sum of subfunctors amounts to splitting the identity of $\mathrm{End}_{\mathcal{F}_k}(M)$ as a sum of orthogonal idempotents. Since $\mathrm{End}_{\mathcal{F}_k}(M)$ is a finite dimensional algebra over the field $k$, the standard theorems on decomposition of the identity as a sum of primitive idempotents apply. Thus $M$ can be split as a direct sum of indecomposable functors, and such a decomposition is unique up to isomorphism. □

After the Krull-Remak-Schmidt theorem, we treat the case of projective covers. Recall (see 2.5.14 in [**AF**] for categories of modules) that in an abelian category $\mathcal{A}$, a subobject $N$ of an object $P$ is called *superfluous* if for any subobject $X$ of $P$, the equality $X + N = P$ implies $X = P$. Similarly, an epimorphism $f : P \to M$ in $\mathcal{A}$ is called superfluous if $\mathrm{Ker}\, f$ is superfluous in $P$, or equivalently, if for any morphism $g : L \to P$ in $\mathcal{A}$, the composition $f \circ g$ is an epimorphism if and only if $g$ is an epimorphism. A *projective cover* of an object $M$ of $\mathcal{A}$ is defined as a pair $(P,p)$, where $P$ is projective and $p$ is a superfluous epimorphism from $P$ to $M$.

**6.7. Proposition.** *Let $M$ be a finitely generated correspondence functor over a commutative ring $k$.*

- *(a) Suppose that $M$ is generated by $M(E)$ where $E$ is a finite. If $(P,p)$ is a projective cover of $M(E)$ in $k\mathcal{C}(E,E)$-*Mod*, then $(L_{E,P},\widetilde{p})$ is a projective cover of $M$ in $\mathcal{F}_k$, where $\widetilde{p} : L_{E,P} \to M$ is obtained from $p : P \to M(E)$ by the adjunction of Lemma 2.3.*
- *(b) If $k$ is a field, then $M$ admits a projective cover.*
- *(c) In particular, when $k$ is a field, let $E$ be a finite set, let $R$ be an order relation on $E$, and let $V$ be a simple $k\,\mathrm{Aut}(E,R)$-module. Let moreover $(P,p)$ be a projective cover of $\mathcal{P}f_R \otimes_{k\,\mathrm{Aut}(E,R)} V$ in $k\mathcal{C}(E,E)$-*Mod*. Then $(L_{E,P},\widetilde{p})$ is a projective cover of the simple correspondence functor $S_{E,R,V}$.*

**Proof :** (a) (This was already proved in Lemme 2 of [**Bo1**].) By Lemma 2.3, the functor $Q \mapsto L_{E,Q}$ maps projectives to projectives. So the functor $L_{E,P}$ is projective. Since $M$ is generated by $M(E)$, and since the evaluation at $E$ of the morphism $\widetilde{p} : L_{E,P} \to M$ is equal to $p : P \to M(E)$, it follows that $\widetilde{p}$ is surjective. If $N$ is any subfunctor of $L_{E,P}$ such that $\widetilde{p}(N) = M$, then in particular $N(E) \subseteq P$

and $p\big(N(E)\big) = M(E)$. Since $p$ is superfluous, it follows that $N(E) = P$, hence $N = L_{E,P}$ since $L_{E,P}$ is generated by its evaluation $P$ at $E$.

(b) The algebra $k\mathcal{C}(E,E)$ is a finite dimensional algebra over the field $k$. Hence any finite dimensional $k\mathcal{C}(E,E)$-module admits a projective cover. Therefore (b) follows from (a).

(c) The evaluation of the simple functor $S_{E,R,V}$ at $E$ is the simple $k\mathcal{C}(E,E)$-module $\mathcal{P}f_R \otimes_{k\,\mathrm{Aut}(E,R)} V$. Hence (c) follows from (a) and (b). ∎

## 7. Bounded type

In this section, we analyze a notion which is more general than finite generation.

**7.1. Definition.** *Let $k$ be a commutative ring and let $M$ be a correspondence functor over $k$.*

- *(a) We say that $M$ has* bounded type *if there exists a finite set $E$ such that $M$ is generated by $M(E)$.*
- *(b) We say that $M$ has a* bounded presentation *if there are projective correspondence functors $P$ and $Q$ of bounded type and an exact sequence of functors*

$$Q \to P \to M \to 0 \,.$$

*Such a sequence is called a* bounded presentation *of $M$.*

Suppose that $M$ has bounded type and let $E$ be a finite set such that $M$ is generated by $M(E)$. It is elementary to see that $M$ is finitely generated if and only if $M(E)$ is a finitely generated $\mathcal{R}_E$-module (using Example 6.2 and Lemma 6.3). Thus an infinite direct sum of copies of a simple functor $S_{E,R,V}$ has bounded type (because it generated by its evaluation at $E$) but is not finitely generated. Also, a typical example of a correspondence functor which does not have bounded type is a direct sum of simple functors $\bigoplus\limits_{n=0}^{\infty} S_{E_n,R_n,V_n}$, where $|E_n| = n$ for each $n$. This is because $S_{E_n,R_n,V_n}$ cannot be generated by a set of cardinality $< n$.

**7.2. Lemma.** *Let $k$ be a commutative ring and let $M$ be a correspondence functor over $k$. Suppose that $M$ has bounded type and let $E$ be a finite set such that $M$ is generated by $M(E)$. For any finite set $F$ with $|F| \geq |E|$, the functor $M$ is generated by $M(F)$.*

**Proof :** Let $i_* \in \mathcal{C}(F,E)$ and $i^* \in \mathcal{C}(E,F)$ be as in Lemma 4.1, so that $i^*i_* = \mathrm{id}_E$. Saying that $M$ is generated by $M(E)$ amounts to saying that $M(X)$ is equal to $k\mathcal{C}(X,E)M(E)$, for any finite set $X$. It follows that

$$\begin{aligned} M(X) &= k\mathcal{C}(X,E)M(E) \\ &= k\mathcal{C}(X,E)i^*i_*M(E) \\ &\subseteq k\mathcal{C}(X,F)i_*M(E) \\ &\subseteq k\mathcal{C}(X,F)M(F) \subseteq M(X) \,, \end{aligned}$$

hence $M(X) = k\mathcal{C}(X,F)M(F)$, i.e. $M$ is generated by $M(F)$. ∎

We are going to prove that any correspondence functor having a bounded presentation is isomorphic to some functor $L_{E,V}$. We first deal with the case of projective functors.

**7.3. Lemma.** *Suppose that a correspondence functor $M$ has bounded type and let $E$ be a finite set such that $M$ is generated by $M(E)$. If $M$ is projective, then for any finite set $F$ with $|F| \geq |E|$, the $\mathcal{R}_F$-module $M(F)$ is projective, and the counit morphism $L_{F,M(F)} \to M$ is an isomorphism.*

**Proof :** By Lemma 7.2, $M$ is generated by $M(F)$. Choosing a set $B$ of generators of $M(F)$ as an $\mathcal{R}_F$-module (e.g. $B = M(F)$), we see that $M$ is also generated by $B$. As in the beginning of the proof of Proposition 6.4, we can apply Yoneda's lemma and obtain a surjective morphism $\bigoplus_{b\in B} k\mathcal{C}(-,F) \to M$. Since $M$ is projective, this morphism splits, and its evaluation at $F$ also splits as a map of $\mathcal{R}_F$-modules. Hence $M(F)$ is isomorphic to a direct summand of a free $\mathcal{R}_F$-module, that is, a projective $\mathcal{R}_F$-module.

By adjunction (Lemma 2.3), there is a morphism $\theta : L_{F,M(F)} \to M$ which, evaluated at $F$, gives the identity map of $M(F)$. As $M$ is generated by $M(F)$, it follows that $\theta$ is surjective, hence split since $M$ is projective. Let $\eta : M \to L_{F,M(F)}$ be a section of $\theta$. Since, on evaluation at $F$, we have $\theta_F = \mathrm{id}_{M(F)}$, the equation $\theta\eta = \mathrm{id}_M$ implies that, on evaluation at $F$, we get $\eta_F = \mathrm{id}_{M(F)}$. Therefore $\eta_F\theta_F = \mathrm{id}_{M(F)}$. Now $\eta\theta : L_{F,M(F)} \to L_{F,M(F)}$ corresponds by adjunction to

$$\mathrm{id}_{M(F)} = \eta_F\theta_F : M(F) \longrightarrow L_{F,M(F)}(F) = M(F)\ .$$

Therefore $\eta\theta$ must be the identity. It follows that $\eta$ and $\theta$ are mutual inverses. Thus $M \cong L_{F,M(F)}$. □

We now prove that any functor with a bounded presentation is an $L_{E,V}$, and conversely. In the case of a noetherian base ring $k$, this result will be improved in Section 10.

**7.4. Theorem.**

(a) *Suppose that a correspondence functor $M$ has a bounded presentation*

$$Q \to P \to M \to 0\ .$$

*Let $E$ be a finite set such that $P$ is generated by $P(E)$ and $Q$ is generated by $Q(E)$. Then for any finite set $F$ with $|F| \geq |E|$, the counit morphism*

$$\eta_{M,F} : L_{F,M(F)} \to M$$

*is an isomorphism.*

(b) *If $E$ is a finite set and $V$ is an $\mathcal{R}_E$-module, then the functor $L_{E,V}$ has a bounded presentation. More precisely, if*

$$W_1 \to W_0 \to V \to 0$$

*is a projective resolution of $V$ as an $\mathcal{R}_E$-module, then*

$$L_{E,W_1} \to L_{E,W_0} \to L_{E,V} \to 0$$

*is a bounded presentation of $L_{E,V}$.*

**Proof :** (a) Consider the commutative diagram

$$\begin{array}{ccccccc}
L_{F,Q(F)} & \longrightarrow & L_{F,P(F)} & \longrightarrow & L_{F,M(F)} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \eta_{Q,F}} & & \downarrow{\scriptstyle \eta_{P,F}} & & \downarrow{\scriptstyle \eta_{M,F}} & & \\
Q & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

where the vertical maps are obtained by the adjunction of Lemma 2.3. This lemma also asserts that the first row is exact. By Lemma 7.3, for any finite set $F$ with

$|F| \geq |E|$, the vertical morphisms $\eta_{Q,F}$ and $\eta_{P,F}$ are isomorphisms. Since the rows of this diagram are exact, it follows that $\eta_{M,F}$ is also an isomorphism.

(b) We use the adjunction of Lemma 2.3. Applying the right exact functor $U \mapsto L_{E,U}$ to the exact sequence $W_1 \to W_0 \to V \to 0$ gives the exact sequence

$$L_{E,W_1} \to L_{E,W_0} \to L_{E,V} \to 0 \ .$$

By Lemma 2.3, $L_{E,W_1}$ and $L_{E,W_0}$ are projective functors, since $W_1$ and $W_0$ are projective $\mathcal{R}_E$-modules. They all have bounded type since they are generated by their evaluation at $E$. ∎

Given a finite set $E$ and an $\mathcal{R}_E$-module $V$, we define an induction procedure as follows. For any finite set $F$, we define the $\mathcal{R}_F$-module

$$V\uparrow_E^F := k\mathcal{C}(F,E) \otimes_{\mathcal{R}_E} V \ .$$

Notice that, by the definition of $L_{E,V}$, we have $L_{E,V}(F) = V\uparrow_E^F$. To end this section, we mention the behavior of the functors $L_{E,V}$ under induction.

**7.5. Proposition.** *Let $E$ be a finite set and $V$ be an $\mathcal{R}_E$-module. If $F$ is a finite set with $|F| \geq |E|$, the equality $L_{E,V}(F) = V\uparrow_E^F$ induces an isomorphism of correspondence functors*

$$L_{F,V\uparrow_E^F} \cong L_{E,V} \ .$$

**Proof :** Let $M = L_{E,V}$. Then by Theorem 7.4, there exists a bounded presentation

$$Q \to P \to M \to 0$$

where $Q = L_{E,W_1}$ is generated by $Q(E)$ and $P = L_{E,W_0}$ is generated by $P(E)$. Hence by Theorem 7.4, for any finite set $F$ with $|F| \geq |E|$, the counit morphism

$$\eta_{M,F} : L_{F,M(F)} \to M$$

is an isomorphism. In other words $\eta_{M,F} : L_{F,V\uparrow_E^F} \to L_{E,V}$ is an isomorphism. ∎

## 8. Exponential behavior and finite length

One of our purposes in this section is to show that, if our base ring $k$ is a field, then every finitely generated correspondence functor has finite length. This is based on a lower bound estimate for the dimension of the evaluations of a simple functor $S_{E,R,V}$, which is proved to behave exponentially. We also prove that the exponential behavior is equivalent to finite generation.

We first need a well-known combinatorial lemma.

**8.1. Lemma.** *Let $E$ be a finite set and let $G$ be a finite set containing $E$.*

*(a) For any finite set $X$, the number $s(X,E)$ of surjective maps $\varphi : X \to E$ is equal to*

$$s(X,E) = \sum_{i=0}^{|E|} (-1)^{|E|-i} \binom{|E|}{i} \, i^{|X|} \, ,$$

*or equivalently*

$$s(X,E) = \sum_{j=0}^{|E|} (-1)^{j} \binom{|E|}{j} \, (|E|-j)^{|X|} \, .$$

*(b) More generally, for any finite set $X$, the number $ss(X,E,G)$ of all maps $\varphi : X \to G$ such that $E \subseteq \varphi(X) \subseteq G$ is equal to*

$$ss(X,E,G) = \sum_{i=0}^{|E|} (-1)^{i} \binom{|E|}{i} (|G|-i)^{|X|} \, .$$

**Proof :** (a) Up to multiplication by $|E|!$, the number $s(X,E)$ is known as a Stirling number of the second kind. Either by Formula (24a) in Section 1.4 of [**St**], or by a direct application of Möbius inversion (i.e. inclusion-exclusion principle in the present case), we have

$$s(X,E) = \sum_{B \subseteq E} (-1)^{|E-B|} |B|^{|X|} \, .$$

Setting $|B| = i$, the first formula in (a) follows.

(b) Applying (a) to each subset $A$ such that $E \subseteq A \subseteq G$, we obtain

$$\begin{aligned} ss(X,E,G) &= \sum_{E \subseteq A \subseteq G} s(X,A) \\ &= \sum_{E \subseteq A \subseteq G} \sum_{B \subseteq A} (-1)^{|A-B|} |B|^{|X|} \\ &= \sum_{B \subseteq G} \Big( \sum_{E \cup B \subseteq A \subseteq G} (-1)^{|A-B|} \Big) |B|^{|X|} \end{aligned}$$

But the inner sum is zero unless $E \cup B = G$. Therefore

$$\begin{aligned} ss(X,E,G) &= \sum_{\substack{B \subseteq G \\ E \cup B = G}} (-1)^{|G-B|} |B|^{|X|} \\ &= \sum_{C \subseteq E} (-1)^{|C|} \big( |G-C| \big)^{|X|} \, , \end{aligned}$$

where the last equality follows by setting $C = G - B$. This proves part (b). □

Now we prove our main lower bound estimate for the dimensions of the evaluations of a simple functor.

**8.2. Theorem.** *Suppose that $k$ is a field and let $S_{E,R,V}$ be a simple correspondence functor, where $E$ is a finite set, $R$ is an order on $E$, and $V$ is a simple $k\,\mathrm{Aut}(E,R)$-module. There exists a positive integer $N$ and a positive real number $c$ such that, for any finite set $X$ of cardinality at least $N$, we have*

$$c|E|^{|X|} \leq \dim\big(S_{E,R,V}(X)\big) \leq \big(2^{|E|}\big)^{|X|}\ .$$

**Proof :** Let $X$ be a finite set. We first prove the upper bound. This is easy and holds for all $X$. Since $S_{E,R,V}$ is a quotient of the representable functor $k\mathcal{C}(-,E)$, we have

$$\dim\big(S_{E,R,V}(X)\big) \leq \dim\big(k\mathcal{C}(X,E)\big) = 2^{|X\times E|} = \big(2^{|E|}\big)^{|X|}\ .$$

Now we want to find first a lower bound for the dimension of the evaluation of the fundamental correspondence functor $\mathbb{S}_{E,R}$. By Lemma 4.9,

$$\mathbb{S}_{E,R}(X) \cong k\mathcal{C}(X,E)R/\operatorname{Ker}(\overline{\omega}_{E,R})\ ,$$

where $\overline{\omega}_{E,R} : k\mathcal{C}(-,E)R \longrightarrow \mathbb{S}_{E,R}$ is the natural morphism. We consider the set

$$A = \{\Gamma_\varphi \mid \varphi \in \Phi\} \subseteq \mathcal{C}(X,E)R\ ,$$

using Notation 5.7, where $\Phi$ is the set of all maps $\varphi : X \to I^{\uparrow}(E,R)$ such that $\mathrm{Im}(\varphi) = E$. We use here Convention 5.6 which identifies $(E,R^{op})$ with the poset of irreducible elements of $I^{\uparrow}(E,R)$, so that $I^{\uparrow}(E,R)$ contains $E$ as a subset. Note that $A$ is a subset of $\mathcal{C}(X,E)R$ because $\Gamma_\varphi$ belongs to $\mathcal{C}(X,E)R$ by Lemma 5.8.

We want to prove that the image of $A$ in $k\mathcal{C}(X,E)R/\operatorname{Ker}(\overline{\omega}_{E,R})$ is linearly independent, from which we will deduce that $|A| \leq \dim\big(\mathbb{S}_{E,R}(X)\big)$. Suppose that $\sum\limits_{\varphi\in\Phi} \lambda_\varphi\Gamma_\varphi$ is zero in $\mathbb{S}_{E,R}(X)$, where $\lambda_\varphi \in k$ for every $\varphi \in \Phi$. In other words, $\sum\limits_{\varphi\in\Phi} \lambda_\varphi\Gamma_\varphi$ is in the kernel of $\overline{\omega}_{E,R}$ and, by Lemma 4.9, we obtain

$$\forall U \in R\mathcal{C}(E,X),\quad \sum_{U\Gamma_\varphi = R} \lambda_\varphi = 0\ .$$

Consider the set $\Psi$ of all maps $\psi : X \to I_{\downarrow}(E,R)$ such that $\mathrm{Im}(\psi) = E$ and choose $U = \Gamma_\psi^{op} \in \mathcal{C}(E,X)$. Since $\Gamma_\psi = \Gamma_\psi R^{op}$ by Lemma 5.8, we get $\Gamma_\psi^{op} = R\Gamma_\psi^{op}$, that is, $\Gamma_\psi^{op} \in R\mathcal{C}(E,X)$. Therefore we have

$$\forall \psi \in \Psi,\quad \sum_{\Gamma_\psi^{op}\Gamma_\varphi = R} \lambda_\varphi = 0\ .$$

Define the matrix $M$ indexed by $\Psi \times \Phi$ by

$$M_{\psi,\varphi} = \begin{cases} 1 & \text{if } \Gamma_\psi^{op}\Gamma_\varphi = R\,, \\ 0 & \text{otherwise}\ . \end{cases}$$

Note that $M$ is a square matrix because both $\Psi$ and $\Phi$ are in bijection with the set of all surjective maps $X \to E$. The previous relations yield $M\lambda = 0$, where $\lambda$ is the column vector with entries $\lambda_\varphi$, for $\varphi \in \Phi$. We are going to prove that $M$ is invertible.

Suppose $M_{\psi,\varphi} = 1$. Then, for any given $x \in X$, we can choose $e = \psi(x)$ and $f = \varphi(x)$ and we obtain $(e,x) \in \Gamma_\psi^{op}$ and $(x,f) \in \Gamma_\varphi$, hence $(e,f) \in \Gamma_\psi^{op}\Gamma_\varphi = R$. In other words $e \leq_R f$, that is, $\psi(x) \leq_R \varphi(x)$. This holds for every $x \in X$ and we obtain $\psi \leq \varphi$ (in the sense that $\psi(x) \leq_R \varphi(x)$ for all $x \in X$). Therefore

$$M_{\psi,\varphi} = 1 \quad\Longrightarrow\quad \psi \leq \varphi\ .$$

This relation $\leq$ is clearly a partial order relation on the set of all surjective maps $X \to E$ and it follows that the matrix $M$ is unitriangular (with an ordering of rows and columns compatible with this partial order). Thus $M$ is invertible (as a square matrix with integer coefficients). Since $M\lambda = 0$, it follows that the column vector $\lambda$ is zero. In other words, $\lambda_\varphi = 0$ for every $\varphi \in \Phi$, proving the linear independence of the image of $A$ in $k\mathcal{C}(X,E)R/\operatorname{Ker}(\overline{\omega}_{E,R})$. Therefore $|A| \leq \dim\big(\mathbb{S}_{E,R}(X)\big)$.

Now we need to estimate $|A|$ and, for simplicity, we write $e = |E|$ and $x = |X|$. By Lemma 8.1, we have

$$|A| = \sum_{i=0}^{e} (-1)^{e-i}\binom{e}{i}\, i^x = e^x + \sum_{i=0}^{e-1} (-1)^{e-i}\binom{e}{i}\, i^x \ .$$

Note that the second sum is negative because the number $|A|$ of surjective maps $X \to E$ is smaller than the number $e^x$ of all maps $X \to E$. Therefore

$$|A| = e^x\Big(1 + \sum_{i=0}^{e-1} (-1)^{e-i}\binom{e}{i}\, \big(\frac{i}{e}\big)^x\Big) \ .$$

Since $\frac{i}{e} \leq \frac{e-1}{e} < 1$, the sum can be made as small as we want, provided $x$ is large enough. Therefore there exists a positive integer $N$ and a positive real number $a$ such that $a\, e^x \leq |A|$ whenever $x \geq N$. In other words, for any finite set $X$ of cardinality at least $N$, we have

$$a\,|E|^{|X|} \leq |A| \leq \dim\big(\mathbb{S}_{E,R}(X)\big) \ , \tag{8.3}$$

giving a lower bound in the case of the fundamental correspondence functor $\mathbb{S}_{E,R}$.

Now we consider the right action of $k\operatorname{Aut}(E,R)$ on each evaluation $\mathbb{S}_{E,R}(X)$, in order to estimate the dimension

$$\dim\big(S_{E,R,V}(X)\big) = \dim\big(\mathbb{S}_{E,R}(X) \otimes_{k\operatorname{Aut}(E,R)} V\big) \ .$$

By Lemma 5.8, $\overline{\omega}_{E,R} : k\mathcal{C}(X,E)R \to \mathbb{S}_{E,R}(X)$ is a homomorphism of $k\operatorname{Aut}(E,R)$-modules. We claim that the $k$-subspace $kA$ generated by $A = \{\Gamma_\varphi \mid \varphi \in \Phi\}$ is a free right $k\operatorname{Aut}(E,R)$-submodule of $k\mathcal{C}(X,E)R$. Let $\sigma \in \operatorname{Aut}(E,R)$. Since

$$\Gamma_\varphi = \{(x,e) \in X \times E \mid e \leq_R \phi(x)\} \qquad \text{and} \qquad \Delta_\sigma = \{(\sigma(f), f) \mid f \in E\} \ ,$$

we obtain

$$\begin{aligned}\Gamma_\varphi\Delta_\sigma &= \{(x,f) \in X \times E \mid \sigma(f) \leq_R \phi(x)\} = \{(x,f) \in X \times E \mid f \leq_R \sigma^{-1}\phi(x)\} \\ &= \Gamma_{\sigma^{-1}\circ\varphi} \ ,\end{aligned}$$

using the fact that $\sigma \in \operatorname{Aut}(E,R)$ preserves the order relation $R$. But $\sigma^{-1}\circ\varphi \neq \varphi$ if $\sigma \neq \mathrm{id}$, because $\varphi$ is surjective onto $E$. Therefore the right action of $\operatorname{Aut}(E,R)$ on $A$ is free, proving the claim.

If we denote by $A'$ the image of $A$ in $\mathbb{S}_{E,R}(X)$, we have seen in the first part of the proof that $A'$ is linearly independent, so that $\overline{\omega}_{E,R}$ maps $kA$ isomorphically onto $kA'$ and $kA'$ is free. Now $kA'$ is an injective $k\operatorname{Aut}(E,R)$-module (because the group algebra $k\operatorname{Aut}(E,R)$ is symmetric, so that projective and injective modules coincide). Therefore

$$\mathbb{S}_{E,R}(X) = kA' \oplus Q$$

for some right $k\operatorname{Aut}(E,R)$-submodule $Q$. It follows that

$$\begin{aligned}\dim\big(S_{E,R,V}(X)\big) &= \dim\big(\mathbb{S}_{E,R}(X) \otimes_{k\operatorname{Aut}(E,R)} V\big) \\ &= \dim(kA' \otimes_{k\operatorname{Aut}(E,R)} V) + \dim(Q \otimes_{k\operatorname{Aut}(E,R)} V) \ .\end{aligned}$$

But since $kA'$ is a free $k\operatorname{Aut}(E,R)$-module, we obtain

$$\dim(kA' \otimes_{k\operatorname{Aut}(E,R)} V) = \dim(kA')\frac{\dim(V)}{\dim(k\operatorname{Aut}(E,R))} = |A|\, b \ ,$$

where we put $b := \frac{\dim(V)}{|\operatorname{Aut}(E,R)|}$ for simplicity. Therefore, from the inequality 8.3, we deduce that

$$ab\,|E|^{|X|} \leq |A|\,b = \dim(kA' \otimes_{k\operatorname{Aut}(E,R)} V) \leq \dim\big(S_{E,R,V}(X)\big)\ .$$

This produces the required lower bound for $\dim\big(S_{E,R,V}(X)\big)$ ∎

We can now characterize finite generation in terms of exponential behavior.

**8.4. Theorem.** *Let $M$ be a correspondence functor over a field $k$. The following are equivalent :*

- *(a) $M$ is finitely generated.*
- *(b) There exists positive real numbers $a,b,r$ such that $\dim(M(X)) \leq a\,b^{|X|}$ for every finite set $X$ with $|X| \geq r$.*

**Proof :** (a) ⇒ (b). Let $M$ be a quotient of $\bigoplus_{i\in I} k\mathcal{C}(-,E)$ for some finite set $E$ and some finite index set $I$. For every finite set $X$, we have

$$\dim(M(X)) \leq |I|\dim(k\mathcal{C}(X,E)) = |I|\,2^{|X\times E|} = |I|\big(2^{|E|}\big)^{|X|}\ .$$

(b) ⇒ (a). Let $P$ and $Q$ be subfunctors of $M$ such that $Q \subseteq P \subseteq M$ and $P/Q$ simple, hence $P/Q \cong S_{E,R,V}$ for some triple $(E,R,V)$. We claim that $|E|$ is bounded above. Indeed, for $|X|$ large enough, we have

$$c|E|^{|X|} \leq \dim(S_{E,R,V}(X))$$

for some $c > 0$, by Theorem 8.2, and

$$\dim(S_{E,R,V}(X)) \leq \dim(M(X)) \leq a\,b^{|X|}$$

by assumption. Therefore, whenever $|X| \geq N$ for some $N$, we have

$$c|E|^{|X|} \leq a\,b^{|X|} \qquad \text{that is,} \qquad c \leq a\big(\frac{b}{|E|}\big)^{|X|}\ .$$

Since $c > 0$, this forces $\dfrac{b}{|E|} \geq 1$ otherwise $a\big(\dfrac{b}{|E|}\big)^{|X|}$ is as small as we want. This shows the bound $|E| \leq b$, proving the claim.

For each set $E$ with $|E| \leq b$, we choose a basis $\{m_i \mid 1 \leq i \leq n_E\}$ of $M(E)$ and we use Yoneda's lemma to construct a morphism $\psi_i^E : k\mathcal{C}(-,E) \to M$ such that, on evaluation at $E$, we have $\psi_{i,E}^E(\Delta_E) = m_i$. Starting from the direct sum of $n_E$ copies of $k\mathcal{C}(-,E)$, we obtain a morphism

$$\psi^E : k\mathcal{C}(-,E)^{n_E} \to M\ ,$$

such that, on evaluation at $E$, the morphism $\psi_E^E : k\mathcal{C}(E,E)^{n_E} \to M(E)$ is surjective, because the basis of $M(E)$ is in the image. Now the sum of all such morphisms $\psi^E$ yields a morphism

$$\psi : \bigoplus_{|E|\leq b} k\mathcal{C}(-,E)^{n_E} \longrightarrow M$$

which is surjective on evaluation at every set $E$ with $|E| \leq b$.

Let $N = \operatorname{Im}(\psi)$ and suppose ab absurdo that $N \neq M$. Let $F$ be a minimal set such that $M(F)/N(F) \neq \{0\}$. Since $\psi$ is surjective on evaluation at every set $E$ with $|E| \leq b$, we must have $|F| > b$. Now $M(F)/N(F)$ is a module for the finite-dimensional algebra $\mathcal{R}_F = k\mathcal{C}(F,F)$ and, by minimality of $F$, inessential relations act by zero on $M(F)/N(F)$. Let $W$ be a simple submodule of $M(F)/N(F)$ as a module for the essential algebra $\mathcal{E}_F$. Associated with $W$, consider the simple functor $S_{F,W}$. (Actually, $W$ is parametrized by a pair $(R,V)$ and $S_{F,W} = S_{F,R,V}$ (see

Theorem 3.11), but we do not need this.) Now the module $W = S_{F,W}(F)$ is isomorphic to a subquotient of $M(F)/N(F)$. By Proposition 2.7, $S_{F,W}$ is isomorphic to a subquotient of $M/N$. By the claim proved above, we obtain $|F| \leq b$. This contradiction shows that $N = M$, that is, $\psi$ is surjective. Therefore $M$ is isomorphic, via $\psi$, to a quotient of $\bigoplus\limits_{|E|\leq b} k\mathcal{C}(-,E)^{n_E}$. By Proposition 6.4, $M$ is finitely generated. ∎

In order to prove that, over a field $k$, any finitely generated correspondence functor has finite length, we need a lemma.

**8.5. Lemma.** *Let $k$ be a field and let $M$ be a finitely generated correspondence functor over $k$.*

*(a) $M$ has a maximal subfunctor.*
*(b) Any subfunctor of $M$ is finitely generated.*

**Proof :** (a) Since $M$ is finitely generated, $M$ is generated by $M(E)$ for some finite set $E$ (Proposition 6.4). Let $N$ be a maximal submodule of $M(E)$ as a $k\mathcal{C}(E,E)$-module. Note that $N$ exists because $M(E)$ is finite-dimensional by Lemma 6.3. Then $M(E)/N$ is a simple $k\mathcal{C}(E,E)$-module. By Proposition 2.7, there exist two subfunctors $F \subseteq G \subseteq M$ such that $G/F$ is simple, $G(E) = M(E)$, and $F(E) = N$. Since $M$ is generated by $M(E)$ and $G(E) = M(E)$, we have $G = M$. Therefore, $F$ is a maximal subfunctor of $M$.

(b) Let $N$ be a subfunctor of $M$. Since $M$ is finitely generated, there exist positive numbers $a, b$ such that, for every large enough finite set $X$, we have

$$\dim(N(X)) \leq \dim(M(X)) \leq a\, b^{|X|} \ ,$$

by Theorem 8.4. The same theorem then implies that $N$ is finitely generated. ∎

Lemma 8.5 fails for other categories of functors. For instance, in the category of biset functors, the Burnside functor is finitely generated and has a maximal subfunctor which is not finitely generated (see [**Bo1**] or [**Bo2**]). Similarly, the following theorem is a specific property of the category of correspondence functors.

**8.6. Theorem.** *Let $k$ be a field and let $M$ be a finitely generated correspondence functor over $k$. Then $M$ has finite length (that is, $M$ has a finite composition series).*

**Proof :** By Lemma 8.5, $M$ has a maximal subfunctor $F_1$ and $F_1$ is again finitely generated. Then $F_1$ has a maximal subfunctor $F_2$ and $F_2$ is again finitely generated. We construct in this way a sequence of subfunctors

$$M = F_0 \supset F_1 \supset F_2 \supset \ldots \qquad (\mathbf{8.7})$$

such that $F_i/F_{i+1}$ is simple whenever $F_i \neq 0$. We claim that the sequence is finite, that is, $F_m = 0$ for some $m$.

Let $F_i/F_{i+1}$ be one simple subquotient, hence $F_i/F_{i+1} \cong S_{E,R,V}$ for some triple $(E,R,V)$. By Theorem 8.4, since $M$ is finitely generated, there exist positive numbers $a, b$ such that, for every large enough finite set $X$, we have

$$\dim(M(X)) \leq a\, b^{|X|} \ .$$

Therefore $\dim(S_{E,R,V}(X)) \leq a\, b^{|X|}$. By Theorem 8.2, there exists some constant $c > 0$ such that $c\,|E|^{|X|} \leq \dim(S_{E,R,V}(X))$ for $|X|$ large enough. So we obtain $c\,|E|^{|X|} \leq a\, b^{|X|}$ for $|X|$ large enough, hence $|E| < b$. This implies that the simple functor $F_i/F_{i+1} \cong S_{E,R,V}$ belongs to a finite set of isomorphism classes of simple functors, because there are finitely many sets $|E|$ with $|E| < b$ and, for any of them, finitely many order relations $R$ on $E$, and then in turn finitely many $k\,\mathrm{Aut}(E,R)$-simple modules $V$ (up to isomorphism).

Therefore, if the series (8.7) of subfunctors $F_i$ was infinite, then some simple functor $S_{E,R,V}$ would occur infinitely many times (up to isomorphism). But then, on evaluation at $E$, the simple $k\mathcal{C}(E,E)$-module $S_{E,R,V}(E)$ would occur infinitely many times in $M(E)$. This is impossible because $M(E)$ is finite-dimensional by Lemma 6.3. □

Theorem 8.6 was obtained independently by Gitlin [**Gi**], using a criterion for finite length proved recently by Wiltshire-Gordon [**WG**].

## 9. Projective functors and duality

This section is devoted to projective correspondence functors, mainly in the case where $k$ is a field. We use duality, which will again appear in Section 14.

Recall that, by Lemma 7.3, if a projective correspondence functor $M$ is generated by $M(E)$, then $M(X)$ is a projective $\mathcal{R}_X$-module, for every set $X$ with $|X| \geq |E|$. Recall also that, by Lemma 7.3 again, a projective correspondence functor $M$ is isomorphic to $L_{E,M(E)}$ whenever $M$ is generated by $M(E)$. Thus if we work with functors having bounded type, we can assume that projective functors have the form $L_{E,V}$ for some $\mathcal{R}_E$-module $V$. In such a case, we can also enlarge $E$ because $L_{E,V} \cong L_{F,V\uparrow_E^F}$ whenever $|F| \geq |E|$ (see Proposition 7.5).

**9.1. Lemma.** *Let $k$ be a commutative ring and consider the correspondence functor $L_{E,V}$ for some finite set $E$ and some $\mathcal{R}_E$-module $V$.*

*(a) $L_{E,V}$ is projective if and only if $V$ is a projective $\mathcal{R}_E$-module.*
*(b) $L_{E,V}$ is finitely generated projective if and only if $V$ is a finitely generated projective $\mathcal{R}_E$-module.*
*(c) $L_{E,V}$ is indecomposable projective if and only if $V$ is an indecomposable projective $\mathcal{R}_E$-module.*

**Proof :** (a) If $L_{E,V}$ is projective, then $V$ is projective by Lemma 7.3. Conversely, if $V$ is projective, then $L_{E,V}$ is projective by Lemma 2.3.

(b) If $V$ is a finitely generated $\mathcal{R}_E$-module, then $L_{E,V}$ is finitely generated by Corollary 6.5. If $L_{E,V}$ is finitely generated, then its evaluation $L_{E,V}(E) = V$ is finitely generated by Lemma 6.3.

(c) By the adjunction property of Lemma 2.3, $\mathrm{End}_{\mathcal{F}_k}(L_{E,V}) \cong \mathrm{End}_{\mathcal{R}_E}(V)$, so $L_{E,V}$ is indecomposable if and only if $V$ is indecomposable. □

Our main duality result has two aspects, which we both include in the following theorem. The notion of symmetric algebra is standard over a field and can be defined over any commutative ring as in [**Br**].

**9.2. Theorem.** *Let $E$ be a finite set.*

*(a) The representable functor $k\mathcal{C}(-,E)$ is isomorphic to its dual.*

*(b) Let $\mathcal{R}_E = k\mathcal{C}(E,E)$ be the $k$-algebra of relations on $E$. Then $\mathcal{R}_E$ is a symmetric algebra. More precisely, let $t : \mathcal{R}_E \to k$ be the $k$-linear form defined, for all basis elements $S \in \mathcal{C}(X,E)$, by the formula*

$$t(S) = \begin{cases} 1 & \text{if } R \cap \Delta_E = \emptyset \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

*Then $t$ is a symmetrizing form on $\mathcal{R}_E$, in the sense that the associated bilinear form $(a,b) \mapsto t(ab)$ is symmetric and induces an isomorphism of $(\mathcal{R}_E,\mathcal{R}_E)$-bimodules between $\mathcal{R}_E$ and its dual $\mathrm{Hom}_k(\mathcal{R}_E,k)$.*

**Proof :** (a) For every finite set $X$, consider the symmetric bilinear form

$$\langle -,- \rangle_X : k\mathcal{C}(X,E) \times k\mathcal{C}(X,E) \longrightarrow k$$

defined, for all basis elements $R, S \in \mathcal{C}(X,E)$, by the formula

$$\langle R,S \rangle = \begin{cases} 1 & \text{if } R \cap S = \emptyset \ , \\ 0 & \text{otherwise.} \end{cases}$$

Then, whenever $U \in \mathcal{C}(Y,X)$, $R \in \mathcal{C}(Y,E)$, and $S \in \mathcal{C}(X,E)$, we have

$$\begin{aligned} R \cap US = \emptyset \quad &\iff \Big( (y,x) \in U, (x,e) \in S \Rightarrow (y,e) \notin R \Big) \\ &\iff \Big( (x,y) \in U^{op}, (y,e) \in R \Rightarrow (x,e) \notin S \Big) \iff U^{op}R \cap S = \emptyset \ . \end{aligned}$$

It follows that $\langle U^{op}R, S \rangle_X = \langle R, US \rangle_Y$. In view of the definition of dual functors (Definition 3.5), this implies that the associated family of linear maps

$$\alpha_X : k\mathcal{C}(X,E) \longrightarrow k\mathcal{C}(X,E)^\natural$$

defines a morphism of correspondence functors $\alpha : k\mathcal{C}(-,E) \longrightarrow k\mathcal{C}(-,E)^\natural$.

To prove that $\alpha$ is an isomorphism, we fix $X$ and we use the complement ${}^{c}R = (X \times E) - R$, for any $R \in \mathcal{C}(X,E)$. Notice that the matrix of $\alpha_X$ relative to the canonical basis $\mathcal{C}(X,E)$ and its dual is the product of two matrices $C$ and $A$, where $C_{R,S} = 1$ if $S = {}^{c}R$ and 0 otherwise, while $A$ is the adjacency matrix of the order relation $\subseteq$. This is because $R \cap S = \emptyset$ if and only if $R \subseteq {}^{c}S$. Clearly $C$ is invertible (it has order 2) and $A$ is unitriangular, hence invertible. Therefore $\alpha_X$ is an isomorphism.

(b) Let $R, S \in \mathcal{C}(E,E)$. Then $t(RS)$ is equal to 1 if $RS \cap \Delta_E = \emptyset$, and $t(RS) = 0$ otherwise. Now

$$\begin{aligned} RS \cap \Delta_E = \emptyset \quad &\iff \quad \forall (e,f) \in E \times E, \ (e,f) \in R \ \text{ implies } \ (f,e) \notin S \\ &\iff \quad R \cap S^{op} = \emptyset \end{aligned}$$

where $S^{op}$ is the opposite relation to $S$. Therefore $t(RS) = \langle R, S^{op} \rangle$, where $\langle -,- \rangle_E$ is the bilinear form on $k\mathcal{C}(E,E)$ defined in (a). Since this bilinear form induces an isomorphism with the dual and since the map $S \mapsto S^{op}$ is an isomorphism (it has order 2), the bilinear form associated with $t$ induces also an isomorphism with the dual.

Since $(R \cap S^{op})^{op} = S \cap R^{op}$ and $\emptyset^{op} = \emptyset$, we have $t(RS) = t(SR)$ for any relations $R$ and $S$ on $E$, hence the bilinear form $(a,b) \mapsto t(ab)$ is symmetric. It is clear that the associated $k$-linear map $\mathcal{R}_E \to \mathrm{Hom}_k(\mathcal{R}_E,k)$ is a morphism of $(\mathcal{R}_E,\mathcal{R}_E)$-bimodules. □

**9.3. Corollary.** *If $k$ is a field, then the correspondence functor $k\mathcal{C}(-,E)$ is both projective and injective.*

**Proof :** Since passing to the dual reverses arrows and since $k\mathcal{C}(-,E)$ is projective, its dual is injective. But $k\mathcal{C}(-,E)$ is isomorphic to its dual, so it is both projective and injective. □

**9.4. Remark.** Corollary 9.3 holds more generally when $k$ is a self-injective ring (see Corollary 14.11).

**9.5. Remark.** If $R$ is an order relation on $E$, then there is a direct sum decomposition

$$k\mathcal{C}(-,E) = k\mathcal{C}(-,E)R \oplus k\mathcal{C}(-,E)(1-R)\,.$$

With respect to the bilinear forms defined in the proof of Theorem 9.2, we have

$$\big(k\mathcal{C}(-,E)R\big)^{\perp} = k\mathcal{C}(-,E)(1-R^{op})$$

because

$$\begin{aligned} U \in \big(k\mathcal{C}(X,E)R\big)^{\perp} &\iff \big\langle U, VR\big\rangle_X = 0 \ \ \forall V \in k\mathcal{C}(X,E)\\ &\iff \big\langle UR^{op}, V\big\rangle_X = 0 \ \ \forall V \in k\mathcal{C}(X,E)\\ &\iff UR^{op} = 0\\ &\iff U(1-R^{op}) = U\\ &\iff U \in k\mathcal{C}(X,E)(1-R^{op})\ . \end{aligned}$$

It follows that

$$k\mathcal{C}(-,E)/\big(k\mathcal{C}(-,E)R\big)^{\perp} = k\mathcal{C}(-,E)/k\mathcal{C}(-,E)(1-R^{op}) \cong k\mathcal{C}(-,E)R^{op}$$

and therefore the bilinear forms $\big\langle -,-\big\rangle_X$ induce perfect pairings

$$k\mathcal{C}(-,E)R \times k\mathcal{C}(-,E)R^{op} \longrightarrow k\ .$$

Thus $\big(k\mathcal{C}(-,E)R\big)^{\natural} \cong k\mathcal{C}(-,E)R^{op}$.

By the Krull-Remak-Schmidt theorem (which holds when $k$ is a field by Proposition 6.6), it is no harm to assume that our functors are indecomposable.

**9.6. Theorem.** *Let $k$ be a field and $M$ be a finitely generated correspondence functor over $k$. The following conditions are equivalent:*

(a) *The functor $M$ is projective and indecomposable.*
(b) *The functor $M$ is projective and admits a unique maximal (proper) subfunctor.*
(c) *The functor $M$ is projective and admits a unique minimal (nonzero) subfunctor.*
(d) *The functor $M$ is injective and indecomposable.*
(e) *The functor $M$ is injective and admits a unique maximal (proper) subfunctor.*
(f) *The functor $M$ is injective and admits a unique minimal (nonzero) subfunctor.*

**Proof :** (a) ⇒ (b). Suppose first that $M$ is projective and indecomposable. Then $M \cong L_{E,V}$ for some finite set $E$ and some indecomposable projective $k\mathcal{C}(E,E)$-module $V$ (Lemma 7.3). Since $k\mathcal{C}(E,E)$ is a finite dimensional algebra over $k$, the module $V$ has a unique maximal submodule $W$. If $N$ is a subfunctor of $M$, then $N(E)$ is a submodule of $V$, so there are two cases: either $N(E) = V$, and then $N = M$, because $M$ is generated by $M(E) = V$, or $N(E) \subseteq W$, and then $N(X) \subseteq J_W(X)$ for any finite set $X$, where

$$J_W(X) = \Big\{ \sum_i \varphi_i \otimes v_i \in k\mathcal{C}(X,E) \otimes_{k\mathcal{C}(E,E)} V \mid \forall \psi \in \mathcal{C}(E,X),\ \sum_i (\psi\varphi_i)\cdot v_i \in W \Big\} .$$

One ckecks easily that the assignment $X \mapsto J_W(X)$ is a subfunctor of $L_{E,V}$, such that $J_W(E) = W$ after the identification $L_{E,V}(E) \cong V$. (This subfunctor is similar to the one introduced in Lemma 2.4.) In particular $J_W$ is a proper subfunctor of $L_{E,V}$. It follows that $J_W$ is the unique maximal proper subfunctor of $L_{E,V}$, as it contains any proper subfunctor $N$ of $L_{E,V}$.

(b) $\Rightarrow$ (a). Suppose that $M$ admits a unique maximal subfunctor $N$. If $M$ splits as a direct sum $M_1 \oplus M_2$ of two nonzero subfunctors $M_1$ and $M_2$, then $M_1$ and $M_2$ are finitely generated. Let $N_1$ be a maximal subfunctor of $M_1$, and $N_2$ be a maximal subfunctor of $M_2$. Such subfunctors exist by Lemma 8.5. Then $M_1 \oplus N_2$ and $N_1 \oplus M_2$ are maximal subfunctors of $M$. This contradiction proves that $M$ is indecomposable.

(a) $\Rightarrow$ (d). If $M$ is a finitely generated projective functor, then there exists a finite set $E$ such that $M$ is isomorphic to a quotient, hence a direct summand, of $\bigoplus_{i\in I} k\mathcal{C}(-,E)$ for some finite set $I$ (Proposition 6.4). Since $k$ is a field, $k\mathcal{C}(-,E)$ is an injective functor (Corollary 9.3), hence so is the direct sum and its direct summand $M$.

(d) $\Rightarrow$ (a). If $M$ is a finitely generated injective functor, then its dual $M^\natural$ is projective, hence injective, and therefore $M \cong (M^\natural)^\natural$ is projective.

(a) $\Rightarrow$ (c). For a finitely generated functor $M$, the duality between $M$ and $M^\natural$ induces an order reversing bijection between the subfunctors of $M$ and the subfunctors of $M^\natural$. If $M$ is projective and indecomposable, then so is $M^\natural$, that is, (a) holds for $M^\natural$. Thus (b) holds for $M^\natural$ and the functor $M^\natural$ has a unique maximal subfunctor. Hence $M$ has a unique minimal subfunctor.

(c) $\Rightarrow$ (a). If $M$ is projective and admits a unique minimal subfunctor, then $M$ is also injective, and its dual $M^\natural$ is projective and admits a unique maximal subfunctor. Hence $M^\natural$ is indecomposable, so $M$ is indecomposable.

It is now clear that (e) and (f) are both equivalent to (a), (b), (c) and (d). □

Finally, we prove that the well-known property of indecomposable projective modules over a symmetric algebra also holds for correspondence functors.

**9.7. Theorem.** *Let $k$ be a field.*

*(a) Let $M$ be a finitely generated projective correspondence functor over $k$. Then $M/\operatorname{Rad}(M) \cong \operatorname{Soc}(M)$.*

*(b) Let $M$ and $N$ be finitely generated correspondence functors over $k$. If $M$ is projective, then $\dim_k \operatorname{Hom}_{\mathcal{F}_k}(M,N) = \dim_k \operatorname{Hom}_{\mathcal{F}_k}(N,M) < +\infty$.*

**Proof :** (a) By Proposition 6.6, we can assume that $M$ is indecomposable. In this case, by Theorem 9.6, both $M/\operatorname{Rad}(M)$ and $\operatorname{Soc}(M)$ are simple functors. By Proposition 6.4, there is a finite set $E$ such that $M$ is a quotient, hence a direct summand, of $F = \bigoplus_{i\in I} k\mathcal{C}(-,E)$ for some finite set $I$. Since $k\mathcal{C}(-,E)^\natural \cong k\mathcal{C}(-,E)$, the dual $M^\natural$ is a direct summand of $F^\natural \cong F$, and both $M$ and $M^\natural$ are generated by their evaluations at $E$. Thus $M \cong L_{E,M(E)}$ and $M^\natural \cong L_{E,M^\natural(E)}$, by Lemma 7.3. As $M$ is a direct summand of $F$ and $M$ is indecomposable, $M$ is a direct summand of $k\mathcal{C}(-,E)$, by the Krull-Remak-Schmidt Theorem (Proposition 6.6). So there is a primitive idempotent $e$ of $k\mathcal{C}(E,E) \cong \mathrm{End}_{\mathcal{F}_k}\big(k\mathcal{C}(-,E)\big)$ such that $M \cong k\mathcal{C}(-,E)e$, and we can assume that $M = k\mathcal{C}(-,E)e$.

If $V$ is a finite dimensional $k$-vector space, and $W$ is a subspace of $V$, set

$$W^{\perp} = \{\varphi \in \mathrm{Hom}_k(V,k) \mid \varphi(W) = 0\} \ .$$

If $N$ is a subfunctor of $M$, the assignment sending a finite set $X$ to $N(X)^{\perp}$ defines a subfunctor $N^{\perp}$ of $M^{\natural}$, and moreover $N \mapsto N^{\perp}$ is an order reversing bijection between the set of subfunctors of $M$ and the set of subfunctors of $M^{\natural}$. In particular $\mathrm{Soc}(M)^{\perp} = \mathrm{Rad}(M^{\natural})$. Hence $\mathrm{Soc}(M)^{\perp}(E) = \big(\mathrm{Soc}(M)(E)\big)^{\perp} = \mathrm{Rad}(M^{\natural})(E)$.

Now $M^{\natural} \neq \mathrm{Rad}(M^{\natural})$, and $M^{\natural}$ is generated by $M^{\natural}(E)$. Hence $\mathrm{Rad}(M^{\natural})(E) \neq M^{\natural}(E)$. It follows that $\mathrm{Soc}(M)(E) \neq 0$. Then $\mathrm{Soc}(M)(E) \subseteq k\mathcal{C}(E,E)e$, and $\mathrm{Soc}(M)(E)$ is a left ideal of $k\mathcal{C}(E,E)$. It follows that $\mathrm{Soc}(M)(E)$ is not contained in the kernel of the map $t$ defined in Theorem 9.2, that is $t\big(\mathrm{Soc}(M)(E)\big) \neq 0$. Hence

$$0 \neq t\big(\mathrm{Soc}(M)(E)\big) = t\big(\mathrm{Soc}(M)(E)e\big) = t\big(e\,\mathrm{Soc}(M)(E)\big) \ ,$$

and in particular $e\,\mathrm{Soc}(M)(E) \neq 0$. Since

$$e\,\mathrm{Soc}(M)(E) \cong \mathrm{Hom}_{\mathcal{F}_k}\big(k\mathcal{C}(-,E)e, \mathrm{Soc}(M)\big) \ ,$$

there is a nonzero morphism from $M = k\mathcal{C}(-,E)e$ to $\mathrm{Soc}(M)$, hence a nonzero morphism from $M/\mathrm{Rad}(M)$ to $\mathrm{Soc}(M)$. Since $M/\mathrm{Rad}(M)$ and $\mathrm{Soc}(M)$ are simple, it is an isomorphism.

(b) First, by Proposition 6.6, both $\mathrm{Hom}_{\mathcal{F}_k}(M,N)$ and $\mathrm{Hom}_{\mathcal{F}_k}(N,M)$ are finite dimensional $k$-vectors spaces.

Now we can again assume that $M$ is an indecomposable projective and injective functor. For a finitely generated functor $N$, set $\alpha(N) = \dim_k \mathrm{Hom}_{\mathcal{F}_k}(M,N)$ and $\beta(N) = \dim_k \mathrm{Hom}_{\mathcal{F}_k}(N,M)$. If $0 \to N_1 \to N_2 \to N_3 \to 0$ is a short exact sequence of finitely generated functors, then $\alpha(N_2) = \alpha(N_1) + \alpha(N_3)$ because $M$ is projective, and $\beta(N_2) = \beta(N_1) + \beta(N_3)$ because $M$ is injective. So, in order to prove (b), as $N$ has finite length, it is enough to assume that $N$ is simple. In that case $\alpha(N) = \dim_k \mathrm{End}_{\mathcal{F}_k}(N)$ if $M/\mathrm{Rad}(M) \cong N$, and $\alpha(N) = 0$ otherwise. Similarly $\beta(N) = \dim_k \mathrm{End}_{\mathcal{F}_k}(N)$ if $\mathrm{Soc}(M) \cong N$, and $\beta(N) = 0$ otherwise. Hence (b) follows from (a). □

## 10. The noetherian case

In this section, we shall assume that the ground ring $k$ is noetherian, in which case we obtain more results about subfunctors. For instance, we shall prove that any subfunctor of a finitely generated functor is finitely generated. It would be interesting to see if the methods developed recently by Sam and Snowden [**SS**] for showing noetherian properties of representations of categories can be applied for proving the results of this section.

Our first results hold without any assumption on $k$.

**10.1. Notation.** *Let $k$ be a commutative ring, let $E$ be a finite set, and let $M$ be a correspondence functor over $k$. We set*

$$\overline{M}(E) = M(E)/\sum_{E' \subset E} k\mathcal{C}(E,E')M(E') ,$$

*where the sum runs over proper subsets $E'$ of $E$.*

Note that if $F$ is any set of cardinality smaller than $|E|$, then there exists a bijection $\sigma : E' \to F$, where $E'$ is a proper subset of $E$. It follows that $k\mathcal{C}(E,F)M(F) = k\mathcal{C}(E,F)R_\sigma M(E') \subseteq \sum\limits_{E' \subset E} k\mathcal{C}(E,E')M(E')$, where $R_\sigma \subseteq F \times E'$ is the graph of $\sigma$.

Note also that $\overline{M}(E)$ is a left module for the essential algebra $\mathcal{E}_E$, because the ideal $I_E = \sum\limits_{|Y|<|E|} k\mathcal{C}(E,Y)k\mathcal{C}(Y,E)$ of the algebra $\mathcal{R}_E = k\mathcal{C}(E,E)$ acts by zero on $\overline{M}(E)$.

**10.2. Lemma.** *Let $k$ be a commutative ring, and let $E$ be a finite set. Let $M$ be a correspondence functor over $k$. If $\mathfrak{p}$ is a prime ideal of $k$, denote by $M_\mathfrak{p}$ the localization of $M$ at $\mathfrak{p}$, defined by $M_\mathfrak{p}(E) = M(E)_\mathfrak{p}$ for every finite set $E$.*

*(a) $M_\mathfrak{p}$ is a correspondence functor over the localization $k_\mathfrak{p}$.*
*(b) If $M$ is finitely generated over $k$, then $M_\mathfrak{p}$ is finitely generated over $k_\mathfrak{p}$.*
*(c) For each finite set $E$, there is an isomorphism of $k_\mathfrak{p}\mathcal{C}(E,E)$-modules*

$$\overline{M}(E)_\mathfrak{p} \cong \overline{M_\mathfrak{p}}(E) .$$

**Proof :** (a) This is straightforward.

(b) If $E$ is a finite set, then clearly $k\mathcal{C}(-,E)_\mathfrak{p} \cong k_\mathfrak{p}\mathcal{C}(-,E)$, because this the localization of a free module (on every evaluation). If $M$ is finitely generated, then there is a finite set $F$ such that $M$ is a quotient of $\bigoplus\limits_{i\in I} k\mathcal{C}(-,F)$ for some finite set $I$. Then $M_\mathfrak{p}$ is a quotient of the functor $\bigoplus\limits_{i\in I} k\mathcal{C}(-,F)_\mathfrak{p} \cong \bigoplus\limits_{i\in I} k_\mathfrak{p}\mathcal{C}(-,F)$, hence it is a finitely generated functor over $k_\mathfrak{p}$.

(c) Since localization is an exact functor, the exact sequence of $k$-modules

$$\bigoplus_{E' \subset E} k\mathcal{C}(E,E')M(E') \to M(E) \to \overline{M}(E) \to 0$$

gives the exact sequence of $k_\mathfrak{p}$-modules

$$\bigoplus_{E' \subset E} \big(k\mathcal{C}(E,E')M(E')\big)_\mathfrak{p} \to M(E)_\mathfrak{p} \to \overline{M}(E)_\mathfrak{p} \to 0 .$$

Now clearly $\big(k\mathcal{C}(E,E')M(E')\big)_\mathfrak{p} = k_\mathfrak{p}\mathcal{C}(E,E')M(E')_\mathfrak{p} = k_\mathfrak{p}\mathcal{C}(E,E')M_\mathfrak{p}(E')$ for each $E' \subset E$. Hence we get an exact sequence

$$\bigoplus_{E' \subset E} k_\mathfrak{p}\mathcal{C}(E,E')M_\mathfrak{p}(E') \to M_\mathfrak{p}(E) \to \overline{M}(E)_\mathfrak{p} \to 0 ,$$

and it follows that $\overline{M}(E)_\mathfrak{p} \cong \overline{M_\mathfrak{p}}(E)$. □

**10.3. Proposition.** *Let $k$ be a commutative ring, let $E$ be a finite set, and let $M$ be a correspondence functor such that $\overline{M}(E) \neq 0$.*

(a) *There exists a prime ideal $\mathfrak{p}$ of $k$ such that $\overline{M_\mathfrak{p}}(E) \neq 0$.*
(b) *If moreover $M(E)$ is a finitely generated $k$-module, then there exist subfunctors $A$ and $B$ of $M_\mathfrak{p}$ such that $\mathfrak{p}M_\mathfrak{p} \subseteq A \subset B$, and a simple module $V$ for the essential algebra $\mathcal{E}_E$ of $E$ over $k(\mathfrak{p})$ such that $B/A \cong S_{E,V}$, where $k(\mathfrak{p}) = k_\mathfrak{p}/\mathfrak{p}k_\mathfrak{p}$.*
(c) *In this case, there exist positive numbers $c$ and $d$ such that*

$$c\,|E|^{|X|} \leq \dim_{k(\mathfrak{p})}(M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p})(X)$$

*whenever $X$ is a finite set such that $|X| \geq d$.*

**Proof :** (a) This follows from the well-known fact that the localization map $\overline{M}(E) \longrightarrow \prod_{\mathfrak{p}\in\mathrm{Spec}(k)} \overline{M}(E)_\mathfrak{p}$ is injective, and from the isomorphism $\overline{M}(E)_\mathfrak{p} \cong \overline{M_\mathfrak{p}}(E)$ of Lemma 10.2.

(b) Set $N = M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ where $\mathfrak{p}$ is the prime ideal obtained in (a). Then $N$ is a correspondence functor over $k(\mathfrak{p})$. Suppose that $\overline{N}(E) = 0$. Then

$$M_\mathfrak{p}(E) = \mathfrak{p}M_\mathfrak{p}(E) + \sum_{E'\subset E} k_\mathfrak{p}\mathcal{C}(E,E')M_\mathfrak{p}(E') \ .$$

Since $M(E)$ is a finitely generated $k$-module, $M_\mathfrak{p}(E)$ is a finitely generated $k_\mathfrak{p}$-module, and Nakayama's lemma implies that

$$M_\mathfrak{p}(E) = \sum_{E'\subset E} k_\mathfrak{p}\mathcal{C}(E,E')M_\mathfrak{p}(E') \ ,$$

that is, $\overline{M_\mathfrak{p}}(E) = 0$. This contradicts (a) and shows that $\overline{N}(E) \neq 0$.

Now $\overline{N}(E)$ is a nonzero module for the essential algebra $\mathcal{E}_E$ of $E$ over $k(\mathfrak{p})$, and it is finite dimensional over $k(\mathfrak{p})$ (because $M_\mathfrak{p}(E)$ is a finitely generated $k_\mathfrak{p}$-module). Hence it admits a simple quotient $V$ as $\mathcal{E}_E$-module. Then $V$ can be viewed as a simple $k(\mathfrak{p})\mathcal{R}_E$-module by inflation, and it is also a quotient of $N(E)$. By Proposition 2.7, there exist subfunctors $A/\mathfrak{p}M_\mathfrak{p} \subset B/\mathfrak{p}M_\mathfrak{p}$ of $N$ such that $B/A$ is isomorphic to the simple functor $S_{E,V}$, proving (b).

(c) By (b) and Theorem 8.2, there exist positive numbers $c$ and $d$ such that

$$c\,|E|^{|X|} \leq \dim_{k(\mathfrak{p})}(B/A)(X)$$

whenever $X$ is a finite set such that $|X| \geq d$. Assertion (c) follows. □

Now we assume that $k$ is noetherian and we can state the critical result.

**10.4. Theorem.** *Let $k$ be a commutative noetherian ring. Let $N$ be a subfunctor of a correspondence functor $M$ over $k$. If $E$ and $Y$ are finite sets such that $M$ is generated by $M(E)$ and $\overline{N}(Y) \neq 0$, then $|Y| \leq 2^{|E|}$.*

**Proof :** Since $M$ is generated by $M(E)$, choosing a set $I$ of generators of $M(E)$ yields a surjection $\Phi : P = \bigoplus_{i\in I} k\mathcal{C}(-,E) \to M$. Let $L = \Phi^{-1}(N)$. Since $\Phi$ induces a surjection $\overline{L}(Y) \to \overline{N}(Y)$, and since $P$ is generated by $P(E)$, we can replace $M$ by $P$ and $N$ by $L$. Hence we now assume that $N$ is a subfunctor of $\bigoplus_{i\in I} k\mathcal{C}(-,E)$.

Since $\overline{N}(Y) \neq 0$, there exists

$$m \in N(Y) - \sum_{Y'\subset Y} k\mathcal{C}(Y,Y')N(Y') \ .$$

Let $N'$ be the subfunctor of $N$ generated by $m$. Then clearly $\overline{N'}(Y) \neq 0$, because

$$m \in N'(Y) - \sum_{Y' \subset Y} k\mathcal{C}(Y, Y')N'(Y') \ .$$

Moreover $N'(Y) = k\mathcal{C}(Y, Y)m$ is a finitely generated $k$-module, and there is a finite subset $S$ of $I$ such that $m \in \bigoplus_{i \in S} k\mathcal{C}(Y, E)$. Therefore $N' \subseteq \bigoplus_{i \in S} k\mathcal{C}(-, E)$. Replacing $N$ by $N'$, we can assume moreover that the set $I$ is finite. In other words, there exists an integer $s \in \mathbb{N}$ such that $N \subseteq k\mathcal{C}(-, E)^{\oplus s}$.

Now by Proposition 10.3, there exists a prime ideal $\mathfrak{p}$ of $k$ such that $\overline{N_\mathfrak{p}}(Y) \neq 0$. Moreover $N(Y)$ is a submodule of $k\mathcal{C}(Y, E)^{\oplus s}$, which is a finitely generated (free) $k$-module. Since $k$ is noetherian, it follows that $N(Y)$ is a finitely generated $k$-module.

By Proposition 10.3, there exist subfunctors $A \subset B$ of $N_\mathfrak{p}$ such that $B/A$ is isomorphic to a simple functor of the form $S_{Y,V}$, where $V$ is a simple module for the essential algebra of $Y$ over $k(\mathfrak{p})$. In particular $Y$ is minimal such that $(B/A)(Y) \neq 0$, thus $(\overline{B/A})(Y) \cong (B/A)(Y) \cong V$.

It follows that $\overline{B}(Y) \neq 0$, and $B$ is a subfunctor of $k_\mathfrak{p}\mathcal{C}(-, E)^{\oplus s}$. In other words, replacing $k$ by $k_\mathfrak{p}$ and $N$ by $B$, we can assume that $k$ is a noetherian local ring, that $\mathfrak{p}$ is the unique maximal ideal of $k$, and that $N$ has a subfunctor $A$ such that $N/A$ is isomorphic to $S_{Y,V}$, where $V$ is a simple module for the essential algebra $\mathcal{E}_Y$ over $k/\mathfrak{p}$.

We claim that there exists an integer $n \in \mathbb{N}$ such that

$$N(Y) \neq A(Y) + \big(\mathfrak{p}^n\mathcal{C}(Y, E)^{\oplus s} \cap N(Y)\big) \ .$$

Indeed $N(Y)$ is a submodule of the finitely generated $k$-module $k\mathcal{C}(Y, E)^{\oplus s}$. By the Artin-Rees lemma (see Theorem 8.5 in [**Ma**]), there exists an integer $l \in \mathbb{N}$ such that for any $n \geq l$

$$\mathfrak{p}^n\mathcal{C}(Y, E)^{\oplus s} \cap N(Y) = \mathfrak{p}^{n-l}\big(\mathfrak{p}^l\mathcal{C}(Y, E)^{\oplus s} \cap N(Y)\big) \ .$$

Let $m_1, \ldots, m_r$ be generators of $N(Y)$ as a $k$-module. Suppose that $n > l$ and that $N(Y) = A(Y) + \big(\mathfrak{p}^n\mathcal{C}(Y, E)^{\oplus s} \cap N(Y)\big)$. Then

$$N(Y) = A(Y) + \mathfrak{p}^{n-l}\big(\mathfrak{p}^l\mathcal{C}(Y, E)^{\oplus s} \cap N(Y)\big) \ .$$

It follows that for each $i \in \{1, \ldots, r\}$, there exist $a_i \in A(Y)$ and scalars $\lambda_{i,j} \in \mathfrak{p}^{n-l}$, for $1 \leq j \leq r$, such that

$$m_i = a_i + \sum_{j=1}^{r} \lambda_{i,j} m_j \ .$$

In other words the sequence $(a_i)_{i=1,\ldots,r}$ is the image of the sequence $(m_i)_{i=1,\ldots,r}$ under the matrix $J = \mathrm{id}_r - \Lambda$, where $\Lambda$ is the matrix of coefficients $\lambda_{i,j}$, and $\mathrm{id}_r$ is the identity matrix of size $r$. Since $\Lambda$ has coefficients in $\mathfrak{p}^{n-l} \subseteq \mathfrak{p}$, the determinant of $J$ is congruent to 1 modulo $\mathfrak{p}$, hence $J$ is invertible. It follows that the $m_i$'s are linear combinations of the $a_j$'s with coefficients in $k$. Hence $m_i \in A(Y)$ for $1 \leq i \leq r$, thus $N(Y) = A(Y)$. This is a contradiction since $(N/A)(Y) \cong V \neq 0$. This proves our claim.

We have obtained that $N \neq A + \big(\mathfrak{p}^n\mathcal{C}(-, E)^{\oplus s} \cap N\big)$. Since $N/A$ is simple, it follows that $\mathfrak{p}^n\mathcal{C}(-, E)^{\oplus s} \cap N = \mathfrak{p}^n\mathcal{C}(-, E)^{\oplus s} \cap A$.

Now we reduce modulo $\mathfrak{p}^n$ and we let respectively $\hat{A}$ and $\hat{N}$ denote the images of $A$ and $N$ in the reduction $(k/\mathfrak{p}^n)\mathcal{C}(-,E)^{\oplus s}$. Then

$$\begin{aligned}\hat{N}/\hat{A} &= \Big(\big(N+\mathfrak{p}^n\mathcal{C}(-,E)^{\oplus s}\big)/\mathfrak{p}^n\mathcal{C}(-,E)^{\oplus s}\Big)\Big/\Big(\big(A+\mathfrak{p}^n\mathcal{C}(-,E)^{\oplus s}\big)/\mathfrak{p}^n\mathcal{C}(-,E)^{\oplus s}\Big)\\ &\cong \Big(N/\big(\mathfrak{p}^n\mathcal{C}(-,E)^{\oplus s}\cap N\big)\Big)\Big/\Big(A/\big(\mathfrak{p}^n\mathcal{C}(-,E)^{\oplus s}\cap A\big)\Big)\\ &\cong N/A\ ,\end{aligned}$$

and this is isomorphic to the simple functor $S_{Y,V}$ over the field $k/\mathfrak{p}$. Hence for any finite set $X$, the module $\hat{N}(X)/\hat{A}(X)$ is a $(k/\mathfrak{p})$-vector space. Moreover, by Proposition 10.3, there exist positive numbers $c$ and $d$ such that the dimension of this vector space is larger than $c\,|Y|^{|X|}$ whenever $|X|\geq d$.

Now for any finite set $X$, the module $(k/\mathfrak{p}^n)\mathcal{C}(X,E)^{\oplus s}$ is filtered by the submodules $\Gamma_j=(\mathfrak{p}^j/\mathfrak{p}^n)\mathcal{C}(X,E)^{\oplus s}$, for $j=1,\ldots,n-1$, and the quotient $\Gamma_j/\Gamma_{j+1}$ is a vector space over $k/\mathfrak{p}$, of dimension $sd_j\,2^{|X||E|}$, where $d_j=\dim_{k/\mathfrak{p}}(\mathfrak{p}^j/\mathfrak{p}^{j+1})$. It follows that, for $|X|\geq d$,

$$c\,|Y|^{|X|}\leq \dim_{k/\mathfrak{p}}\big(\hat{N}(X)/\hat{A}(X)\big)\leq s\big(\sum_{j=0}^{n-1}d_j\big)2^{|X||E|}\ .$$

As $|X|$ tends to infinity, this forces $|Y|\leq 2^{|E|}$, completing the proof of Theorem 10.4. □

**10.5. Corollary.** *Let $k$ be a commutative noetherian ring and let $N$ be a subfunctor of a correspondence functor $M$ over $k$.*

- *(a) If $E$ is a finite set such that $M$ is generated by $M(E)$ and if $F$ is a finite set with $|F|\geq 2^{|E|}$, then $N$ is generated by $N(F)$.*
- *(b) If $M$ has bounded type, then $N$ has bounded type. In particular, over $k$, any correspondence functor of bounded type has a bounded presentation.*
- *(c) If $M$ is finitely generated, then $N$ is finitely generated. In particular, over $k$, any finitely generated correspondence functor is finitely presented.*

**Proof :** (a) Let $E$ be a finite set such that $M$ is generated by $M(E)$. If $X$ is a finite set such that $\overline{N}(X)\neq 0$, then $|X|\leq 2^{|E|}$, by Theorem 10.4. For each integer $e\leq 2^{|E|}$, let $[e]=\{1,\ldots,e\}$ and choose a subset $S_e$ of $N([e])$ which maps to a generating set of $\overline{N}([e])$ as a $k$-module. Each $i\in S_e$ yields a morphism $\psi_i:k\mathcal{C}(-,[e])\to N$. Let

$$Q=\bigoplus_{e\leq 2^{|E|}}\bigoplus_{i\in S_e}k\mathcal{C}(-,[e])\qquad\text{and}\qquad \Psi=\sum_{\substack{e\leq 2^{|E|}\\ i\in S_e}}\psi_i:Q\to N\ .$$

Then by construction the induced map

$$\overline{\Psi}_X:\overline{Q}(X)\to\overline{N}(X)$$

is surjective, for any finite set $X$, because either $\overline{N}(X)=0$ or $|X|=e\leq 2^{|E|}$. Suppose that $\Psi:Q\to N$ is not surjective and let $A$ be a set of minimal cardinality such that $\Psi_A:Q(A)\to N(A)$ is not surjective. Let $l\in N(A)-\operatorname{Im}\Psi_A$. Since the map $\overline{\Psi}_A$ is surjective, there is an element $q\in Q(A)$ and elements $l_e\in N([e])$ and $R_e\in\mathcal{C}(A,[e])$, for $e<|A|$, such that

$$l=\Psi_A(q)+\sum_{e<|A|}R_el_e\ .$$

The minimality of $A$ implies that the map $\Psi_{[e]} : Q([e]) \to N([e])$ is surjective for each $e < |A|$, so there are elements $q_e \in Q([e])$, for $e < |A|$, such that $\Psi_{[e]}(q_e) = l_e$. It follows that $l = \Psi_A\big(q + \sum\limits_{e<|A|} R_e q_e\big)$, thus $l \in \operatorname{Im}\Psi_A$. This contradiction proves that the morphism $\Psi : Q \to N$ is surjective.

Now let $F$ be a set with $|F| \geq 2^{|E|}$. For each $e \leq 2^{|E|}$, the representable functor $k\mathcal{C}(-,[e])$ is generated by its evaluation at $[e]$, hence also by its evaluation at $F$, because $k\mathcal{C}(-,[e])$ is a direct summand of $k\mathcal{C}(-,F)$ by Corollary 4.2. Therefore $Q$ is generated by $Q(F)$. Since $\Psi : Q \to N$ is surjective, it follows that $N$ is generated by $N(F)$.

(b) This follows clearly from (a).

(c) If now $M$ is finitely generated, then the same argument applies, but we can assume moreover that all the set $S_e$ appearing in the proof of (a) are *finite*, since for any finite set $X$, the module $N(X)$ is finitely generated, being a submodule of the finitely generated module $M(X)$. It follows that the functor $Q$ of the proof of (a) is finitely generated and this proves (c). □

It follows from (b) and Theorem 7.4 that, whenever $k$ is noetherian, any correspondence functor of bounded type is isomorphic to $L_{F,V}$ for some $F$ and $V$. We shall return to this in Theorem 10.9 below.

**10.6. Notation.** *We denote by $\mathcal{F}_k^b$ the full subcategory of $\mathcal{F}_k$ consisting of correspondence functors having bounded type and by $\mathcal{F}_k^f$ the full subcategory of $\mathcal{F}_k$ consisting of finitely generated functors.*

**10.7. Corollary.** *Let $k$ be a commutative noetherian ring. Then the categories $\mathcal{F}_k^b$ and $\mathcal{F}_k^f$ are abelian full subcategories of $\mathcal{F}_k$.*

**Proof :** Any quotient of a functor of bounded type has bounded type and any quotient of a finitely generated functor is finitely generated. When $k$ is noetherian, any subfunctor of a functor of bounded type has bounded type and any subfunctor of a finitely generated functor is finitely generated, by Corollary 10.5. □

Recall from Lemma 2.4 that for any finite set $E$ and any $\mathcal{R}_E$-module $V$, we have defined a subfunctor $J_{E,V}$ of $L_{E,V}$ by setting

$$J_{E,V}(X) = \Big\{ \sum_i R_i \underset{\mathcal{R}_E}{\otimes} v_i \mid R_i \in \mathcal{C}(X,E),\ v_i \in V,\ \forall S \in \mathcal{C}(E,X),\ \sum_i (SR_i)v_i = 0 \Big\}\,.$$

Moreover $J_{E,V}(E) = 0$ and $S_{E,V} = L_{E,V}/J_{E,V}$.

We have seen in Proposition 7.5 that $L_{F,V\uparrow_E^F} \cong L_{E,V}$ whenever $|F| \geq |E|$. The subfunctor $J_{F,V\uparrow_E^F}$ vanishes at $F$, hence also at $E$, so that $J_{F,V\uparrow_E^F} \subset J_{E,V}$. When $k$ is noetherian, we show that this decreasing sequence reaches zero.

**10.8. Theorem.** *Let $k$ be a commutative noetherian ring, let $E$ be a finite set, and let $V$ be an $\mathcal{R}_E$-module. For any finite set $F$ such that $|F| \geq 2^{|E|}$, we have $J_{F,V\uparrow_E^F} = 0$.*

**Proof :** Let $F$ be a finite set. By Proposition 7.5, there is an isomorphism $L_{F,V\uparrow_E^F} \cong L_{E,V}$. Thus $J_{F,V\uparrow_E^F}$ is isomorphic to a subfunctor of $L_{E,V}$. Since $J_{F,V\uparrow_E^F}(F) = 0$, it follows from Corollary 4.4 that $J_{F,V\uparrow_E^F}(X) = 0$ for any finite set $X$ with $|X| \leq |F|$.

We now assume that $J_{F,V\uparrow_E^F} \neq 0$ and we prove that $|F| < 2^{|E|}$. Let $Y$ be a set of minimal cardinality such that $J_{F,V\uparrow_E^F}(Y) \neq 0$. Then $|Y| > |F|$. Moreover $\overline{J_{F,V\uparrow_E^F}}(Y) \cong J_{F,V\uparrow_E^F}(Y) \neq 0$, hence $|Y| \leq 2^{|E|}$ by Theorem 10.4, because $J_{F,V\uparrow_E^F}$

is (isomorphic to) a subfunctor of $L_{E,V}$, which is generated by $L_{E,V}(E) = V$. It follows that $|F| < |Y| \leq 2^{|E|}$. □

We now show that, over a noetherian ring, any correspondence functor of bounded type is isomorphic to $L_{F,V}$ for some $F$ and $V$, or also isomorphic to $S_{G,W}$ for some $G$ and $W$ (where the symbol $S_{G,W}$ refers to Notation 2.5).

**10.9. Theorem.** *Let $k$ be a commutative noetherian ring. Let $M$ be a correspondence functor over $k$ generated by $M(E)$, for some finite set $E$.*

*(a) For any finite set $F$ such that $|F| \geq 2^{|E|}$, the counit morphism $\eta_{M,F} : L_{F,M(F)} \to M$ is an isomorphism.*
*(b) For any finite set $G$ such that $|G| \geq 2^{2^{|E|}}$, we have $M \cong L_{G,M(G)}$ and $J_{G,M(G)} = 0$, hence $M \cong S_{G,M(G)}$.*

**Proof :** (a) If $M$ is generated by $M(E)$, then there is a set $I$ and a surjective morphism $P = \bigoplus_{i \in I} k\mathcal{C}(-,E) \to M$. If $F$ is a finite set with $|F| \geq 2^{|E|}$, then by Corollary 10.5 the kernel $K$ of this morphism is generated by $K(F)$. Then $K$ is in turn covered by a projective functor $Q$ and we have a bounded presentation

$$Q \to P \to M \to 0$$

with both $Q$ and $P$ generated by their evaluation at $F$. By Theorem 7.4, the counit morphism $\eta_{M,F} : L_{F,M(F)} \to M$ is an isomorphism.

(b) For any finite set $F$ such that $|F| \geq 2^{|E|}$, we have $M \cong L_{F,M(F)}$ by (a). For any finite set $G$ such that $|G| \geq 2^{|F|}$, that is, $|G| \geq 2^{2^{|E|}}$, we obtain $J_{G,M(F)\uparrow_F^G} = 0$ by Theorem 10.8. It follows that

$$M \cong L_{F,M(F)} \cong L_{G,M(F)\uparrow_F^G} = S_{G,M(F)\uparrow_F^G} \, .$$

Finally, notice that, by the definition of $L_{F,M(F)}$, we have $M(G) \cong L_{F,M(F)}(G) = M(F)\uparrow_F^G$, so we obtain $M \cong L_{G,M(G)} = S_{G,M(G)}$. □

Other kinds of stabilizations also occur, as the next theorems show.

**10.10. Theorem.** *Let $k$ be a commutative noetherian ring, let $M$ and $N$ be correspondence functors over $k$, and let $E$ and $F$ be finite sets.*

*(a) If $M$ is generated by $M(E)$, then for $|F| \geq 2^{|E|}$, the evaluation map at $F$*

$$\mathrm{Hom}_{\mathcal{F}_k}(M,N) \to \mathrm{Hom}_{\mathcal{R}_F}\big(M(F),N(F)\big)$$

*is an isomorphism.*
*(b) If $M$ has bounded type, then for any integer $i \in \mathbb{N}$, there exists an integer $n_i$ such that the evaluation map*

$$\mathrm{Ext}^i_{\mathcal{F}_k}(M,N) \to \mathrm{Ext}^i_{\mathcal{R}_F}\big(M(F),N(F)\big)$$

*is an isomorphism whenever $|F| \geq n_i$.*

**Proof :** (a) By Theorem 10.9, we have an isomorphism $M \cong L_{F,M(F)}$ for $|F| \geq 2^{|E|}$. Hence

$$\mathrm{Hom}_{\mathcal{F}_k}(M,N) \cong \mathrm{Hom}_{\mathcal{F}_k}\big(L_{F,M(F)},N\big) \cong \mathrm{Hom}_{\mathcal{R}_F}\big(M(F),N(F)\big) \, ,$$

where the last isomorphism comes from the adjunction of Lemma 2.3, and is given by evaluation at $F$.

(b) This assertion will follow from (a) by *décalage* and induction on $i$. If $M$ is generated by $M(E)$, then there is an exact sequence of correspondence functors

$$0 \to L \to P \to M \to 0$$

where $P$ is projective and generated by $P(E)$. This gives an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{F}_k}(M,N) \to \mathrm{Hom}_{\mathcal{F}_k}(P,N) \to \mathrm{Hom}_{\mathcal{F}_k}(L,N) \to \mathrm{Ext}^1_{\mathcal{F}_k}(M,N) \to 0 \ ,$$

and isomorphisms $\mathrm{Ext}^i_{\mathcal{F}_k}(M,N) \cong \mathrm{Ext}^{i-1}_{\mathcal{F}_k}(L,N)$ for $i \geq 2$.

Now $L$ has bounded type by Corollary 10.5, and $P(F)$ is a projective $\mathcal{R}_F$-module by Lemma 7.3, whenever $|F|$ is large enough. It follows that there is also an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{R}_F}\big(M(F),N(F)\big) \to \mathrm{Hom}_{\mathcal{R}_F}\big(P(F),N(F)\big) \to \\ \mathrm{Hom}_{\mathcal{R}_F}\big(L(F),N(F)\big) \to \mathrm{Ext}^1_{\mathcal{R}_F}\big(M(F),N(F)\big) \to 0$$

and isomorphisms $\mathrm{Ext}^i_{\mathcal{R}_F}\big(M(F),N(F)\big) \cong \mathrm{Ext}^{i-1}_{\mathcal{R}_F}\big(L(F),N(F)\big)$ for $i \geq 2$, whenever $F$ is large enough.

Now by (a), the exact sequences

$$0 \to \mathrm{Hom}_{\mathcal{F}_k}(M,N) \to \mathrm{Hom}_{\mathcal{F}_k}(P,N) \to \mathrm{Hom}_{\mathcal{F}_k}(L,N)$$

and

$$0 \to \mathrm{Hom}_{\mathcal{R}_F}\big(M(F),N(F)\big) \to \mathrm{Hom}_{\mathcal{R}_F}\big(P(F),N(F)\big) \to \mathrm{Hom}_{\mathcal{R}_F}\big(L(F),N(F)\big)$$

are isomorphic for $F$ large enough. It follows that

$$\mathrm{Ext}^1_{\mathcal{F}_k}(M,N) \cong \mathrm{Ext}^1_{\mathcal{R}_F}\big(M(F),N(F)\big) \ .$$

Similarly, for each $i \geq 2$, when $F$ is large enough (depending on $i$), there are isomorphisms $\mathrm{Ext}^i_{\mathcal{F}_k}(M,N) \cong \mathrm{Ext}^i_{\mathcal{R}_F}\big(M(F),N(F)\big)$. □

There is also a stabilization result involving the Tor groups.

**10.11. Theorem.** *Let $k$ be a commutative noetherian ring, and $E$ be a finite set. If $F$ is a finite set with $|F| \geq 2^{2^{|E|}}$, then for any finite set $X$ and any left $\mathcal{R}_E$-module $V$, we have*

$$\mathrm{Tor}_1^{\mathcal{R}_F}\big(k\mathcal{C}(X,F), V\!\uparrow_E^F\big) = 0 \ .$$

**Proof :** Let $V$ be a left $\mathcal{R}_E$-module and $s : Q \to V$ be a surjective morphism of $\mathcal{R}_E$-modules, where $Q$ is projective. Let $K$ denote the kernel of the surjective morphism

$$L_{E,s} : L_{E,Q} \to L_{E,V} \ .$$

Since $L_{E,Q}$ is generated by $L_{E,Q}(E) \cong Q$, it follows from Corollary 10.5 that $K$ is generated by $K(G)$ whenever $G$ is a finite set with $|G| \geq 2^{|E|}$. Now by Theorem 10.9, the counit $L_{F,K(F)} \to K$ is an isomorphism whenever $F$ is a finite set with $|F| \geq 2^{|G|}$. Hence if $|F| \geq 2^{2^{|E|}}$, we have an exact sequence of correspondence functors

$$0 \to L_{F,W_F} \to L_{E,Q} \to L_{E,V} \to 0 \ , \qquad \textbf{(10.12)}$$

where $W_F = K(F)$, and where the middle term $L_{E,Q}$ is projective. Evaluating this sequence at $F$, we get the exact sequence of $\mathcal{R}_F$-modules

$$0 \to W_F \to k\mathcal{C}(F,E) \otimes_{\mathcal{R}_E} Q \to k\mathcal{C}(F,E) \otimes_{\mathcal{R}_E} V \to 0 \ ,$$

where the middle term is projective.

Let $X$ be a finite set. Applying the functor $k\mathcal{C}(X,F)\otimes_{\mathcal{R}_F}(-)$ to this sequence yields the exact sequence

$$0\to \mathrm{Tor}_1^{\mathcal{R}_F}\big(k\mathcal{C}(X,F),V\!\uparrow_E^F\big)\to \\ k\mathcal{C}(X,F)\otimes_{\mathcal{R}_F}W_F\to k\mathcal{C}(X,E)\otimes_{\mathcal{R}_E}Q\to k\mathcal{C}(X,E)\otimes_{\mathcal{R}_E}V\to 0\;,$$

because $k\mathcal{C}(X,F)\otimes_{\mathcal{R}_F}k\mathcal{C}(F,E)\cong k\mathcal{C}(X,E)$ by Corollary 4.5, as $|F|\geq|E|$. On the other hand, evaluating at $X$ the exact sequence 10.12 gives the exact sequence

$$0\to k\mathcal{C}(X,F)\otimes_{\mathcal{R}_F}W_F\to k\mathcal{C}(X,E)\otimes_{\mathcal{R}_E}Q\to k\mathcal{C}(X,E)\otimes_{\mathcal{R}_E}V\to 0\;.$$

In both latter exact sequences, the maps are exactly the same. It follows that

$$\mathrm{Tor}_1^{\mathcal{R}_F}\big(k\mathcal{C}(X,F),V\!\uparrow_E^F\big)=0\;,$$

as was to be shown. ∎

As a final approach to stabilization, we introduce the following definition.

**10.13. Definition.** *Let $\mathcal{G}_k$ denote the following category:*

- *The objects of $\mathcal{G}_k$ are pairs $(E,U)$ consisting of a finite set $E$ and a left $\mathcal{R}_E$-module $U$.*
- *A morphism $\varphi:(E,U)\to(F,V)$ in $\mathcal{G}_k$ is a morphism of $\mathcal{R}_E$-modules $U\to k\mathcal{C}(E,F)\otimes_{\mathcal{R}_F}V$.*
- *The composition of morphisms $\varphi:(E,U)\to(F,V)$ and $\psi:(F,V)\to(G,W)$ is the morphism obtained by composition*

$$U\xrightarrow{\varphi}k\mathcal{C}(E,F)\otimes_{\mathcal{R}_F}V\xrightarrow{\mathrm{id}\otimes\psi}k\mathcal{C}(E,F)\otimes_{\mathcal{R}_F}k\mathcal{C}(F,G)\otimes_{\mathcal{R}_G}W\xrightarrow{\mu\otimes\mathrm{id}}k\mathcal{C}(E,G)\otimes_{\mathcal{R}_G}W\;,$$

*where $\mu:k\mathcal{C}(E,F)\otimes_{\mathcal{R}_F}k\mathcal{C}(F,G)\to k\mathcal{C}(E,G)$ is the composition in the category $k\mathcal{C}$.*

- *The identity morphism of $(E,U)$ is the canonical isomorphism*

$$U\to k\mathcal{C}(E,E)\otimes_{\mathcal{R}_E}U$$

*resulting from the definition $\mathcal{R}_E=k\mathcal{C}(E,E)$.*

One can check easily that $\mathcal{G}_k$ is a $k$-linear category.

**10.14. Theorem.** *Let $k$ be a commutative ring. Let $\mathbb{L}:\mathcal{G}_k\to\mathcal{F}_k^b$ be the assignment sending $(E,U)$ to $L_{E,U}$, and $\varphi:(E,U)\to(F,V)$ to the morphism $L_{E,U}\to L_{F,V}$ associated by adjunction to $\varphi:U\to L_{F,V}(E)$.*

*(a) $\mathbb{L}$ is a fully faithful $k$-linear functor.*
*(b) $\mathbb{L}$ is an equivalence of categories if $k$ is noetherian.*
*(c) $\mathcal{G}_k$ is an abelian category if $k$ is noetherian.*

**Proof :** It is straightforward to check that $\mathbb{L}$ is a $k$-linear functor. It is moreover fully faithful, since

$$\mathrm{Hom}_{\mathcal{F}_k^b}(L_{E,U},L_{F,V})\cong\mathrm{Hom}_{\mathcal{R}_E}\big(U,k\mathcal{C}(E,F)\otimes_F V\big)\cong\mathrm{Hom}_{\mathcal{G}_k}\big((E,U),(F,V)\big)\;.$$

Finally, if $k$ is noetherian, then any correspondence functor $M$ of bounded type is isomorphic to a functor of the form $L_{E,U}$, by Theorem 10.9, for $E$ large enough and $U=M(E)$. Hence $\mathbb{L}$ is essentially surjective, so it is an equivalence of categories. In particular, $\mathcal{G}_k$ is abelian by Corollary 10.7. ∎

# Part 2

# FUNCTORS AND LATTICES

## 11. Functors associated to lattices

A fundamental construction associates a correspondence functor $F_T$ to any finite lattice $T$. This is one of our main tools for the analysis of correspondence functors. Throughout this section, $k$ is an arbitrary commutative ring.

**11.1. Definition.** *Let $T$ be a finite lattice. For a finite set $X$, we define $F_T(X)$ to be the free $k$-module with basis the set $T^X$ of all functions from $X$ to $T$ :*

$$F_T(X) = k(T^X) \ .$$

*For two finite sets $X$ and $Y$ and a correspondence $R \subseteq Y \times X$, we define a map $F_T(R) : F_T(X) \to F_T(Y)$ as follows : to a function $\varphi : X \to T$, we associate the function $F_T(R)(\varphi) : Y \to T$, also simply denoted by $R\varphi$, defined by*

$$(R\varphi)(y) = \bigvee_{(y,x)\in R} \varphi(x) \ ,$$

*with the usual rule that a join over the empty set is equal to $\hat{0}$. The map*

$$F_T(R) : F_T(X) \to F_T(Y)$$

*is the unique $k$-linear extension of this construction. More generally, if $\alpha = \sum\limits_{R\in\mathcal{C}(Y,X)} \alpha_R R$ is any element of $k\mathcal{C}(Y,X)$, where $\alpha_R \in k$, we set*

$$F_T(\alpha) = \sum_{R\in\mathcal{C}(Y,X)} \alpha_R F_T(R) \ .$$

**11.2. Proposition.** *The assignment sending a finite set $X$ to $F_T(X)$ and $\alpha \in k\mathcal{C}(Y,X)$ to $F_T(\alpha) : F_T(X) \to F_T(Y)$ is a correspondence functor.*

**Proof :** First it is clear that if $X$ is a finite set and $\Delta_X \in \mathcal{C}(X,X)$ is the identity correspondence, then for any $\varphi : X \to T$ and any $y \in X$

$$(\Delta_X\varphi)(y) = \bigvee_{(y,x)\in\Delta_X} \varphi(x) = \varphi(y) \ ,$$

hence $\Delta_X\varphi = \varphi$ and $F_T(\Delta_X)$ is the identity map of $F_T(X)$.

Now if $X$, $Y$, and $Z$ are finite sets, if $R \in \mathcal{C}(Y,X)$ and $S \in \mathcal{C}(Z,Y)$, then for any $\varphi : X \to T$ and any $z \in Z$

$$\begin{aligned} \big(S(R\varphi)\big)(z) &= \bigvee_{(z,y)\in S} (R\varphi)(y) \\ &= \bigvee_{(z,y)\in S} \bigvee_{(y,x)\in R} \varphi(x) \\ &= \bigvee_{(z,x)\in SR} \varphi(x) \\ &= (SR\varphi)(z) \ . \end{aligned}$$

By linearity, it follows that $F_T(\beta) \circ F_T(\alpha) = F_T(\beta\,\alpha)$, for any $\beta \in k\mathcal{C}(Z,Y)$ and any $\alpha \in k\mathcal{C}(Y,X)$. □

We now establish the link between the action of correspondences on functions $\varphi : X \to T$ (as in Definition 11.1 above) and the correspondences $\Gamma_\varphi$ defined in Notation 5.7.

**11.3. Lemma.** *Let $T$ be a finite lattice, let $E = \mathrm{Irr}(T)$, and assume that $T$ is distributive. Then, for any finite sets $X$, $Y$, any correspondence $S \in \mathcal{C}(Y,X)$, and any function $\varphi : X \to T$, we have $\Gamma_{S\varphi} = S\Gamma_\varphi$, where $\Gamma_\varphi$ is defined in Notation 5.7.*

**Proof :** Let $y \in Y$ and $e \in E$. Then

$$\begin{aligned}(y,e) \in \Gamma_{S\varphi} \quad &\iff e \leq_T (S\varphi)(y) \iff e \leq_T \bigvee_{(y,x)\in S} \varphi(x)\\ &\iff e = e \wedge \big( \bigvee_{(y,x)\in S} \varphi(x)\big)\ .\end{aligned}$$

But, since $T$ is distributive, the latter equality is equivalent to $e = \bigvee_{(y,x)\in S} \big(e \wedge \varphi(x)\big)$. Now, since $e$ is irreducible, this is in turn equivalent to

$$\begin{aligned}&\exists x \in X\ ,\ (y,x) \in S \ \text{ and }\ e \wedge \varphi(x) = e \iff \exists x \in X\ ,\ (y,x) \in S \ \text{ and } e \leq_T \varphi(x)\\ &\iff \exists x \in X\ ,\ (y,x) \in S \ \text{ and } (x,e) \in \Gamma_\varphi \iff (y,e) \in S\Gamma_\varphi\ .\end{aligned}$$

This completes the proof. ∎

**11.4. Proposition.** *Let $(E,R)$ be a finite poset.*

*(a) For any finite set $X$*

$$\big\{\Gamma_\varphi \mid \varphi : X \to I^\uparrow(E,R)\big\} = \big\{S \in \mathcal{C}(X,E) \mid SR = S\big\} = \mathcal{C}(X,E)R\ .$$

*(b) The correspondence functor $F_{I^\uparrow(E,R)}$ is isomorphic to $k\mathcal{C}(-,E)R$. In particular $F_{I^\uparrow(E,R)}$ is a projective object of $\mathcal{F}_k$.*

*(c) The correspondence functor $F_{I_\downarrow(E,R)}$ is isomorphic to $k\mathcal{C}(-,E)R^{op}$. In particular $F_{I_\downarrow(E,R)}$ is a projective object of $\mathcal{F}_k$.*

**Proof :** (a) This is a restatement of Lemma 5.8.

(b) The map

$$F_{I^\uparrow(E,R)}(X) \longrightarrow k\mathcal{C}(X,E)R\,, \qquad \varphi \mapsto \Gamma_\varphi$$

is an isomorphism of correspondence functors, by (a) and Lemma 11.3.

(c) follows from (b) and the obvious equality $I_\downarrow(E,R) = I^\uparrow(E,R^{op})$. ∎

We now introduce a suitable category $\mathcal{L}$ of lattices, as well as its $k$-linearization $k\mathcal{L}$ (see Definition 2.1), such that the assignment $T \mapsto F_T$ becomes a $k$-linear functor from $k\mathcal{L}$ to $\mathcal{F}_k$.

**11.5. Definition.** *Let $\mathcal{L}$ denote the following category :*

- *The objects of $\mathcal{L}$ are the finite lattices.*
- *For any two lattices $T$ and $T'$, the set $\mathcal{L}(T',T)$ is the set of all maps $f : T \to T'$ which commute with joins, i.e. such that*

  $$f(\bigvee_{a\in A} a) = \bigvee_{a\in A} f(a)\ ,$$

  *for any subset $A$ of $T$.*
- *The composition of morphisms in $\mathcal{L}$ is the composition of maps.*

**11.6. Remark.** Morphisms in $\mathcal{L}$ preserve the order relation, but they are generally not morphisms of lattices in the sense that they need not commute with the meet operation. On the other hand, the case $A = \emptyset$ in Definition 11.5 shows that a morphism $f : T \to T'$ in $\mathcal{L}$ always maps $\hat{0} \in T$ to $\hat{0} \in T'$.

Conversely, if $f : T \to T'$ satisfies $f(\hat{0}) = \hat{0}$ and $f(a \vee b) = f(a) \vee f(b)$ for all $a, b \in T$, then $f$ is a morphism in $\mathcal{L}$.

Let $f : T \to T'$ be a morphism in the category $\mathcal{L}$. For a finite set $X$, let $F_{f,X} : F_T(X) \to F_{T'}(X)$ be the $k$-linear map sending the function $\varphi : X \to T$ to the function $f \circ \varphi : X \to T'$.

**11.7. Theorem.**

*(a) Let $f : T \to T'$ be a morphism in the category $\mathcal{L}$. Then the collection of maps $F_{f,X} : F_T(X) \to F_{T'}(X)$, for all finite sets $X$, yields a natural transformation $F_f : F_T \to F_{T'}$ of correspondence functors.*

*(b) The assignment sending a lattice $T$ to $F_T$, and a morphism $f : T \to T'$ in $\mathcal{L}$ to $F_f : F_T \to F_{T'}$, yields a functor $\mathcal{L} \to \mathcal{F}_k$. This functor extends uniquely to a $k$-linear functor*

$$F_? : k\mathcal{L} \to \mathcal{F}_k \ .$$

*(c) The functor $F_?$ is fully faithful.*

**Proof :** (a) Let $X$ and $Y$ be finite sets, let $\varphi : X \to T$ be a function, and let $R \in \mathcal{C}(Y, X)$ be a correspondence. Then $F_{T'}(R)\big(F_{f,X}(\varphi)\big) = F_{T'}(R)(f \circ \varphi) = R(f \circ \varphi)$ and $F_{f,Y}\big(F_T(R)(\varphi)\big) = F_{f,Y}(R\varphi) = f \circ R\varphi$. We show that they are equal by evaluating at any $y \in Y$ :

$$\begin{aligned} R(f \circ \varphi)(y) &= \bigvee_{(y,x)\in R} (f \circ \varphi)(x) \\ &= \bigvee_{(y,x)\in R} f\big(\varphi(x)\big) \\ &= f\big(\bigvee_{(y,x)\in R} \varphi(x)\big) \\ &= (f \circ R\varphi)(y) \ , \end{aligned}$$

hence $R(f \circ \varphi) = f \circ R\varphi$, which proves (a).

(b) It follows that the assignment $T \mapsto F_T$ is a functor $\mathcal{L} \to \mathcal{F}_k$. Since $k\mathcal{L}$ is the $k$-linearization of $\mathcal{L}$, this functor extends uniquely to a $k$-linear functor $F_? : k\mathcal{L} \to \mathcal{F}_k$.

(c) Let $S$ and $T$ be finite lattices, and $\Phi : F_S \to F_T$ be a morphism of functors. Thus, for any finite set $X$, we have a morphism of $k$-modules $\Phi_X : F_S(X) \to F_T(X)$ such that for any finite set $Y$ and any correspondence $R \subseteq (Y \times X)$, the diagram

$$\begin{array}{ccc} F_S(X) & \xrightarrow{\Phi_X} & F_T(X) \\ {\scriptstyle F_S(R)}\downarrow & & \downarrow{\scriptstyle F_T(R)} \\ F_S(Y) & \xrightarrow{\Phi_Y} & F_T(Y) \end{array}$$

is commutative. In other words, for any function $\alpha : X \to S$

$$R\Phi_X(\alpha) = \Phi_Y(R\alpha) \ . \qquad \textbf{(11.8)}$$

Taking $X = S$ and $\alpha = \mathrm{id}_S$ in this relation, and setting

$$\varphi = \Phi_S(\mathrm{id}_S) = \sum_{\lambda:S\to T} u_\lambda \lambda \ ,$$

where $u_\lambda \in k$, this gives

$$R\varphi = \Phi_Y(R\,\mathrm{id}_S) \ ,$$

for any $Y$ and any $R \subseteq (Y \times S)$.

Given a function $\beta : Y \to S$ and taking $R = \Omega_\beta := \{(y, \beta(y)) \mid y \in Y\}$, one can check easily that $\Omega_\beta\,\mathrm{id}_S = \beta$. It follows that

$$\Phi_Y(\beta) = \Omega_\beta\,\varphi \ . \tag{11.9}$$

Hence $\Phi$ is entirely determined by $\varphi$. Now Condition 11.8 is fulfilled if and only if, for any finite sets $X$ and $Y$, any correspondence $R \subseteq (Y \times X)$, and any function $\alpha : X \to S$, we have

$$R\Omega_\alpha(\varphi) = \Omega_{R\alpha}(\varphi) \ .$$

In other words

$$\sum_\lambda u_\lambda R\Omega_\alpha(\lambda) = \sum_\lambda u_\lambda \Omega_{R\alpha}(\lambda) \ .$$

Hence Condition 11.8 is satisfied if and only if, for any finite sets $X$ and $Y$, any correspondence $R \subseteq (Y \times X)$, any function $\alpha : X \to S$, and any function $\psi : Y \to T$, we have

$$\sum_{R\Omega_\alpha(\lambda)=\psi} u_\lambda = \sum_{\Omega_{R\alpha}(\lambda)=\psi} u_\lambda \ . \tag{11.10}$$

But for $y \in Y$

$$\begin{aligned} R\Omega_\alpha(\lambda)(y) &= \bigvee_{(y,s)\in R\Omega_\alpha} \lambda(s) \\ &= \bigvee_{(y,x)\in R} \lambda\alpha(x) \ . \end{aligned}$$

On the other hand

$$\begin{aligned} \Omega_{R\alpha}(\lambda)(y) &= \bigvee_{(y,s)\in \Omega_{R\alpha}} \lambda(s) \\ &= \lambda\big(R\alpha(y)\big) \\ &= \lambda\big( \bigvee_{(y,x)\in R} \alpha(x)\big) \ . \end{aligned}$$

Now take $X = S$ and $\alpha = \mathrm{id}_S$ in 11.10. Then let $Y = \mathcal{B}(S)$ be the set of subsets of $S$ and let $R \subseteq (Y \times S)$ be the set of pairs $(A, s)$, where $A \subseteq S$ and $s \in A$.

Then for a given map $\lambda : S \to T$, let us *define* $\psi : Y \to T$ by $\psi = \Omega_{R\,\mathrm{id}_S}(\lambda)$, in other words

$$\forall A \subseteq S,\ \psi(A) = \lambda(\bigvee_{s\in A} s) \ .$$

Suppose that there exists $\lambda' : S \to T$ such that $\Omega_{R\,\mathrm{id}_S}(\lambda') = \psi$. Then for $A \subseteq S$

$$\psi(A) = \lambda'(\bigvee_{s\in A} s) \ .$$

Taking $A = \{s\}$, it follows that $\lambda' = \lambda$. Hence in 11.10 with our specific choices, the right hand side is simply equal to $u_\lambda$.

On the other hand the left hand side is equal to the sum of $u_{\lambda'}$, for all $\lambda'$ such that $R\lambda' = \psi$, that is, satisfying

$$\forall A \subseteq S,\ \psi(A) = \bigvee_{s\in A} \lambda'(s) \ .$$

Again, taking $A = \{s\}$, it follows that $\lambda' = \lambda$. With our specific choices, the left hand side of 11.10 is then equal to $u_\lambda$ if and only if $R\lambda = \Omega_{R\,\mathrm{id}_S}(\lambda)$, that is, for any $A \subseteq S$,

$$\bigvee_{s\in A} \lambda(s) = \lambda(\bigvee_{s\in A} s) \; .$$

If this condition is not satisfied, then the left hand side of 11.10 is zero (empty sum). In other words $u_\lambda = 0$ if $\lambda$ is not a morphism in the category $\mathcal{L}$. It follows that $\varphi = \sum\limits_{\lambda} u_\lambda \lambda$ is a morphism in $k\mathcal{L}$, from $S$ to $T$. We claim that the image of this morphism via the functor $F_?$ is equal to $\Phi$ and this will prove that the functor $F_? : k\mathcal{L} \to \mathcal{F}_k$ is full. To prove the claim, notice that, for any function $\beta : Y \to S$, we have

$$F_{\varphi,Y}(\beta) = \sum_\lambda u_\lambda F_{\lambda,Y}(\beta) = \sum_\lambda u_\lambda(\lambda\circ\beta) = \sum_\lambda u_\lambda \Omega_\beta\, \lambda = \Omega_\beta\, \varphi = \Phi_Y(\beta) \; ,$$

using the equation 11.9. This proves the claim and completes the proof that $F_?$ is full.

It remains to show that the functor $F_?$ is faithful. So let $\varphi$ and $\psi$ be two linear combinations of morphisms $S \to T$ in $\mathcal{L}$, which induce the same morphism $\theta = F_\varphi = F_\psi : F_S \to F_T$. Evaluating this morphism at the set $S$ gives a map $\theta_S : F_S(S) \to F_T(S)$, and moreover

$$\theta_S(\mathrm{id}_S) = F_{\varphi,S}(\mathrm{id}_S) = \varphi\circ \mathrm{id}_S = \varphi \in F_T(S) = k(T^S) \; .$$

For the same reason, $\theta_S(\mathrm{id}_S) = F_{\psi,S}(\mathrm{id}_S) = \psi$, hence $\varphi = \psi$. This completes the proof of Theorem 11.7. □

The connection between finite lattices and correspondence functors also has the following rather remarkable feature.

**11.11. Theorem.** *Let $T$ be a finite lattice. The functor $F_T$ is projective in $\mathcal{F}_k$ if and only if $T$ is distributive.*

**Proof :** Let $\mathcal{B}(T)$ be the lattice of subsets of $T$. Let $\upsilon : \mathcal{B}(T) \to T$ be the morphism in the category $\mathcal{L}$ defined by

$$\forall A \subseteq T,\; \upsilon(A) = \bigvee_{t\in A} t \; .$$

This morphism induces a morphism of functors $F_\upsilon : F_{\mathcal{B}(\mathcal{T})} \to F_T$, and $F_\upsilon$ is surjective : indeed, if $X$ is a finite set and $\alpha : X \to T$ is a function, and if we define $\hat\alpha : X \to \mathcal{B}(T)$ by

$$\forall x \in X,\; \hat\alpha(x) = \{\alpha(x)\} \; ,$$

then, for any $x \in X$

$$F_\upsilon(\hat\alpha)(x) = (\upsilon\circ\hat\alpha)(x) = \bigvee_{t\in\hat\alpha(x)} t = \alpha(x) \; ,$$

thus $F_\upsilon(\hat\alpha) = \alpha$, so $F_\upsilon$ is surjective.

Now if $F_T$ is projective, then the morphism $F_\upsilon$ splits and there exists a morphism $\Phi : F_T \to F_{\mathcal{B}(T)}$ such that $F_\upsilon \circ \Phi$ is the identity morphism of $F_T$. It follows from Theorem 11.7 that $\Phi$ is of the form $\sum\limits_{\sigma\in M} u_\sigma F_\sigma$, where $M$ is a finite set of morphisms $\sigma : T \to \mathcal{B}(T)$ in $\mathcal{L}$, and $u_\sigma \in k$. Moreover $F_\upsilon \circ \Phi$ is then equal to $\sum\limits_{\sigma\in M} u_\sigma F_{\upsilon\circ\sigma}$, hence there exists at least one such $\sigma \in M$ such that $\upsilon\circ\sigma$ is equal to the identity of $T$. This means that

$$\forall t \in T,\; t = \bigvee_{x\in\sigma(t)} x \; .$$

In particular $\sigma(t) \subseteq [\hat{0}, t]_T$ for any $t \in T$. Then for $r, s \in T$

$$[\hat{0}, r \wedge s]_T = [\hat{0}, r]_T \cap [\hat{0}, s]_T \supseteq \sigma(r) \cap \sigma(s) \supseteq \sigma(r \wedge s) ,$$

because $\sigma$ is order-preserving. It follows that

$$r \wedge s \geq \bigvee_{x \in \sigma(r) \cap \sigma(s)} x \geq \bigvee_{x \in \sigma(r \wedge s)} x = r \wedge s ,$$

hence

$$r \wedge s = \bigvee_{x \in \sigma(r) \cap \sigma(s)} x .$$

Now, since $\sigma$ preserves joins, we obtain, for all $r, s, t \in T$,

$$\begin{aligned} t \wedge (r \vee s) &= \bigvee_{x \in \sigma(t) \cap \sigma(r \vee s)} x \\ &= \bigvee_{x \in \sigma(t) \cap \big(\sigma(r) \cup \sigma(s)\big)} x \\ &= \bigvee_{x \in \big(\sigma(t) \cap \sigma(r)\big) \cup \big(\sigma(t) \cap \sigma(s)\big)} x \\ &= \Big(\bigvee_{x \in \sigma(t) \cap \sigma(r)} x\Big) \vee \Big(\bigvee_{x \in \sigma(t) \cap \sigma(s)} x\Big) \\ &= (t \wedge r) \vee (t \wedge s) . \end{aligned}$$

In other words the lattice $T$ is distributive.

Conversely, by Theorem 3.4.1 in [**St**], any finite distributive lattice $T$ is isomorphic to the lattice $I_\downarrow(E, R)$ of lower ideals of a finite poset $(E, R)$. By Proposition 11.4, the associated functor $F_T$ is projective in $\mathcal{F}_k$. This completes the proof of Theorem 11.11. □

## 12. Quotients of functors associated to lattices

We now introduce, for any finite lattice $T$, a subfunctor of $F_T$ naturally associated with the set of irreducible elements of $T$.

**12.1. Notation.** *Let $T$ be a finite lattice and let $(E, R)$ be the full subposet of its irreducible elements. For a finite set $X$, let $H_T(X)$ denote the $k$-submodule of $F_T(X) = k(T^X)$ generated by all functions $\varphi : X \to T$ such that $E \nsubseteq \varphi(X)$.*

**12.2. Proposition.** *Let $T$ be a finite lattice and let $(E, R)$ be the full subposet of its irreducible elements.*

(a) *The assignment sending a finite set $X$ to $H_T(X) \subseteq F_T(X)$ is a subfunctor $H_T$ of $F_T$.*
(b) *The evaluation $(F_T/H_T)(X)$ has a $k$-basis consisting of (the classes of) all functions $\varphi : X \to T$ such that $E \subseteq \varphi(X)$.*

**Proof :** (a) Let $X$ and $Y$ be finite sets, let $Q \in \mathcal{C}(Y, X)$ be a correspondence, and let $\varphi : X \to T$ be a function. Then

**(12.3)** $$(Q\varphi)(Y) \cap E \subseteq \varphi(X) \cap E .$$

Indeed, if $e \in E$ and $e = (Q\varphi)(y)$, for $y \in Y$, then

$$e = \bigvee_{(y,x) \in Q} \varphi(x) \ .$$

As $e$ is irreducible in $T$, there exists $x \in X$ such that $(y,x) \in Q$ and $e = \varphi(x)$, and 12.3 follows.

In particular, if $\varphi(X) \cap E$ is a proper subset of $E$, then $(Q\varphi)(Y) \cap E$ is a proper subset of $E$. Hence $H_T$ is a subfunctor of $F_T$.

(b) This follows from the definitions of $F_T$ and $H_T$. □

The quotient functor $F_T/H_T$ plays a important role in the rest of this paper, in particular for the description of the fundamental functors $\mathbb{S}_{E,R}$ in Section 17. Note that the module $(F_T/H_T)(X) = F_T(X)/H_T(X)$ has a $k$-basis consisting of (the classes of) all maps $\varphi : X \to T$ such that $\varphi(X) \supseteq E$. We now give another characterization of $H_T(X)$.

**12.4. Proposition.** *Let $T = I^{\uparrow}(E,R)$ for a finite poset $(E,R)$ and let $X$ be a finite set.*

*(a) Under the isomorphism $F_T \to k\mathcal{C}(-,E)R$ of Proposition 11.4, $H_T(X)$ is isomorphic to the $k$-submodule of $k\mathcal{C}(X,E)R$ generated by the correspondences $S$ which have no retraction, that is, for which there is no $U \in \mathcal{C}(E,X)$ such that $US = R$.*

*(b) Under the isomorphism $F_T \to k\mathcal{C}(-,E)R$ of Proposition 11.4, the image of $F_T(X)/H_T(X)$ is a free $k$-module with basis consisting of all the correspondences $S \in \mathcal{C}(X,E)R$ which have a retraction $U \in \mathcal{C}(E,X)$.*

**Proof :** By Proposition 11.4, the functor $F_T$ is isomorphic to the functor $k\mathcal{C}(-,E)R$ by sending, for a finite set $X$, a function $\varphi : X \to I^{\uparrow}(E,R)$ to the correspondence $\Gamma_\varphi = \{(x,e) \in X \times E \mid e \in \varphi(x)\}$.

(a) The set $E^{\uparrow}$ of irreducible elements of the lattice $I^{\uparrow}(E,R)$ is the set of principal upper ideals

$$[e,\cdot[_R = \big\{f \in E \mid (e,f) \in R\big\} \ ,$$

for $e \in E$. Let $\varphi : X \to I^{\uparrow}(E,R)$ be such that $\varphi \notin H_T(X)$, that is, $\varphi(X) \supseteq E^{\uparrow}$. Then, for each $e \in E$, there exists $x_e \in X$ such that $\varphi(x_e) = [e,\cdot[_R$. Let $U \in \mathcal{C}(E,X)$ be defined by

$$U = \big\{(e,x_e) \mid e \in E\big\} \subseteq E \times X \ .$$

Then for any $e \in E$

$$(U\varphi)(e) = \bigcup_{(e,x) \in U} \varphi(x) = \varphi(x_e) = [e,\cdot[_R \ .$$

By Lemma 11.3, it follows that

$$U\Gamma_\varphi = \Gamma_{U\varphi} = \big\{(e,f) \in E \times E \mid f \in [e,\cdot[_R\big\} = R \ ,$$

so $\Gamma_\varphi$ has a retraction.

Conversely, let $S \in \mathcal{C}(X,E)R$ be a correspondence such that there exists a correspondence $U \in \mathcal{C}(X,E)$ with $US = R$. Then $S = \Gamma_\varphi$, where $\varphi : X \to I^{\uparrow}(E,R)$ is the function defined by $\varphi(x) = \big\{e \in E \mid (x,e) \in S\big\}$, for any $x \in X$. It follows that $US = \Gamma_{U\varphi} = R$, or in other words

$$\forall e,f \in E,\ (e,f) \in R \iff \exists x \in X,\ (e,x) \in U,\ (x,e) \in S \ .$$

As $\Delta_E \subseteq R$, for any $e \in E$, there exists $x_e \in X$ such that $(e,x_e) \in U$ and $(x_e,e) \in S$. Moreover if $(x_e,f) \in S$, then $(e,f) \in R$, and conversely, if $(e,f) \in R$, then $(x_e, f) \in SR = S$. In other words, $f \in \varphi(x_e)$ if and only if $(e, f) \in R$. It follows that $\varphi(x_e) = [e, \cdot[_R$, hence $\varphi(X) \supseteq E^\uparrow$. This proves that $\varphi \notin H_T(X)$.

(b) This follows from (a). ∎

**12.5. Remark.** In the special case when $R = \Delta_E$ is the equality relation, then $\mathcal{C}(X, E)R = \mathcal{C}(X, E)$ and a retraction of $S \in \mathcal{C}(X, E)$ is a correspondence $U \in \mathcal{C}(E, X)$ such that $US = \mathrm{id}_E$ (a retraction in the usual sense). Moreover, if $S \in \mathcal{C}(X, E)$ has a retraction, then $S$ is a monomorphism in the category $\mathcal{C}$. It can be shown conversely that any monomorphism in the category $\mathcal{C}$ has a retraction. Thus in this case, the evaluation $F_T(X)/H_T(X)$ of the quotient functor $F_T/H_T$ has a $k$-basis consisting of all the monomorphisms in $\mathcal{C}(X, E)$.

In order to deal with quotients of the functor $F_T$, we need information on morphisms starting from $F_T$. But we first need a lemma.

**12.6. Lemma.** *Let $T$ be a finite lattice, let $(E, R)$ be the full subposet of its irreducible elements, and let $\iota : E \to T$ denote the inclusion map. If $\varphi : X \to T$ is a function, then $\Gamma_\varphi\, \iota = \varphi$ and $\Gamma_\varphi R^{op} = \Gamma_\varphi$, where $\Gamma_\varphi$ is defined in Notation 5.7.*

**Proof :** By definition, the map $\Gamma_\varphi\, \iota : X \to T$ satisfies

$$\forall x \in X, \;\; (\Gamma_\varphi\, \iota)(x) = \bigvee_{(x,e)\in\Gamma_\varphi} \iota(e) = \bigvee_{e \leq_T \varphi(x)} e = \varphi(x) \;,$$

as any element $t$ of $T$ is equal to the join of the irreducible elements of $T$ smaller than $t$. Thus we have $\Gamma_\varphi\, \iota = \varphi$.

The equality $\Gamma_\varphi R^{op} = \Gamma_\varphi$ was proved in Lemma 5.8. ∎

**12.7. Proposition.** *Let $T$ be a finite lattice, let $(E, R)$ be the full subposet of its irreducible elements, and let $\iota : E \to T$ denote the inclusion map. Let $M$ be a correspondence functor.*

*(a) The $k$-linear map*

$$\mathrm{Hom}_{\mathcal{F}_k}(F_T, M) \longrightarrow M(E) \,, \quad \Phi \mapsto \Phi_E(\iota)$$

*is injective. Its image is contained in the $k$-submodule*

$$R^{op}M(E) = \{m \in M(E) \mid R^{op}m = m\} \;.$$

*(b) If $T$ is distributive, then the image of the above map is equal to $R^{op}M(E)$, so that $\mathrm{Hom}_{\mathcal{F}_k}(F_T, M) \cong R^{op}M(E)$ as $k$-modules.*

**Proof :** By Lemma 12.6, for any $\Phi \in \mathrm{Hom}_{\mathcal{F}_k}(F_T, M)$ and any map $\varphi : X \to T$, we have

$$\Phi_X(\varphi) = \Phi_X(\Gamma_\varphi\, \iota) = \Gamma_\varphi\, \Phi_E(\iota) \;.$$

This shows that $\Phi$ is entirely determined by $\Phi_E(\iota)$, proving the injectivity of the map $\Phi \mapsto \Phi_E(\iota)$.

In the special case where $\varphi = \iota$, we have

$$\Gamma_\iota = \{(x, e) \in E \times E \mid e \leq_T \iota(x)\} = \{(x, e) \in E \times E \mid e \leq_R x\} = R^{op} \;.$$

Moreover $R^{op}\iota = \Gamma_\iota\, \iota = \iota$, so its image $\Phi_E(\iota)$ must also be invariant by $R^{op}$, proving that the image of the map $\Phi \mapsto \Phi_E(\iota)$ is contained in $R^{op}M(E)$.

(b) Since $T$ is distributive, we have

$$\Gamma_{Q\varphi} = Q\Gamma_\varphi$$

by Lemma 11.3. Now given $m \in R^{op}M(E)$, we can define $\Phi : F_T \to M$ by setting

$$\Phi_X(\varphi) = \Gamma_\varphi m \ , \quad \forall \varphi : X \to T \ .$$

This is indeed a natural transformation of functors since

$$\Phi_Y(Q\varphi) = \Gamma_{Q\varphi} m = Q\Gamma_\varphi m = Q\Phi_X(\varphi)$$

for any correspondence $Q \subseteq Y \times X$. Moreover,

$$\Phi_E(\iota) = \Gamma_\iota m = R^{op} m = m \ ,$$

because $m \in R^{op}M(E)$ by assumption and $R^{op}$ is idempotent. Thus $m$ is indeed in the image of the map $\Phi \mapsto \Phi_E(\iota)$. □

When $k$ is a field, we can now give some information on simple functors $S_{F,Q,V}$ appearing as quotients of $F_T$. As usual, we prove a more general result over an arbitrary commutative ring $k$, involving the not necessarily simple functors $S_{F,Q,V}$.

**12.8. Theorem.** *Let $T$ be a finite lattice and let $(E, R)$ be the full subposet of its irreducible elements. Let $(F, Q)$ be a poset and let $V$ be a $k\,\mathrm{Aut}(F, Q)$-module generated by a single element (e.g. a simple module).*

*(a) If $S_{F,Q,V}$ is isomorphic to a quotient of $F_T$ , then $|F| \leq |E|$.*
*(b) Assume that $F = E$. If $S_{E,Q,V}$ is isomorphic to a quotient of $F_T$, then there exists a permutation $\sigma \in \Sigma_E$ such that $R^{op} \subseteq {}^{\sigma}Q$.*
*(c) Assume that $F = E$ and that $T$ is distributive. Then $S_{E,Q,V}$ is isomorphic to a quotient of $F_T$ if and only if there exists a permutation $\sigma \in \Sigma_E$ such that $R^{op} \subseteq {}^{\sigma}Q$.*

**Proof :** (a) If $S_{F,Q,V}$ is isomorphic to a quotient of $F_T$, then $\mathrm{Hom}_{\mathcal{F}_k}(F_T, S_{F,Q,V}) \neq \{0\}$, so we have $S_{F,Q,V}(E) \neq \{0\}$ by Proposition 12.7. But we know that $F$ is a minimal set for $S_{F,Q,V}$, so $|F| \leq |E|$.

(b) If $S_{E,Q,V}$ is isomorphic to a quotient of $F_T$, there exists a nonzero morphism $\Phi : F_T \to S_{E,Q,V}$. By Proposition 12.7, $\Phi_E(\iota) = m \neq 0 \in R^{op}S_{E,Q,V}(E)$. Now

$$S_{E,Q,V}(E) = \mathcal{P}_E f_Q \otimes_{k\,\mathrm{Aut}(E,Q)} V$$

and $\mathcal{P}_E f_Q$ is a free $k$-module with basis $\{\Delta_\sigma f_Q \mid \sigma \in \Sigma_E\}$, by Proposition 3.10. Thus we can write

$$m = \sum_{\sigma \in \Sigma_E} \lambda_\sigma \Delta_\sigma f_Q \otimes v \qquad (\lambda_\sigma \in k) \ .$$

Since $m \in R^{op}S_{E,Q,V}(E)$, we have $R^{op}m = m$ and so there exists $\sigma \in \Sigma_E$ such that $R^{op}\Delta_\sigma f_Q \neq 0$. Hence $R^{op}\Delta_\sigma f_Q \Delta_\sigma^{-1} \neq 0$, that is, $R^{op} \subseteq {}^{\sigma}Q$, by Lemma 3.8.

(c) One implication follows from (b). Assume now that there exists a permutation $\sigma \in \Sigma_E$ such that $R^{op} \subseteq {}^{\sigma}Q$. We first note that $S_{E,Q,V}$ is generated by $f_Q \otimes v \in S_{E,Q,V}(E) = \mathcal{P}_E f_Q \otimes_{k\,\mathrm{Aut}(E,Q)} V$, where $v$ is a generator of $V$. This follows from the definition of $S_{E,Q,V}$ as a quotient of $L_{E,\mathcal{P}_E f_Q \otimes V}$ and the fact that any functor $L_{E,W}$ is generated by $L_{E,W}(E) = W$ by definition.

Also $S_{E,Q,V} \cong S_{E,{}^{\sigma}Q,{}^{\sigma}V}$ for any $\sigma \in \Sigma_E$, by construction. Since $R^{op}$ is contained in a conjugate of $Q$, we can assume that $R^{op} \subseteq Q$. This is equivalent to $R^{op} f_Q = f_Q$, by Lemma 3.8.

Thus $m = f_Q \otimes v \in S_{E,Q,V}(E)$ is invariant under left multiplication by $R^{op}$. By Proposition 12.7 and the assumption that $T$ is distributive, there exists a morphism $\Phi : F_T \to S_{E,Q,V}$ such that $\Phi_E(\iota) = f_Q \otimes v$. Since this is nonzero and generates $S_{E,Q,V}$, this functor is isomorphic to a quotient of $F_T$. □

## 13. The fundamental functor associated to a poset

The fundamental functor $\mathbb{S}_{E,R}$ associated to a poset $(E,R)$ was introduced in Definition 4.7 and some description of its evaluations is given in Lemma 4.9. One of our important goals is to give a more precise description of its evaluations, but this will be fully achieved only in Section 17. We prepare the ground by proving several main results on $\mathbb{S}_{E,R}$. By Lemma 3.14, we already know that $E$ is a minimal set for $\mathbb{S}_{E,R}$ and that $\mathbb{S}_{E,R}(E)$ is the fundamental module $\mathcal{P}_E f_R$, which is described in Proposition 3.10.

The following theorem establishes the link between the quotient functor $F_T/H_T$ and the fundamental correspondence functors.

**13.1. Theorem.** *Let $T$ be a finite lattice, let $(E,R)$ be the full subposet of its irreducible elements, and let $\iota : E \to T$ denote the inclusion map.*

*(a) There is a unique homomorphism of $k\mathcal{C}(E,E)$-module*

$$\theta : (F_T/H_T)(E) \to \mathcal{P}_E f_{R^{op}}$$

*sending $\iota + H_T(E)$ to $f_{R^{op}}$. Moreover $\theta$ is an isomorphism.*

*(b) There exists a unique surjective morphism of correspondence functors*

$$\Theta_T : F_T/H_T \to \mathbb{S}_{E,R^{op}}$$

*such that $\theta$ is equal to the composition of $\Theta_{T,E} : (F_T/H_T)(E) \to \mathbb{S}_{E,R^{op}}(E)$ with the canonical isomorphism $\mathbb{S}_{E,R^{op}}(E) \stackrel{\cong}{\to} \mathcal{P}_E f_{R^{op}}$.*

*(c) The functor $F_T$ is generated by $\iota \in F_T(E)$ and $\mathbb{S}_{E,R^{op}}$ is generated by $\Theta_{T,E}(\iota) \in \mathbb{S}_{E,R^{op}}(E)$.*

**Proof :** (a) The $k\mathcal{C}(E,E)$-module $(F_T/H_T)(E)$ has a $k$-basis consisting of the classes of maps $\varphi : E \to T$ such that $\varphi(E) \supseteq E$, in other words the classes of maps $d_\sigma = \iota\sigma^{-1} : E \to T$, where $\sigma \in \Sigma_E$ (the reason for inverting $\sigma$ here will become clear below). By a small abuse, we view the elements $d_\sigma$ as basis elements of $(F_T/H_T)(E)$.

By Proposition 3.10, the $k\mathcal{C}(E,E)$-module $\mathcal{P}_E f_{R^{op}}$ has a $k$-basis $\{\Delta_\sigma f_{R^{op}} \mid \sigma \in \Sigma_E\}$. We define

$$\theta : (F_T/H_T)(E) \to \mathcal{P}_E f_{R^{op}} \ , \qquad \theta(d_\sigma) = \Delta_\sigma f_{R^{op}} \ ,$$

extended $k$-linearly. We only have to check that $\theta$ is a homomorphism of $k\mathcal{C}(E,E)$-modules.

Let $Q \in \mathcal{C}(E,E)$. Then the map $Qd_\sigma$ is defined by

$$\forall e \in E,\ (Qd_\sigma)(e) = \bigvee_{(e,f)\in Q} d_\sigma(f) = \bigvee_{(e,f)\in Q} \sigma^{-1}(f) \ .$$

The image of $Qd_\sigma$ in $(F_T/H_T)(E)$ is nonzero if and only if there exists a permutation $\rho \in \Sigma_E$ such that $(Qd_\sigma)(e) = \rho^{-1}(e)$, for any $e \in E$, i.e.

$$\forall e \in E, \quad \bigvee_{(e,f)\in Q} \sigma^{-1}(f) = \rho^{-1}(e) \ .$$

Since $\rho^{-1}(e)$ is irreducible in $T$, this is equivalent to the following two conditions :

$$\begin{cases} \forall e \in E,\ \exists f_e \in E \text{ such that } (e,f_e) \in Q \text{ and } \sigma^{-1}(f_e) = \rho^{-1}(e) \ , \\ (e,f) \in Q \implies \sigma^{-1}(f) \leq_T \rho^{-1}(e) \ . \end{cases}$$

The first of these conditions gives $f_e = \sigma\rho^{-1}(e)$, and $\big(e, \sigma\rho^{-1}(e)\big) \in Q$ for any $e \in E$. In other words $\Delta_{(\sigma\rho^{-1})^{-1}} \subseteq Q$, i.e. $\Delta \subseteq \Delta_{\sigma\rho^{-1}}Q$. For the second condition, note that we have equivalences

$$\sigma^{-1}(f) \leq_T \rho^{-1}(e) \iff \sigma^{-1}(f) \leq_R \rho^{-1}(e) \iff \big(\rho^{-1}(e), \sigma^{-1}(f)\big) \in R^{op} \ .$$

So the second condition is equivalent to $Q \subseteq \Delta_\rho R^{op} \Delta_{\sigma^{-1}}$, that is, $Q \subseteq \Delta_{\rho\sigma^{-1}}\, {}^\sigma R^{op}$, or equivalently $\Delta_{\sigma\rho^{-1}}Q \subseteq {}^\sigma R^{op}$.

Finally, the class of $Qd_\sigma$ in $(F_T/H_T)(E)$ is zero, unless there exists a permutation $\tau = \rho\sigma^{-1} \in \Sigma_E$ such that

$$\Delta \subseteq \Delta_{\tau^{-1}}Q \subseteq {}^\sigma R^{op} \ ,$$

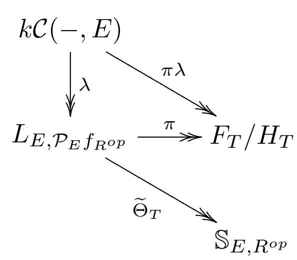
and in that case $Qd_\sigma = d_\rho = d_{\tau\sigma}$, hence $\theta(Qd_\sigma) = \theta(d_\rho) = \Delta_\rho f_{R^{op}}$. On the other hand, by Proposition 3.10, $Q\theta(d_\sigma) = Q\Delta_\sigma f_{R^{op}}$ is zero, unless there exists a permutation $\tau \in \Sigma_E$ such that $\Delta \subseteq \Delta_{\tau^{-1}}Q \subseteq {}^\sigma R^{op}$, in which case

$$Q\theta(d_\sigma) = Q\Delta_\sigma f_{R^{op}} = \Delta_{\tau\sigma} f_{R^{op}} = \Delta_\rho f_{R^{op}} = \theta(Qd_\sigma) \ .$$

It follows that the map $\theta : (F_T/H_T)(E) \to \mathcal{P}_E f_{R^{op}}$ is a homomorphism of $k\mathcal{C}(E,E)$-modules. It sends $\iota = d_{\mathrm{id}}$ to $\Delta_{\mathrm{id}} f_{R^{op}} = f_{R^{op}}$. Such a homomorphism is unique, since $(F_T/H_T)(E)$ is generated by $\iota$ as a $k\mathcal{C}(E,E)$-module. Moreover $\theta$ is clearly an isomorphism, which completes the proof of (a).

(b) Observe first that the functor $F_T$ is generated by $\iota \in F_T(E)$ : indeed, if $X$ is a finite set and $\varphi : X \to T$ is any map, recall that $\Gamma_\varphi\, \iota = \varphi$, where $\Gamma_\varphi$ is the correspondence defined in Lemma 12.6.

By Assertion (a), the map $\theta^{-1} : \mathcal{P}_E f_{R^{op}} \to (F_T/H_T)(E)$ is an isomorphism of $k\mathcal{C}(E,E)$-modules, mapping $f_{R^{op}}$ to $\iota$. By Lemma 2.3, there is a unique morphism of correspondence functors $\pi : L_{E,\mathcal{P}_E f_{R^{op}}} \to F_T/H_T$ such that $\pi_E = \theta^{-1}$. Since $\pi_E$ is surjective, and since $F_T/H_T$ is generated by the image of $\iota$ in $(F_T/H_T)(E)$, it follows that $\pi$ is surjective. Moreover, since $L_{E,\mathcal{P}_E f_{R^{op}}}(E) \cong \mathcal{P}_E f_{R^{op}}$, there exists a unique morphism of correspondence functors $\lambda : k\mathcal{C}(-,E) \to L_{E,\mathcal{P}_E f_{R^{op}}}$ such that $\lambda_E(\Delta_E) = f_{R^{op}}$ in $\mathcal{P}_E f_{R^{op}}$. The morphism $\lambda$ is surjective, because $L_{E,\mathcal{P}_E f_{R^{op}}}$ is generated by $f_{R^{op}} \in L_{E,\mathcal{P}_E f_{R^{op}}}(E)$. So we have a diagram




where $\widetilde{\Theta}_T$ is the canonical surjection $L_{E,\mathcal{P}_E f_{R^{op}}} \to S_{E,R^{op},\mathcal{P}_E f_{R^{op}}} = \mathbb{S}_{E,R^{op}}$. Moreover, the composition $\widetilde{\Theta}_T\lambda$ is equal to the morphism

$$\omega_{E,R^{op}} : k\mathcal{C}(-E) \longrightarrow \mathbb{S}_{E,R}$$

given by Lemma 4.9. All we have to do is to show that $\widetilde{\Theta}_T$ factors through $\pi$, i.e. that $\operatorname{Ker}\pi \subseteq \operatorname{Ker}\widetilde{\Theta}_T$. Equivalently, we have to show that $\operatorname{Ker}\pi\lambda \subseteq \operatorname{Ker}\omega_{E,R^{op}}$.

To see this, let $X$ be a finite set, and let $u = \sum\limits_{S \in \mathcal{C}(X,E)} u_S S$ be an element of $k\mathcal{C}(X,E)$, where $u_S \in k$. Since $\pi_X\lambda_X(S) = \pi_X(S \otimes f_{R^{op}}) = S\pi_E(f_{R^{op}}) = S\iota$, we obtain that $u \in \operatorname{Ker}\pi_X\lambda_X$ if and only if

$$\textbf{(13.2)} \qquad \forall \varphi : X \to T,\ \varphi(X) \supseteq E \implies \sum_{\substack{S \in \mathcal{C}(X,E) \\ S\iota = \varphi}} u_S = 0 \ .$$

On the other hand, by Lemma 4.9, the element $u$ belongs to $\operatorname{Ker}\omega_{E,R^{op},X}$ if and only if for any $V \in \mathcal{C}(E,X)$,

$$\sum_{\Delta \subseteq VS \subseteq R^{op}} u_S = 0\ .$$

**13.3. Claim.** *Let $Q \in \mathcal{C}(E,E)$. Then $\Delta \subseteq Q \subseteq R^{op}$ if and only if $Q\iota = \iota$.*

Postponing the proof of this claim, we have $u \in \operatorname{Ker}\omega_{E,R^{op},X}$ if and only if

$$\forall V \in \mathcal{C}(E,X), \quad \sum_{\substack{S \in \mathcal{C}(X,E) \\ VS\iota = \iota}} u_S = 0\ .$$

Equivalently

$$\forall V \in \mathcal{C}(E,X), \quad \sum_{\substack{\varphi : X \to T \\ V\varphi = \iota}} \ \sum_{\substack{S \in \mathcal{C}(X,E) \\ S\iota = \varphi}} u_S = 0\ .$$

Now if $V\varphi = \iota$, then $\varphi \notin H_T(X)$, otherwise we would have $\iota \in H_T(E)$. Hence $\varphi(X) \supseteq E$. Now Condition 13.2 shows that $\operatorname{Ker}\pi_X\lambda_X \subseteq \operatorname{Ker}\omega_{E,R^{op},X}$.

It remains to prove Claim 13.3. Suppose first that $\Delta \subseteq Q \subseteq R^{op}$. Then for any $e \in E$,

$$(Q\iota)(e) = \bigvee_{(e,f) \in Q} \iota(f) = \bigvee_{(e,f) \in Q} f\ .$$

Since $(e,e) \in Q$, it follows that $(Q\iota)(e) \geq_T e$. On the other hand since $Q \subseteq R^{op}$, if $(e,f) \in Q$, then $f \leq_R e$, hence $f \leq_T e$. Thus $(Q\iota)(e) \leq_T e$. It follows that $(Q\iota)(e) = e$ for any $e \in E$, i.e. $Q\iota = \iota$.

Conversely, if $Q\iota = \iota$, then

$$\forall e \in E, \quad \bigvee_{(e,f) \in Q} f = e\ .$$

As $e$ is irreducible, it follows that $(e,e) \in Q$ for any $e \in E$, i.e. $\Delta \subseteq Q$. Moreover if $(e,f) \in Q$, then $f \leq_T e$, hence $f \leq_R e$ and so $Q \subseteq R^{op}$. This proves the claim and completes the proof of (b).

(c) At the beginning of the proof of (b), we have already noticed that $F_T$ is generated by $\iota \in F_T(E)$. Since $\Theta_T : F_T/H_T \to \mathbb{S}_{E,R^{op}}$ is surjective, $\mathbb{S}_{E,R^{op}}$ is generated by $\Theta_{T,E}(\iota)$. This completes the proof of Theorem 13.1. ∎

Whenever we need it, we shall view $\Theta_{T,X}$ as a map $F_T(X) \to \mathbb{S}_{E,R^{op}}(X)$ having $H_T(X)$ in its kernel. Thus we can evaluate $\Theta_{T,X}$ on any element of $F_T(X)$, that is, on any map $\varphi : X \to T$. The following result will be our main tool for analyzing evaluations of fundamental functors and simple functors (see Section 17).

**13.4. Theorem.** *Let $T$ be a finite lattice, let $(E,R)$ be the full subposet of its irreducible elements, and let $X$ be a finite set. The kernel of the map*

$$\Theta_{T,X} : F_T(X) \to \mathbb{S}_{E,R^{op}}(X)$$

*is equal to the set of linear combinations $\sum\limits_{\varphi : X \to T} \lambda_\varphi \varphi$, where $\lambda_\varphi \in k$, such that for any map $\psi : X \to I^{\uparrow}(E,R)$*

$$\sum_{\substack{\varphi \\ \Gamma_\psi^{op}\Gamma_\varphi = R^{op}}} \lambda_\varphi = 0\ .$$

*Here $\Gamma_\varphi = \{(x,e) \in X \times E \mid e \in \varphi(x)\} \subseteq \mathcal{C}(X,E)$, as in Notation 5.7.*

**Proof :** As in the proof of Theorem 13.1 above, there is a sequence of surjective morphisms of correspondence functors

$$k\mathcal{C}(-,E) \xrightarrow{\lambda} L_{E,\mathcal{P}_E f_{R^{op}}} \xrightarrow{\pi} F_T/H_T \xrightarrow{\Theta_T} \mathbb{S}_{E,R^{op}}$$

The generator $\iota \in (F_T/H_T)(E)$ is the image under $\pi_E$ of the element $f_{R^{op}} \in \mathcal{P}_E f_{R^{op}} = L_{E,\mathcal{P}_E f_{R^{op}}}(E)$. Therefore, any map $\varphi : X \to T$, viewed as an element of $(F_T/H_T)(X)$, is the image under $\pi_X$ of the element $\Gamma_\varphi \otimes f_{R^{op}} \in L_{E,\mathcal{P}_E f_{R^{op}}}(X)$, because

$$\pi_X(\Gamma_\varphi \otimes f_{R^{op}}) = \Gamma_\varphi \pi_E(f_{R^{op}}) = \Gamma_\varphi\, \iota = \varphi \ ,$$

by Lemma 12.6. Then $\Gamma_\varphi \otimes f_{R^{op}}$ is in turn the image under $\lambda_X$ of $\Gamma_\varphi \in k\mathcal{C}(X,E)$.

Recall that the composition $\Theta_T \circ \pi \circ \lambda$ is equal to the morphism $\omega_{E,R^{op}}$ of Lemma 4.9. Therefore the image of $\varphi : X \to T$ under the map $\Theta_{T,X}$ is equal to $\omega_{E,R^{op}}(\Gamma_\varphi)$. It follows that the linear combination $u = \sum\limits_{\varphi:X\to T} \lambda_\varphi \varphi$ lies in $\operatorname{Ker}\Theta_{T,X}$ if and only if $\sum\limits_{\varphi:X\to T} \lambda_\varphi \Gamma_\varphi$ belongs to $\operatorname{Ker}\omega_{E,R^{op}}$. By Lemma 4.9

$$(\mathbf{13.5}) \qquad \forall S \in \mathcal{C}(E,X), \quad \sum_{\Delta \subseteq S\Gamma_\varphi \subseteq R^{op}} \lambda_\varphi = 0 \ .$$

But $\Gamma_\varphi R^{op} = \Gamma_\varphi$ by Lemma 5.8. Multiplying on the right by $R^{op}$ the inclusions $\Delta \subseteq S\Gamma_\varphi \subseteq R^{op}$, we see that the sum runs over all $\varphi$ satisfying $S\Gamma_\varphi = R^{op}$. It follows that Condition 13.5 is equivalent to

$$(\mathbf{13.6}) \qquad \forall S \in \mathcal{C}(E,X), \quad \sum_{S\Gamma_\varphi = R^{op}} \lambda_\varphi = 0 \ .$$

This holds for $S$ if and only if it holds for $R^{op}S$, so we can assume that $R^{op}S = S$. Now $R^{op}S = S$ if and only if $S^{op}R = S^{op}$. By Lemma 5.8, there exists a map $\psi : X \to I^{\uparrow}(E,R)$ such that $S^{op} = \Gamma_\psi$, that is,

$$S = \Gamma_\psi^{op} = \big\{(e,x) \mid e \in \psi(x)\big\} \ .$$

Thus the condition $S\Gamma_\varphi = R^{op}$ becomes $\Gamma_\psi^{op}\Gamma_\varphi = R^{op}$. ∎

**13.7. Corollary.** *Let $X$ be a finite set. The kernel of the map*

$$\overline{\omega}_{E,R^{op},X} : k\mathcal{C}(X,E)R^{op} \to \mathbb{S}_{E,R^{op}}(X)$$

*is equal to the set of linear combinations $\sum\limits_{\varphi:X\to T} \lambda_\varphi \Gamma_\varphi$, where $\lambda_\varphi \in k$, such that for any map $\psi : X \to I^{\uparrow}(E,R)$*

$$\sum_{\substack{\varphi \\ \Gamma_\psi^{op}\Gamma_\varphi = R^{op}}} \lambda_\varphi = 0 \ .$$

**Proof :** For the lattice $T$, we choose $T = I_{\downarrow}(E,R)$, so that $F_T \cong k\mathcal{C}(-,E)R^{op}$ by Proposition 11.4. Since $R^{op} f_{R^{op}} = f_{R^{op}}$, the morphism $\lambda : k\mathcal{C}(-,E) \to L_{E,\mathcal{P}_E f_{R^{op}}}$ factorizes through $\overline{\lambda} : k\mathcal{C}(-,E)R^{op} \to L_{E,\mathcal{P}_E f_{R^{op}}}$. Moreover the composite

$$k\mathcal{C}(-,E)R^{op} \xrightarrow{\overline{\lambda}} L_{E,\mathcal{P}_E f_{R^{op}}} \xrightarrow{\pi} F_T/H_T \xrightarrow{\Theta_T} \mathbb{S}_{E,R^{op}}$$

is equal to the morphism $\overline{\omega}_{E,R^{op}}$ of Lemma 4.9, and $\pi \circ \overline{\lambda}$ maps $\Gamma_\varphi$ to $\varphi$ for any map $\varphi : X \to T$ (because again $\Gamma_\varphi\, \iota = \varphi$). It follows that $\sum\limits_{\varphi:X\to T} \lambda_\varphi \varphi$ is in the kernel of $\Theta_{T,X}$ if and only if $\sum\limits_{\varphi:X\to T} \lambda_\varphi \Gamma_\varphi$ is in the kernel of $\overline{\omega}_{E,R^{op},X}$. But by Lemma 5.8, any element $V \in \mathcal{C}(X,E)R^{op}$ has the form $V = \Gamma_\varphi$ for some map $\varphi : X \to T$. It follows that any element in $k\mathcal{C}(X,E)R^{op}$ can be written $\sum_{\varphi:X\to T}\lambda_\varphi\Gamma_\varphi$ and the result follows from Theorem 13.4. □

Now we want to characterize the condition $\Gamma_\psi^{op}\Gamma_\varphi = R^{op}$ which appears in both Theorem 13.4 and Corollary 13.7. We use the following notation.

**13.8. Notation.** *Let $T$ be a finite lattice, let $(E,R)$ be the full subposet of its irreducible elements, let $\psi : X \to I^\uparrow(E,R)$ be any map, and let $\varphi : X \to T$ be any map. We define the function $\wedge\psi : X \to T$ by*

$$\forall x \in X,\ \ \wedge\psi(x) = \bigwedge_{e\in\psi(x)} e\ ,$$

*where $\bigwedge$ is the meet in the lattice $T$.*

*Moreover, the notation $\varphi \leq \wedge\psi$ means that $\varphi(x) \leq_T \wedge\psi(x)$ for all $x \in X$.*

**13.9. Lemma.** *Let $T$ be a finite lattice, let $(E,R)$ be the full subposet of its irreducible elements, and let $X$ be a finite set. Let $\varphi : X \to T$ be a map and let $\Gamma_\varphi = \{(x,e) \in X\times E \mid e \leq_T \varphi(x)\}$ be the associated correspondence. Let $\psi : X \to I^\uparrow(E,R)$ be a map and let $\Gamma_\psi^{op} = \big\{(e,x) \in E\times X \mid e \in \psi(x)\big\}$ be the associated correspondence. The following conditions are equivalent.*

*(a)* $\Gamma_\psi^{op}\varphi = \iota$.
*(b)* $\Gamma_\psi^{op}\Gamma_\varphi\iota = \iota$.
*(c)* $\Delta \subseteq \Gamma_\psi^{op}\Gamma_\varphi \subseteq R^{op}$.
*(d)* $\Gamma_\psi^{op}\Gamma_\varphi = R^{op}$.
*(e)* $\varphi \leq \wedge\psi$ *and* $\forall e \in E$, $\exists x \in X$ *such that* $\varphi(x) = e$ *and* $\psi(x) = [e,\cdot[_E$.
*(f)* $\forall t \in T$, $\psi\big(\varphi^{-1}(t)\big) \subseteq [t,\cdot[_T \cap E$ *and* $\forall e \in E$, $\psi\big(\varphi^{-1}(e)\big) = [e,\cdot[_E$.

**Proof :** (a) ⇔ (b). By Lemma 12.6, we have $\varphi = \Gamma_\varphi\iota$.

(b) ⇔ (c). This follows from Claim 13.3.

(c) ⇔ (d). If (c) holds, multiply on the right by $R^{op}$ and use the equality $\Gamma_\varphi R^{op} = \Gamma_\varphi$ of Lemma 5.8 to obtain (d). On the other hand, it is clear that (d) implies (c).

(d) ⇒ (e). Suppose that $\Gamma_\psi^{op}\Gamma_\varphi = R^{op}$ and let $x \in X$. Then for all $f \leq_T \varphi(x)$ and for all $e \in \psi(x)$, we have $(e,x) \in \Gamma_\psi^{op}$ and $(x,f) \in \Gamma_\varphi$, hence $(e,f) \in R^{op}$, that is, $f \leq_R e$, hence $f \leq_T e$. Therefore $\varphi(x) = \bigvee_{f\leq_T\varphi(x)} f \leq_T e$, whenever $e \in \psi(x)$. Thus

$$\forall x \in X,\ \ \varphi(x) \leq_T \bigwedge_{e\in\psi(x)} e = \wedge\psi(x)\ ,$$

that is, $\varphi \leq \wedge\psi$. This shows that the first property in (e) holds.

Since $(e,e) \in R^{op}$, there exists $x_e \in X$ such that $e \leq_T \varphi(x_e)$ and $e \in \psi(x_e)$. Then for all $f \leq_T \varphi(x_e)$, we have $(e,x_e) \in \Gamma_\psi^{op}$ and $(x_e,f) \in \Gamma_\varphi$, hence $(e,f) \in R^{op}$, that is, $f \leq_R e$, or in other words $f \leq_T e$. Thus again $\varphi(x_e) = \bigvee_{f\leq_T\varphi(x_e)} f \leq_T e$, hence $\varphi(x_e) = e$. Moreover, if $g \in E$ with $g \in \psi(x_e)$, then $(g,x_e) \in \Gamma_\psi^{op}$ and $(x_e,e) \in \Gamma_\varphi$, hence $(g,e) \in R^{op}$, that is, $e \leq_T g$. Therefore $\psi(x_e) \subseteq [e,\cdot[_E$. But we also have $[e,\cdot[_E \subseteq \psi(x_e)$, as $e \in \psi(x_e)$ and $\psi(x_e)$ is an upper ideal of $E$. Thus $\psi(x_e) = [e,\cdot[_E$. This shows that the second property in (e) holds.

(e) ⇒ (d). For any $e \in E$, there exists $x_e \in X$ such that $\varphi(x_e) = e$ and $\psi(x_e) = [e,\cdot[_E$. If now $(f,e) \in R^{op}$, then $e \leq_R f$, hence $f \in \psi(x_e)$. Since we also have $e \leq_R \varphi(x_e)$, we obtain $(f,x_e) \in \Gamma_\psi^{op}$ and $(x_e,e) \in \Gamma_\varphi$. Thus $R^{op} \subseteq \Gamma_\psi^{op}\Gamma_\varphi$.

Moreover if $(f,e) \in \Gamma_\psi^{op}\Gamma_\varphi$, then there exists $x \in X$ such that $f \in \psi(x)$ and $e \leq_T \varphi(x)$. Since $\varphi \leq \wedge\psi$, we have $\varphi(x) \leq_T \bigwedge_{f\in\psi(x)} f$. It follows that $e \leq_T f$, hence $e \leq_R f$, that is, $(f,e) \in R^{op}$. Thus $\Gamma_\psi^{op}\Gamma_\varphi \subseteq R^{op}$. Therefore we obtain $\Gamma_\psi^{op}\Gamma_\varphi = R^{op}$.

(e) $\Leftrightarrow$ (f). We are going to slightly abuse notation by setting, for any subset $Y$ of $X$, $\psi(Y) = \bigcup_{x\in Y} \psi(x)$. Taking $t = \varphi(x)$, the first condition in (e) is equivalent to

$$\forall t \in T,\ e \in \psi\big(\varphi^{-1}(t)\big) \Longrightarrow t \leq_T e\ ,$$

which in turn is equivalent to

$$\forall t \in T,\ \psi\big(\varphi^{-1}(t)\big) \subseteq [t,\cdot[_T \cap E\ .$$

In particular $\psi\big(\varphi^{-1}(e)\big) \subseteq [e,\cdot[_E$ for all $e \in E$ because $[e,\cdot[_T \cap E = [e,\cdot[_E$. But the second condition in (e) says that $e$ must belong to $\psi\big(\varphi^{-1}(e)\big)$, so we get $\psi\big(\varphi^{-1}(e)\big) = [e,\cdot[_E$. This shows that the second condition in (e) is equivalent to

$$\forall e \in E,\ \ \psi\big(\varphi^{-1}(e)\big) = [e,\cdot[_E\ .$$

This completes the proof of Lemma 13.9. □

Condition (d) will play an important role in the proof of Theorem 14.16, condition (f) will be essential in the proof of Theorem 17.5, and condition (e) will be a main tool used in the proof of Theorem 17.10.

## 14. Duality and opposite lattices

By Theorem 13.1, any fundamental functor $\mathbb{S}_{E,R}$ is isomorphic to a quotient of some functor associated to a lattice. One of our main purposes in this section is to use duality to realize $\mathbb{S}_{E,R}$ as a subfunctor of some functor associated to a lattice. This requires to define a duality between $F_T$ and $F_{T^{op}}$.

Let $F$ be a correspondence functor over $k$. Recall from Definition 3.5, that the *dual* $F^\natural$ of $F$ is the correspondence functor defined on a finite set $X$ by

$$F^\natural(X) = \mathrm{Hom}_k\big(F(X), k\big)\ .$$

If $Y$ is a finite set and $R \subseteq Y \times X$, then the map $F^\natural(R) : F^\natural(X) \to F^\natural(Y)$ is defined by

$$\forall \alpha \in F^\natural(X),\ \ F^\natural(R)(\alpha) = \alpha \circ F(R^{op})\ .$$

Recall that $\mathcal{L}$ denotes the category of finite lattices (Definition 11.5), and $k\mathcal{L}$ its $k$-linearization (Definition 2.1). For any finite lattice $T = (T, \vee, \wedge)$, denote by $T^{op} = (T, \wedge, \vee)$ the *opposite* lattice, i.e. the set $T$ ordered with the opposite partial order. For simplicity throughout this section, we write $\leq$ for $\leq_T$ and $\leq^{op}$ for $\leq_{T^{op}}$.

**14.1. Lemma.** *The assignment $T \mapsto T^{op}$ extends to an isomorphism $\mathcal{L} \to \mathcal{L}^{op}$, and to a $k$-linear isomorphism $k\mathcal{L} \to k\mathcal{L}^{op}$.*

**Proof :** Let $f : T_1 \to T_2$ be a morphism in the category $\mathcal{L}$. For any $t \in T_2$, let $f^{op}(t)$ denote the join in $T_1$ of all the elements $x$ such that $f(x) \leq t$, i.e.

$$f^{op}(t) = \bigvee_{f(x)\leq t} x \; . \tag{14.2}$$

Then $f\big(f^{op}(t)\big) = \bigvee_{f(x)\leq t} f(x) \leq t$, so $f^{op}(t)$ is actually the greatest element of $f^{-1}\big([\hat{0}, t]_{T_2}\big)$, i.e. $f^{-1}\big([\hat{0}, t]_{T_2}\big) = [\hat{0}, f^{op}(t)]_{T_1}$. In other words,

$$\forall t_1 \in T_1,\ \forall t_2 \in T_2,\ \ f(t_1) \leq t_2 \iff t_1 \leq f^{op}(t_2) \; , \tag{14.3}$$

that is, the pair $(f, f^{op})$ is an adjoint pair of functors between the posets $T_1$ and $T_2$, viewed as categories. In those terms, saying that $f$ is a morphism in $\mathcal{L}$ is equivalent to saying that $f$ commutes with colimits in $T_1$ and $T_2$. Hence $f^{op}$ commutes with limits, that is, $f^{op}$ commutes with the meet operation, i.e. it is a morphism of lattices $T_2^{op} \to T_1^{op}$.

In more elementary terms, for any subset $A \subseteq T_2$,

$$\begin{aligned}
[\hat{0}, f^{op}(\bigwedge_{t\in A} t)]_{T_1} &= f^{-1}([\hat{0}, \bigwedge_{t\in A} t]_{T_2})\\
&= f^{-1}(\bigcap_{t\in A} [\hat{0}, t]_{T_2})\\
&= \bigcap_{t\in A} f^{-1}\big([\hat{0}, t]_{T_2}\big)\\
&= \bigcap_{t\in A} [\hat{0}, f^{op}(t)]_{T_1}\\
&= [\hat{0}, \bigwedge_{t\in A} f^{op}(t)]_{T_1} \; .
\end{aligned}$$

It follows that $f^{op}\big(\bigwedge_{t\in A} t\big) = \bigwedge_{t\in A} f^{op}(t)$, i.e. $f^{op}$ is a morphism $T_2^{op} \to T_1^{op}$ in $\mathcal{L}$.

Now denoting by $\leq^{op}$ the opposite order relations on both $T_1$ and $T_2$, Equation 14.3 reads

$$\forall t_2 \in T_2,\ \forall t_1 \in T_1,\ \ f^{op}(t_2) \leq^{op} t_1 \iff t_2 \leq^{op} f(t_1) \; ,$$

which shows that the same construction applied to the morphism $f^{op} : T_2^{op} \to T_1^{op}$ yields $(f^{op})^{op} = f$. This proves that the map $f \mapsto f^{op}$ is a bijection from $\mathcal{L}(T_2, T_1)$ to $\mathcal{L}(T_1^{op}, T_2^{op})$ (recall our notational convention of Section 2 on morphisms in a category).

Now if $f : T_1 \to T_2$ and $g : T_2 \to T_3$ are morphisms in $\mathcal{L}$, the adjunction 14.3 easily implies that $(gf)^{op} = f^{op}g^{op}$. It is clear moreover that $(\mathrm{id}_T)^{op} = \mathrm{id}_{T^{op}}$ for any finite lattice $T$. Hence the assignment $T \mapsto T^{op}$ and $f \mapsto f^{op}$ is an isomorphism $\mathcal{L} \to \mathcal{L}^{op}$, which extends linearly to an isomorphism $k\mathcal{L} \to k\mathcal{L}^{op}$. ∎

**14.4. Definition.** *Let $T$ be a finite lattice and let $X$ be a finite set. For two functions $\varphi : X \to T$ and $\psi : X \to T^{op}$, set*

$$(\varphi, \psi)_X = \begin{cases} 1 & \textit{if } \varphi \leq \psi, \ \textit{ i.e. if } \varphi(x) \leq_T \psi(x), \ \forall x \in X, \\ 0 & \textit{otherwise.} \end{cases}$$

*This definition extends uniquely to a $k$-bilinear form*

$$(-,-)_X : F_T(X) \times F_{T^{op}}(X) \to k \; .$$

*This bilinear form induces a k-linear map* $\Psi_{T,X} : F_{T^{op}}(X) \to (F_T)^\natural(X)$ *defined by* $\Psi_{T,X}(\psi)(\varphi) = (\varphi, \psi)_X$.

We need some notation.

**14.5. Notation.** *Let $T$ be a finite lattice, $X$ and $Y$ finite sets, $Q \subseteq Y \times X$ a correspondence, and $\psi : X \to T^{op}$ a map. We denote by $Q \star \psi$ the action of the correspondence $Q$ on $\psi$. In other words, $Q \star \psi$ is the map $F_{T^{op}}(Q)(\psi) : Y \to T^{op}$. Recall that it is defined by*

$$\forall y \in Y, \ \ (Q \star \psi)(y) = \bigwedge_{(y,x)\in Q} \psi(x) \ ,$$

*because the join in $T^{op}$ is the meet in $T$.*

**14.6. Lemma.**

(a) *With the notation 14.5, the family of bilinear forms in Definition 14.4 satisfy*

$$(\varphi, Q \star \psi)_Y = (Q^{op}\varphi, \psi)_X \ .$$

(b) *The family of maps $\Psi_{T,X} : F_{T^{op}}(X) \to (F_T)^\natural(X)$ form a morphism of correspondence functors $\Psi_T : F_{T^{op}} \to (F_T)^\natural$.*

**Proof :** (a) We have

$$\begin{aligned} \varphi \le Q \star \psi \quad &\iff \quad \forall y \in Y, \ \varphi(y) \le_T Q \star \psi(y) \\ &\iff \quad \forall y \in Y, \ \varphi(y) \le_T \bigwedge_{(y,x)\in Q} \psi(x) \\ &\iff \quad \forall (y,x) \in Q, \ \varphi(y) \le_T \psi(x) \\ &\iff \quad \forall x \in X, \ \bigvee_{(x,y)\in Q^{op}} \varphi(y) \le_T \psi(x) \\ &\iff \quad Q^{op}\varphi \le \psi \ . \end{aligned}$$

(b) The equation in part (a) also reads

$$\Psi_{T,X}(\psi)(Q^{op}\varphi) = \Psi_{T,Y}(Q \star \psi)(\varphi) \ ,$$

that is $Q\Psi_{T,X}(\psi) = \Psi_{T,Y}(Q \star \psi)$. □

**14.7. Remark.** Let $T = I_\downarrow(E, R^{op})$ be the lattice corresponding to a poset $(E, R^{op})$. Then $T^{op} = I_\downarrow(E, R^{op})^{op}$ is isomorphic, via complementation, to the lattice $I_\downarrow(E, R)$. Using the isomorphisms of Proposition 11.4

$$F_T = F_{I_\downarrow(E,R^{op})} \cong k\mathcal{C}(-,E)R \ , \qquad F_{T^{op}} = F_{I_\downarrow(E,R)} \cong k\mathcal{C}(-,E)R^{op} \ ,$$

we can transport the bilinear forms $(-,-)_X$ defined in 14.4 and obtain a pairing

$$k\mathcal{C}(-,E)R \times k\mathcal{C}(-,E)R^{op} \longrightarrow k \ .$$

It is easy to check, using complementation, that this pairing coincides with the one obtained in Remark 9.5.

**14.8. Notation.** *Let $T$ be a finite lattice, $X$ a finite set, and $\varphi : X \to T$ a map. We denote by $\varphi^\star$ the element of $F_{T^{op}}(X)$ defined by*

$$\varphi^\star = \sum_{\substack{\rho:X\to T\\ \rho\leq\varphi}} \mu(\rho,\varphi)\rho^\circ \ ,$$

*where $\rho^\circ$ is the function $\rho$, viewed as a map $X \to T^{op}$, and where $\mu(\rho,\varphi)$ is the Möbius function of the poset of maps from $X$ to $T$, in which $\rho \leq \varphi$ if and only if $\rho(x) \leq \varphi(x)$ in $T$ for any $x \in X$. Recall that $\mu(\rho,\varphi)$ can be computed as follows :*

$$\mu(\rho,\varphi) = \prod_{x\in X} \mu_T\big(\rho(x),\varphi(x)\big) \ ,$$

*where $\mu_T$ is the Möbius function of the poset $T$.*

**14.9. Theorem.** *Let $T$ be a finite lattice.*

(a) *Let $X$ be a finite set. The bilinear form (14.4) is nondegenerate, in the strong sense, namely it induces an isomorphism*

$$\Psi_{T,X} : F_{T^{op}}(X) \to (F_T)^\natural(X) \ .$$

*More precisely, $\{\varphi^\star \mid \varphi : X \to T\}$ is the dual basis, in $F_{T^{op}}(X)$, of the $k$-basis of functions $X \to T$, in $F_T(X)$.*

(b) *$\Psi_T : F_{T^{op}} \to (F_T)^\natural$ is an isomorphism of correspondence functors.*

(c) *The functor $T \mapsto F_{T^{op}}$ and the functor $T \mapsto (F_T)^\natural$ are naturally isomorphic functors from $k\mathcal{L}$ to $\mathcal{F}_k^{op}$. More precisely, the family of isomorphisms $\Psi_T$, for finite lattices $T$, form a natural transformation $\Psi$ between the functor $T \mapsto F_{T^{op}}$ and the functor $T \mapsto (F_T)^\natural$.*

**Proof :** (a) The set $\{\rho^\circ \mid \rho^\circ : X \to T^{op}\}$ is a $k$-basis of the free $k$-module $F_{T^{op}}(X)$. It follows that $\{\varphi^\star \mid \varphi : X \to T\}$ is also a $k$-basis of $F_{T^{op}}(X)$, because the integral matrix of Möbius coefficients $\mu(\rho,\phi)$ is unitriangular, hence invertible over $\mathbb{Z}$. Actually its inverse is the adjacency matrix of the order relation $\rho \leq \varphi$ on the set of maps $X \to T$.

Now, for any two functions $\varphi, \lambda : X \to T$,

$$(\lambda,\varphi^\star)_X = \sum_{\substack{\rho:X\to T\\ \rho\leq\varphi}} \mu(\rho,\varphi)(\lambda,\rho^\circ)_X = \sum_{\substack{\rho:X\to T\\ \lambda\leq\rho\leq\varphi}} \mu(\rho,\varphi) = \delta_{\lambda,\varphi} \ , \qquad (\mathbf{14.10})$$

where $\delta_{\lambda,\varphi}$ is the Kronecker symbol (the last equality coming from the definition of the Möbius function). This shows that $\{\varphi^\star \mid \varphi : X \to T\}$ is the dual basis, in $F_{T^{op}}(X)$, of the $k$-basis of functions $X \to T$, in $F_T(X)$.

(b) This follows immediately from (a). Another way of seeing this is to build an explicit inverse $\Phi_T$ of $\Psi_T$. For each finite set $X$, we define a linear map $\Phi_{T,X} : (F_T)^\natural(X) \to F_{T^{op}}(X)$ by setting

$$\forall \alpha \in (F_T)^\natural(X), \ \ \Phi_{T,X}(\alpha) = \sum_{\varphi:X\to T} \alpha(\varphi)\varphi^\star \ .$$

Then, for any function $\lambda : X \to T$,

$$\big(\Psi_{T,X}\Phi_{T,X}(\alpha)\big)(\lambda) = \big(\lambda, \Phi_{T,X}(\alpha)\big)_X = \sum_{\varphi:X\to T} \alpha(\varphi)(\lambda,\varphi^\star)_X = \alpha(\lambda) \ ,$$

so $\Psi_{T,X}\Phi_{T,X}$ is the identity map of $(F_T)^\natural(X)$.

On the other hand, $\Psi_{T,X}$ is injective, because if $\Psi_{T,X}(\beta) = 0$, then we write $\beta = \sum\limits_{\varphi:X\to T} a_\varphi \varphi^\star$, where $a_\varphi \in k$, and then for all $\lambda : X \to T$, we get

$$0 = \Psi_{T,X}\big(\sum_{\varphi:X\to T} a_\varphi \varphi^\star\big)(\lambda) = (\lambda, \sum_{\varphi:X\to T} a_\varphi \varphi^\star)_X = \sum_{\varphi:X\to T} a_\varphi (\lambda, \varphi^\star)_X = a_\lambda \ ,$$

so that $\beta = 0$. Therefore $\Psi_{T,X}$ is an isomorphism and $\Phi_{T,X}$ is its inverse.

(c) Let $T'$ be another finite lattice, and let $\Psi_{T'} : F_{T'^{op}} \to (F_{T'})^\natural$ be the corresponding morphism. Let moreover $f : T \to T'$ be a morphism in $\mathcal{L}$. We claim that for any finite set $X$, the square

$$\begin{array}{ccc} F_{T^{op}}(X) & \xrightarrow{\Psi_{T,X}} & (F_T)^\natural(X) \\ {\scriptstyle F_{f^{op}}}\uparrow & & \uparrow{\scriptstyle (F_f)^\natural} \\ F_{T'^{op}}(X) & \xrightarrow[\Psi_{T',X}]{} & (F_{T'})^\natural(X) \end{array}$$

is commutative: indeed, for any functions $\psi : X \to T'^{op}$ and $\varphi : X \to T$,

$$\big((F_f)^\natural \Psi_{T',X}(\psi)\big)(\varphi) = \Psi_{T',X}(\psi)(f \circ \varphi) = (f \circ \varphi, \psi)_X \ ,$$

whereas

$$\big(\Psi_{T,X} F_{f^{op}}(\psi)\big)(\varphi) = \big(\varphi, F_{f^{op}}(\psi)\big)_X = (\varphi, f^{op} \circ \psi)_X \ .$$

Now by 14.3, we have that

$$\begin{aligned} f \circ \varphi \leq \psi \quad &\iff \quad \forall x \in X,\ \ f\big(\varphi(x)\big) \leq \psi(x) \\ &\iff \quad \forall x \in X,\ \ \varphi(x) \leq f^{op}\big(\psi(x)\big) \\ &\iff \quad \varphi \leq f^{op} \circ \psi \ , \end{aligned}$$

which proves our claim. This shows that the isomorphisms $\Psi_T$, for finite lattices $T$, form a natural transformation $\Psi$ of the functor $T \mapsto F_{T^{op}}$ to the functor $T \mapsto (F_T)^\natural$ from $k\mathcal{L}$ to $\mathcal{F}_k^{op}$. This completes the proof of Theorem 14.9. □

**14.11. Corollary.** *Let $k$ be a self-injective ring. Then for any distributive lattice $T$, the functor $F_T$ is projective and injective in $\mathcal{F}_k$.*

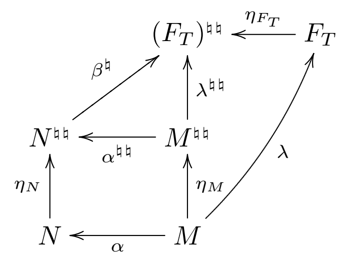
**Proof :** Since $T$ is distributive, the functor $F_T$ is projective by Theorem 11.11, without further assumption on $k$.

If $k$ is self-injective, the functor sending a $k$-module $A$ to its $k$-dual $\mathrm{Hom}_k(A, k)$ is exact. It follows that the functor $M \mapsto M^\natural$ is an exact contravariant endofunctor of the category $\mathcal{F}_k$ (where $M^\natural$ denotes the dual correspondence functor defined in 3.5). Let $\alpha : M \to N$ be an injective morphism in $\mathcal{F}_k$, and let $\lambda : M \to F_T$ be any morphism. Then $\alpha^\natural : N^\natural \to M^\natural$ is surjective, and we have the following diagram with exact row in $\mathcal{F}_k$

$$\begin{array}{ccccc} & & (F_T)^\natural & & \\ & & \downarrow{\scriptstyle \lambda^\natural} & & \\ N^\natural & \xrightarrow{\alpha^\natural} & M^\natural & \longrightarrow & 0 \end{array}$$

Now $(F_T)^\natural \cong F_{T^{op}}$ by Theorem 14.9, and $T^{op}$ is distributive. Hence $F_{T^{op}}$ is projective in $\mathcal{F}_k$, and there exists a morphism $\beta : (F_T)^\natural \to N^\natural$ such that $\alpha^\natural \circ \beta = \lambda^\natural$.

Dualizing once again the previous diagram yields the commutative diagram




where for any functor $M$, we denote by $\eta_M$ the canonical morphism from $M$ to $M^{\natural\natural}$. Now $\eta_{F_T}$ is an isomorphism, because for any finite set $X$, the module $F_T(X)$ is a finitely generated free $k$-module. Let $\varepsilon : N \to F_T$ be defined by $\varepsilon = \eta_{F_T}^{-1} \circ \beta^\natural \circ \eta_N$. Then

$$\varepsilon \circ \alpha = \eta_{F_T}^{-1} \circ \beta^\natural \circ \eta_N \circ \alpha = \eta_{F_T}^{-1} \circ \lambda^{\natural\natural} \circ \eta_M = \eta_{F_T}^{-1} \circ \eta_{F_T} \circ \lambda = \lambda \; .$$

Thus for any injective morphism $\alpha : M \to N$ and any morphism $\lambda : M \to F_T$, there exists a morphism $\varepsilon : N \to F_T$ such that $\varepsilon \circ \alpha = \lambda$. Hence $F_T$ is injective in $\mathcal{F}_k$. ∎

We now want to study the subfunctor generated by a specific element of $F_{T^{op}}(E)$ which will be defined below. We need some more notation.

**14.12. Notation.** *Let $T$ be a finite lattice. If $t \in T$, let $r(t)$ denote the join of all the elements of $T$ strictly smaller than $t$, i.e.*

$$r(t) = \bigvee_{s<t} s$$

Thus $r(t) = t$ if and only if $t$ is not irreducible. If $t$ is irreducible, then $r(t)$ is the unique maximal element of $[\hat{0}, t[$.

**14.13. Notation.** *Let $T$ be a finite lattice and let $(E, R)$ be the full subposet of its irreducible elements. If $A \subseteq E$, let $\eta_A : E \to T$ be the map defined by*

$$\forall e \in E, \;\; \eta_A(e) = \begin{cases} r(e) & \text{if } e \in A \\ e & \text{if } e \notin A \end{cases} \; .$$

*Moreover, let $\gamma_T$ denote the element of $F_{T^{op}}(E)$ defined by*

$$\gamma_T = \sum_{A \subseteq E} (-1)^{|A|} \eta_A^\circ \; ,$$

*where $\eta_A^\circ$ denotes the function $\eta_A$, viewed as a map $E \to T^{op}$.*

We now show that this element $\gamma_T$ has another characterization. Recall that we use a star $\star$, as in Notation 14.5, for the action of a correspondence on evaluations of $F_{T^{op}}$.

**14.14. Lemma.** *Let $T$ be a finite lattice, let $(E, R)$ be the full subposet of its irreducible elements, and let $\iota : E \to T$ be the inclusion map.*

(a) *The element $\gamma_T$ is equal to $\iota^*$ (using the notation defined in 14.8).*
(b) *$R \star \gamma_T = \gamma_T$.*

**Proof :** (a) By definition, $\iota^\star = \sum_{\rho \leq \iota} \mu(\rho, \iota) \rho^\circ$, where $\rho^\circ$ is the function $\rho$, viewed as a map $X \to T^{op}$, and where $\mu$ is the Möbius function of the poset of functions from $X$ to $T$ (see Notation 14.8). Furthermore

$$\mu(\rho, \iota) = \prod_{e \in E} \mu_T\big(\rho(e), \iota(e)\big) \; ,$$

where $\mu_T$ is the Möbius function of the poset $T$. Now $\mu_T\big(\rho(e),\iota(e)\big)=\mu_T\big(\rho(e),e\big)$ is equal to 0 if $\rho(e)<r(e)$, because in that case the interval $]\rho(e),e[_T$ has a greatest element $r(e)$. Moreover $\mu_T\big(\rho(e),e\big)$ is equal to -1 if $\rho(e)=r(e)$, and to +1 if $\rho(e)=e$. It follows that the only maps $\rho$ appearing in the sum above are of the form $\rho=\eta_A$ for some subset $A\subseteq E$ and $\mu(\eta_A,\iota)=(-1)^{|A|}$. Therefore

$$\iota^\star=\sum_{A\subseteq E}(-1)^{|A|}\eta_A^\circ=\gamma_T\ .$$

(b) For any $A\subseteq E$ and any $e\in E$,

$$(R*\eta_A)(e)=\bigwedge_{(e,e')\in R}\eta_A(e')=\bigwedge_{e\leq e'}\eta_A(e')=\eta_A(e)\ ,$$

since $e<e'$ implies $\eta_A(e)\leq e\leq r(e')\leq\eta_A(e')$. Therefore $R\star\gamma_T=\gamma_T$. ∎

Our aim is to show that the subfunctor $\langle\gamma_T\rangle$ of $F_{T^{op}}$ generated by $\gamma_T$ is isomorphic to the fundamental correspondence functor $\mathbb{S}_{E,R}$. We first show that $\langle\gamma_T\rangle$ is independent of the choice of $T$.

**14.15. Lemma.** *Let $f:T\to T'$ be a morphism in $\mathcal{L}$, let $E=\mathrm{Irr}(T)$, and let $E'=\mathrm{Irr}(T')$. Suppose that the restriction of $f$ to $E$ is an isomorphism of posets $f_|:E\xrightarrow{\cong}E'$.*

*(a) The map $f$ is surjective and $fr(e)=rf(e)$ for any $e\in E$.*

*(b) The map $f^{op}:T'^{op}\to T^{op}$ restricts to a bijection $f^{op}_|:E'\xrightarrow{\cong}E$, which is inverse to $f_|$. Moreover $f^{op}r(e')=rf^{op}(e')$ for any $e'\in E'$.*

*(c) $f^{op}:T'^{op}\to T^{op}$ induces an injective morphism $F_{f^{op}}:F_{T'^{op}}\to F_{T^{op}}$ and an isomorphism $\langle\gamma_{T'}\rangle\cong\langle\gamma_T\rangle$.*

**Proof :** (a) Since any element of $T'$ is a join of irreducible elements, which are in the image of $f$, and since $f$ commutes with joins, the map $f$ is surjective.

Let $e\in E$. By assumption $f(e)\in E'$. The condition $r(e)<e$ implies $f\big(r(e)\big)\leq f(e)$. Moreover $r(e)=\bigvee_{\substack{e_1\in E\\ e_1<e}}e_1$, hence $fr(e)=\bigvee_{\substack{e_1\in E\\ e_1<e}}f(e_1)$. Thus if $fr(e)=f(e)$, then there exists $e_1<e$ such that $f(e_1)=f(e)$, contradicting the assumption on $f$. It follows that $fr(e)\leq rf(e)$.

Now $rf(e)=\bigvee_{\substack{e'\in E'\\ e'<f(e)}}e'$, and each $e'\in E'$ with $e'<f(e)$ can be written $e'=f(e_1)$, for $e_1\in E$ with $e_1<e$. It follows that $rf(e)\leq\bigvee_{\substack{e_1\in E\\ e_1<e}}f(e_1)=fr(e)$. Thus $rf(e)=fr(e)$, as was to be shown.

(b) Recall from Equation 14.2 that $f^{op}$ is defined by $f^{op}(t')=\bigvee_{f(t)\leq t'}t$. Let $e'\in E'$. Then there exists $e\in E$ such that $f(e)=e'$. Let $t\in T$ be such that $f(t)\leq e'$ and write $t=\bigvee_{\substack{e_1\in E\\ e_1\leq t}}e_1$. For each $e_1\in E$ with $e_1\leq t$, we have $f(e_1)\leq f(t)\leq e'=f(e)$, hence $e_1\leq e$, and $t\leq e$. It follows that $f^{op}(e')=\bigvee_{f(t)\leq e'}t=e$, so $f^{op}_|$ is a bijection $E'\to E$, inverse to $f_|$. This proves the first statement in (b).

Now let $e\in E$, and set $e'=f(e)\in E'$. First we have $r(e')\leq e'$, thus $f^{op}r(e')\leq f^{op}(e')=e$. If $f^{op}r(e')=e$, then $\bigvee_{f(t)\leq r(e')}t=e$, hence

$$f(e)\leq r(e')<e'=f(e)\ ,$$

a contradiction. Thus $f^{op}r(e') \leq r(e) = rf^{op}(e')$. But we also have

$$rf^{op}(e') = r(e) = \bigvee_{\substack{e_1 \in E \\ e_1 < e}} e_1 = \bigvee_{\substack{e_1 \in E \\ f(e_1) < f(e)}} e_1 \leq \bigvee_{\substack{t \in T \\ f(t) \leq rf(e)}} t = f^{op}rf(e) = f^{op}r(e') ,$$

so $f^{op}r(e') = rf^{op}(e')$, which proves the second statement in (b).

(c) Since $f$ is surjective by (a), so is the morphism $F_f : F_T \to F_{T'}$. By duality and Theorem 14.9, the morphism $F_{f^{op}} : F_{T'^{op}} \to F_{T^{op}}$ can be identified with the dual of $F_f$ and is therefore injective. This proves the first statement in (c).

Now for any $B \subseteq E'$, consider the map $\eta^\circ_B : E' \to T'^{op}$. Then for any $e' \in E'$

$$f^{op}\eta^\circ_B(e') = \begin{cases} f^{op}(e') & \text{if } e' \notin B, \\ f^{op}r(e') = rf^{op}(e') & \text{if } e' \in B. \end{cases}$$

Hence $f^{op} \circ \eta^\circ_B = \eta^\circ_{f_|^{op}(B)} \circ f_|^{op}$, and therefore $f^{op} \circ \gamma_{T'} = \gamma_T \circ f_|^{op}$. It follows that

$$F_{f^{op}}(\gamma_{T'} \circ f_|) = f^{op} \circ \gamma_{T'} \circ f_| = \gamma_T \circ f_|^{op} \circ f_| = \gamma_T \ .$$

Therefore the injective morphism $F_{f^{op}}$ maps the subfunctor $\langle\, \gamma_{T'} \circ f_| \,\rangle$ isomorphically to the subfunctor $\langle \gamma_T \rangle$. But since $f_| : E \to E'$ is a bijection, the subfunctor $\langle\, \gamma_{T'} \circ f_| \,\rangle$ of $F_{T'^{op}}$ is equal to the subfunctor $\langle \gamma_{T'} \rangle$. This proves the second statement in (c). ∎

Recall that we use a star $\star$, as in Notation 14.5, for the action of a correspondence on evaluations of $F_{T^{op}}$. We now come to our main result.

**14.16. Theorem.** *Let $T$ be a finite lattice and let $(E,R)$ be the full subposet of its irreducible elements.*

- *(a) The subfunctor $\langle \gamma_T \rangle$ of $F_{T^{op}}$ generated by $\gamma_T$ is isomorphic to $\mathbb{S}_{E,R}$.*
- *(b) In other words, for any finite set $X$, the module $\mathbb{S}_{E,R}(X)$ is isomorphic to the $k$-submodule of $F_{T^{op}}(X)$ generated by the elements $S \star \gamma_T$, for $S \subseteq X \times E$.*

**Proof :** In view of Lemma 14.15, it suffices to prove the result in the case when $T$ is the lattice $I_\downarrow(E,R)$: indeed, for any other lattice $T'$ with the same poset $(E,R)$ of irreducible elements, the inclusion $E \subseteq T'$ extends to a unique map of lattices $f : T = I_\downarrow(E,R) \to T'$ which induces the identity $E = \mathrm{Irr}(T) \xrightarrow{=} E = \mathrm{Irr}(T')$. This result is not difficult and will be proved later in an appendix (Proposition 21.4). Then $\langle \gamma_{T'} \rangle$ is isomorphic to $\langle \gamma_T \rangle$ by Lemma 14.15, so we now assume that $T = I_\downarrow(E,R)$.

By Yoneda's lemma, there is a unique morphism of correspondence functors $\xi : k\mathcal{C}(-,E) \to F_{T^{op}}$ such that $\xi_E(\Delta_E) = \gamma_T$, and the image of $\xi$ is the subfunctor $\langle \gamma_T \rangle$ of $F_{T^{op}}$ generated by $\gamma_T$.

Then for any finite set $X$, the surjection $\xi_X : k\mathcal{C}(X,E) \to \langle \gamma_T \rangle(X)$ maps the correspondence $S \in \mathcal{C}(X,E)$ to $S \star \gamma_T \in \langle \gamma_T \rangle(X)$. Since $R \star \gamma_T = \gamma_T$ by Lemma 14.14, the morphism $\xi$ factorizes as

$$k\mathcal{C}(-,E) \twoheadrightarrow k\mathcal{C}(-,E)R \xrightarrow{\overline{\xi}} \langle \gamma_T \rangle \ ,$$

where the left hand side morphism is right multiplication by $R$.

By Proposition 11.4, for a finite set $X$, the set $\mathcal{C}(X,E)R$ is the set of correspondences of the form

$$\Gamma_\psi = \{(x,e) \in X \times E \mid e \in \psi(x)\} \ ,$$

where $\psi$ is a map from $X$ to $I^\uparrow(E,R)$.

We want to prove that, for any finite set $X$, the kernel of the surjection

$$\overline{\xi}_X : k\mathcal{C}(X,E)R \to \langle\gamma_T\rangle(X)$$

is equal to the kernel of the surjection

$$\overline{\omega}_{E,R,X} : k\mathcal{C}(X,E)R \to \mathbb{S}_{E,R}(X) .$$

The kernel of the surjection $\overline{\xi}_X$ is the set of linear combinations $\sum\limits_{\psi:X\to I^{\uparrow}(E,R)} \lambda_\psi \Gamma_\psi$, with $\lambda_\psi \in k$, such that

$$\sum_{\psi:X\to I^{\uparrow}(E,R)} \lambda_\psi \Gamma_\psi \star \gamma_T = 0 .$$

Equivalently,

$$\forall \varphi : X \to T,\ \ (\varphi, \sum_\psi \lambda_\psi \Gamma_\psi \star \gamma_T)_X = \sum_\psi \lambda_\psi (\Gamma_\psi^{op}\varphi, \gamma_T)_X = 0 ,$$

where $(-,-)_X$ is the bilinear form of 14.4, using also Equation 14.6.

Now $\gamma_T = \iota^\star$ by Lemma 14.14 and we use $\iota^\star$ instead. By Equation 14.10, we have

$$(\Gamma_\psi^{op}\varphi, \gamma_T)_X = (\Gamma_\psi^{op}\varphi, \iota^\star) = \delta_{\Gamma_\psi^{op}\varphi,\iota}$$

and therefore we obtain the condition

$$\forall \varphi : X \to T, \qquad \sum_{\substack{\psi:X\to I^{\uparrow}(E,R)\\ \Gamma_\psi^{op}\varphi=\iota}} \lambda_\psi = 0 .$$

For $\varphi : X \to T$ and $\psi : X \to I^{\uparrow}(E,R)$, we know from Lemma 13.9 that the property $\Gamma_\psi^{op}\varphi = \iota$ is equivalent to $\Gamma_\psi^{op}\Gamma_\varphi = R^{op}$.

It follows that $u = \sum\limits_{\psi:X\to I^{\uparrow}(E,R)} \lambda_\psi \Gamma_\psi$ is in the kernel of $\overline{\xi}_X$ if and only if

$$\forall \varphi : X \to I_{\downarrow}(E,R), \qquad \sum_{\substack{\psi\\ \Gamma_\psi^{op}\Gamma_\varphi=R^{op}}} \lambda_\psi = 0 .$$

But the condition $\Gamma_\psi^{op}\Gamma_\varphi = R^{op}$ is in turn is equivalent to $\Gamma_\varphi^{op}\Gamma_\psi = R$, and moreover $I_{\downarrow}(E,R) = I^{\uparrow}(E,R^{op})$ and $I^{\uparrow}(E,R) = I_{\downarrow}(E,R^{op})$. This is where we use that the lattice $T$ is equal to $I_{\downarrow}(E,R)$.

Therefore $u = \sum\limits_{\psi:X\to I_{\downarrow}(E,R^{op})} \lambda_\psi \Gamma_\psi$ is in the kernel of $\overline{\xi}_X$ if and only if

$$\forall \varphi : X \to I^{\uparrow}(E,R^{op}), \qquad \sum_{\substack{\psi\\ \Gamma_\varphi^{op}\Gamma_\psi=R}} \lambda_\psi = 0 .$$

By Corollary 13.7, this is equivalent to requiring that $u \in \operatorname{Ker}\overline{\omega}_{E,R,X}$. It follows that $\operatorname{Ker}\overline{\xi}_X = \operatorname{Ker}\overline{\omega}_{E,R,X}$. Consequently, the images of $\overline{\xi}_X$ and $\overline{\omega}_{E,R,X}$ are isomorphic, that is, $\langle\gamma_T\rangle \cong \mathbb{S}_{E,R}$. This completes the proof of Theorem 14.16. □

Since we now know that the subfunctor $\langle\gamma_T\rangle$ of $F_{T^{op}}$ is isomorphic to $\mathbb{S}_{E,R}$, we use again duality to obtain more.

**14.17. Theorem.** *Let $T$ be a finite lattice and let $(E,R)$ be the full subposet of its irreducible elements. We consider orthogonal $k$-submodules with respect to the family of $k$-bilinear forms $(-,-)_X$ defined in 14.4.*

(a) $\langle\gamma_T\rangle^\perp = \operatorname{Ker}\Theta_T$ *(as subfunctors of $F_T$).*
(b) $F_T/\langle\gamma_T\rangle^\perp \cong \mathbb{S}_{E,R^{op}}$.
(c) $\langle\gamma_T\rangle^{\perp\perp} \cong \mathbb{S}^\natural_{E,R^{op}}$.
(d) *There is a canonical injective morphism $\alpha_{E,R} : \mathbb{S}_{E,R} \longrightarrow \mathbb{S}^\natural_{E,R^{op}}$.*

**Proof :** (a) Let $\sum\limits_{\varphi:X\to T}\lambda_\varphi\varphi \in F_T(X)$, where $X$ is a finite set. Then

$$\begin{aligned}
&\sum_\varphi \lambda_\varphi\varphi \in \langle\gamma_T\rangle(X)^\perp \iff \big(\sum_\varphi \lambda_\varphi\varphi, Q\star\gamma_T\big)_X = 0 \quad \forall Q\in\mathcal{C}(X,E)\\
&\iff \big(\sum_\varphi \lambda_\varphi\varphi, Q\star\gamma_T\big)_X = 0 \quad \forall Q\in\mathcal{C}(X,E)R \qquad (\text{because } R\star\gamma_T=\gamma_T)\\
&\iff \sum_\varphi \lambda_\varphi\big(Q^{op}\varphi, \gamma_T\big)_E = 0 \quad \forall Q\in\mathcal{C}(X,E)R \qquad (\text{by 14.6})\\
&\iff \sum_{\substack{\varphi\\ Q^{op}\varphi=\iota}} \lambda_\varphi = 0 \quad \forall Q\in\mathcal{C}(X,E)R \qquad (\text{by 14.10 and Lemma 14.14})\\
&\iff \sum_{\substack{\varphi\\ \Gamma_\psi^{op}\varphi=\iota}} \lambda_\varphi = 0 \quad \forall \psi : X\to I^\uparrow(E,R) \qquad (\text{by Proposition 11.4})\\
&\iff \sum_{\substack{\varphi\\ \Gamma_\psi^{op}\Gamma_\varphi=R^{op}}} \lambda_\varphi = 0 \quad \forall \psi : X\to I^\uparrow(E,R) \qquad (\text{by Lemma 13.9})\\
&\iff \sum_{\varphi:X\to T} \lambda_\varphi\varphi \in \operatorname{Ker}\Theta_{T,X} \quad (\text{by Theorem 13.4})
\end{aligned}$$

Therefore $\langle\gamma_T\rangle(X)^\perp = \operatorname{Ker}\Theta_{T,X}$.

(b) This follows immediately from (a) and Theorem 13.1.

(c) This follows immediately from (b) and duality.

(d) There is an obvious inclusion $\langle\gamma_T\rangle \subseteq \langle\gamma_T\rangle^{\perp\perp}$. Now we have $\langle\gamma_T\rangle \cong \mathbb{S}_{E,R}$ by Theorem 14.16 and $\langle\gamma_T\rangle^{\perp\perp} \cong \mathbb{S}^\natural_{E,R^{op}}$ by (c). Thus we obtain a canonical injective morphism $\mathbb{S}_{E,R} \longrightarrow \mathbb{S}^\natural_{E,R^{op}}$. □

**14.18. Remark.** We will prove later that $\alpha_{E,R} : \mathbb{S}_{E,R} \longrightarrow \mathbb{S}^\natural_{E,R^{op}}$ is an isomorphism (see Theorem 18.1). This is easy to prove if $k$ is a field, because the inclusion $\langle\gamma_T\rangle \subseteq \langle\gamma_T\rangle^{\perp\perp}$ must be an equality since the pairing 14.4 is nondegenerate, by Theorem 14.9.

We end this section with a description of the dual of a simple functor. We assume that $k$ is a field and we let $S_{E,R,V}$ be a simple correspondence functor over $k$. Part (e) of Theorem 14.17 suggests that the index $R$ must become $R^{op}$ after applying duality. We now show that this is indeed the case.

**14.19. Theorem.** *Let $k$ be a field. The dual $S^\natural_{E,R,V}$ of the simple functor $S_{E,R,V}$ is isomorphic to $S_{E,R^{op},V^\natural}$, where $V^\natural$ denotes the ordinary dual of the $k\operatorname{Aut}(E,R)$-module $V$.*

**Proof :** We use Section 3 and consider the $\mathcal{P}_E$-simple module

$$T_{R,V} = \mathcal{P}_E f_R \otimes_{k\operatorname{Aut}(E,R)} V\ ,$$

which is also a simple $\mathcal{R}_E$-module (see Theorem 3.11). Recall that $\mathcal{P}_E$ is a quotient algebra of the algebra $\mathcal{R}_E = k\mathcal{C}(E,E)$. Then we have $S_{E,R,V} = S_{E,T_{R,V}}$. Clearly the dual $S^\natural_{E,T_{R,V}}$ is again a simple functor and its minimal set is $E$ again. Moreover, by evaluation at $E$, we find that

$$S^\natural_{E,T_{R,V}}(E) \cong S_{E,T_{R,V}}(E)^\natural \cong T^\natural_{R,V} \ .$$

Here the action of a relation $Q \in \mathcal{R}_E$ on a $\mathcal{R}_E$-module $W^\natural$ is defined by

$$(Q\cdot\alpha)(w) = \alpha(Q^{op}\cdot w) \ , \ \forall \alpha\in W^\natural \ , \ \forall w\in W \ .$$

We are going to define a nondegenerate pairing

$$\langle -,-\rangle : T_{R,V}\times T_{R^{op},V^\natural} \longrightarrow k \ ,$$

satisfying $\langle Q\cdot x\,,\,y\rangle = \langle x\,,\,Q^{op}\cdot y\rangle$ for all $x\in T_{R,V}$, $y\in T_{R^{op},V^\natural}$, and $Q\in\mathcal{R}_E$. This will induce an isomorphism of $\mathcal{R}_E$-modules

$$T^\natural_{R,V}\cong T_{R^{op},V^\natural} \ .$$

It will then follow that

$$S^\natural_{E,T_{R,V}} \cong S_{E,T_{R^{op},V^\natural}} \ , \qquad \text{that is} \ , \qquad S^\natural_{E,R,V}\cong S_{E,R^{op},V^\natural} \ ,$$

as required.

By Proposition 3.10, $\mathcal{P}_E f_R$ has a $k$-basis $\{\Delta_\sigma f_R \mid \sigma\in\Sigma_E\}$, where $\Sigma_E$ is the group of all permutations of $E$. Moreover, it is a free right $k\operatorname{Aut}(E,R)$-module and it follows that we can write

$$T_{R,V} = \mathcal{P}_E f_R \otimes_{k\operatorname{Aut}(E,R)} V = \bigoplus_{\sigma\in[\Sigma_E/\operatorname{Aut}(E,R)]} \Delta_\sigma f_R\otimes V \ ,$$

where $[\Sigma_E/\operatorname{Aut}(E,R)]$ denotes a set of representatives of the left cosets of $\operatorname{Aut}(E,R)$ in $\Sigma_E$. Noticing that $\operatorname{Aut}(E,R^{op})=\operatorname{Aut}(E,R)$, we have a similar decomposition

$$T_{R^{op},V^\natural} = \mathcal{P}_E f_{R^{op}} \otimes_{k\operatorname{Aut}(E,R)} V^\natural = \bigoplus_{\tau\in[\Sigma_E/\operatorname{Aut}(E,R)]} \Delta_\tau f_{R^{op}}\otimes V^\natural \ .$$

We define the pairing

$$\langle -,-\rangle : T_{R,V}\times T_{R^{op},V^\natural}\longrightarrow k \ , \qquad \langle \Delta_\sigma f_R\otimes v\,,\,\Delta_\tau f_{R^{op}}\otimes\alpha\rangle = \delta_{\sigma,\tau}\,\alpha(v) \ ,$$

where $\sigma,\tau\in[\Sigma_E/\operatorname{Aut}(E,R)]$, $v\in V$, $\alpha\in V^\natural$.

By choosing dual bases of $V$ and $V^\natural$ respectively, we easily find dual bases of $T_{R,V}$ and $T_{R^{op},V^\natural}$ respectively, and it follows that this pairing is nondegenerate. We are left with the proof of the required property of this pairing. We consider the action of a relation $Q\in\mathcal{R}_E$. For its action on $\mathcal{P}_E f_R$, it suffices to consider its image in the quotient algebra $\mathcal{P}_E$, so we can assume that $Q$ is a permuted order, and even an order relation, as the case where $Q$ is a permutation is clear.

By Lemma 3.8, any order relation $Q\in\mathcal{P}_E$ acts on $\mathcal{P}_E f_R$ via

$$Q\cdot\Delta_\sigma f_R = Q\cdot {}^\sigma f_R\,\Delta_\sigma = Q\cdot f_{{}^\sigma R}\,\Delta_\sigma = \begin{cases} \Delta_\sigma f_R & \text{if } Q\subseteq {}^\sigma R \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

It follows that

$$\langle Q\cdot\Delta_\sigma f_R\otimes v\,,\,\Delta_\tau f_{R^{op}}\otimes\alpha\rangle = \begin{cases} \alpha(v) & \text{if } Q\subseteq {}^\sigma R \ \text{ and } \ \sigma=\tau \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

On the other hand

$$\langle \Delta_\sigma f_R\otimes v\,,\,Q^{op}\cdot\Delta_\tau f_{R^{op}}\otimes\alpha\rangle = \begin{cases} \alpha(v) & \text{if } Q^{op}\subseteq {}^\tau R^{op} \ \text{ and } \ \sigma=\tau \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

When $\sigma=\tau$, the conditions $Q\subseteq {}^\sigma R$ and $Q^{op}\subseteq {}^\tau R^{op}$ are equivalent. Therefore, we obtain the required equality

$$\langle Q\cdot\Delta_\sigma f_R\otimes v\,,\,\Delta_\tau f_{R^{op}}\otimes\alpha\rangle = \langle \Delta_\sigma f_R\otimes v\,,\,Q^{op}\cdot\Delta_\tau f_{R^{op}}\otimes\alpha\rangle \ ,$$

from which it follows that we have an isomorphism of $\mathcal{R}_E$-modules $T^\natural_{R,V} \cong T_{R^{op},V^\natural}$. This completes the proof. □

## 15. The case of a total order

In this section, we consider in full detail the case of a totally ordered lattice. For a (non negative) integer $n \in \mathbb{N}$, we denote by $\underline{n}$ the set $\{0, 1, \ldots, n\}$, linearly ordered by $0 < 1 < \ldots < n$. Then $\underline{n}$ is a lattice, with least element 0 and greatest element $n$. Moreover $x \vee y = \sup(x, y)$ and $x \wedge y = \inf(x, y)$, for any $x, y \in \underline{n}$. We denote by $[n] = \{1, \ldots, n\}$ the set of irreducible elements of $\underline{n}$, viewed as a full subposet of $\underline{n}$. If $n = 0$ (in which case $[n] = \emptyset$) and if $n = 1$ (in which case $[n] = \{1\}$), we recover the cases already considered in Examples 4.11 and 4.12. The purpose of this section is to treat the general case.

Let $n \in \mathbb{N}$ and let $T$ be a finite lattice. By Definition 11.5 and Remark 11.6, a map $f : \underline{n} \to T$ is a morphism in $\mathcal{L}$ if and only if $f(0) = \hat{0}$ and $f$ is order-preserving, i.e. if $f(i) \leq f(j)$ for any $(i, j) \in \underline{n}$ with $i \leq j$.

**15.1. Notation.** *Let $n \in \mathbb{N}$, let $A \subseteq [n]$, and let $l = |A|$. We denote by $s_A : \underline{n} \to \underline{l}$ the map defined by*

$$\forall j \in \underline{n},\ s_A(j) = \big|]0, j] \cap A\big| \ .$$

*If moreover $C \subseteq [l]$, and $A = \{a_1, a_2, \ldots, a_l\}$ in increasing order, let $i_{A,C} : \underline{l} \to \underline{n}$ denote the map defined by*

$$\forall j \in \underline{l},\ i_{A,C}(j) = \begin{cases} 0 & \text{if } j = 0\ , \\ a_j & \text{if } j \notin C\ , \\ a_j - 1 & \text{if } j \in C\ . \end{cases}$$

*Clearly all the functions $s_A$ and $i_{A,C}$ are order-preserving and map 0 to 0, so they are morphisms in the category $\mathcal{L}$. Set finally*

$$i_A = \sum_{C \subseteq [l]} (-1)^{|C|} i_{A,C} \in k\mathcal{L}(\underline{n}, \underline{l})\ .$$

In view of Section 11, $i_A$ induces, for every finite set $X$, a $k$-linear map

$$F_{i_A} : F_{\underline{l}}(X) \longrightarrow F_{\underline{n}}(X)\ ,$$

where $F_{\underline{l}}$ and $F_{\underline{n}}$ are the correspondence functors associated to the lattices $\underline{l}$ and $\underline{n}$.

**15.2. Lemma.** *Let $n \in \mathbb{N}$, let $A \subseteq [n]$, and let $l = |A|$. Let $X$ be a finite set, and $\varphi : X \to \underline{l}$ be any function. Then $F_{i_A}(\varphi) = 0$ unless $\varphi(X) \supseteq [l]$.*

**Proof :** We have

$$\begin{aligned} F_{i_A}(\varphi) &= \sum_{C \subseteq [l]} (-1)^{|C|} i_{A,C} \circ \varphi \\ &= \sum_{\psi : X \to \underline{n}} \Big( \sum_{\substack{C \subseteq [l] \\ i_{A,C} \circ \varphi = \psi}} (-1)^{|C|} \Big) \psi\ . \end{aligned}$$

Moreover

$$\forall x \in X,\ i_{A,C} \circ \varphi(x) = \begin{cases} 0 & \text{if } \varphi(x) = 0 \\ a_{\varphi(x)} & \text{if } \varphi(x) \notin C \\ a_{\varphi(x)} - 1 & \text{if } \varphi(x) \in C \end{cases}\ .$$

For a given function $\psi : X \to \underline{n}$, setting

$$\begin{aligned} Z_\psi &= \psi^{-1}(0)\ , \\ U_\psi &= \{x \in X - Z_\psi \mid\ \psi(x) = a_{\varphi(x)} - 1\}\ , \\ V_\psi &= \{x \in X - Z_\psi \mid \psi(x) = a_{\varphi(x)}\}\ , \end{aligned}$$

we have $i_{A,C} \circ \varphi = \psi$ if and only if the following three conditions hold :

$$X = Z_\psi \sqcup U_\psi \sqcup V_\psi\ , \quad \big(x \in U_\psi \iff \varphi(x) \in C\big)\ , \text{ and } \big(x \in V_\psi \iff \varphi(x) \notin \{0\} \sqcup C\big)\ .$$

In other words

$$\varphi(U_\psi) \subseteq C \subseteq \{1, \ldots, l\} - \varphi(V_\psi)\ .$$

So if $\psi : X \to \underline{n}$ appears in $F_{i_A}(\varphi)$ with a nonzero coefficient, then

$$\textbf{(15.3)} \qquad \varphi(U_\psi) \subseteq [l] - \varphi(V_\psi)\ ,$$

i.e. $\varphi(U_\psi) \subseteq [l]$ and $\varphi(U_\psi) \cap \varphi(V_\psi) = \emptyset$. But if the inclusion in (15.3) is proper, then the coefficient of $\psi$ in $F_{i_A}(\varphi)$ is equal to

$$\sum_{\varphi(U_\psi) \subseteq C \subseteq [l] - \varphi(V_\psi)} (-1)^{|C|} = 0\ .$$

Hence the coefficient of $\psi$ in $F_{i_A}(\varphi)$ is nonzero if and only if

$$\textbf{(15.4)} \qquad [l] = \varphi(U_\psi) \sqcup \big(\varphi(V_\psi) \cap [l]\big)\ .$$

In particular, if $[l] \nsubseteq \varphi(X)$, there is no such $\psi$ and $F_{i_A}(\varphi) = 0$. □

**15.5. Notation.** *Let $n \in \mathbb{N}$ and let $A$ and $B$ be subsets of $[n]$ such that $|A| = |B|$. We write $f_{A,B} = i_A s_B \in \mathrm{End}_{k\mathcal{L}}(\underline{n})$.*

**15.6. Lemma.** *Let $n \in \mathbb{N}$, let $A, B, C, D$ be subsets of $[n]$ with $l = |A| = |B|$ and $m = |C| = |D|$. Then :*

*(a) If $C \nsubseteq B$, then $s_B i_C = 0$ in $k\mathcal{L}(\underline{l}, \underline{m})$.*

*(b) In $k\mathcal{L}(\underline{n}, \underline{m})$*

$$i_A s_B i_C = \begin{cases} 0 & \text{if } B \neq C\ , \\ i_A & \text{if } B = C\ . \end{cases}$$

*(c) In $\mathrm{End}_{k\mathcal{L}}(\underline{n})$*

$$f_{A,B} f_{C,D} = \begin{cases} 0 & \text{if } B \neq C\ , \\ f_{A,D} & \text{if } B = C\ . \end{cases}$$

**Proof :** (a) By definition

$$\begin{aligned} s_B i_C &= \sum_{G \subseteq [m]} (-1)^{|G|} s_B i_{C,G} \\ &= \sum_{\psi : \underline{m} \to \underline{l}} \Big( \sum_{\substack{G \subseteq [m] \\ s_B i_{C,G} = \psi}} (-1)^{|G|} \Big) \psi\ . \end{aligned}$$

Write $C = \{c_1, \ldots, c_m\}$ in increasing order. For $j \in \underline{l}$

$$s_B i_{C,G}(j) = \begin{cases} 0 & \text{if } j = 0\ , \\ |]0, c_j] \cap B| & \text{if } j \notin G\ , \\ |]0, c_j - 1] \cap B| & \text{if } j \in G\ . \end{cases}$$

Set

$$U_\psi = \{j \in [m] \mid \psi(j) = |]0, c_j - 1] \cap B|\} \text{ and } V_\psi = \{j \in [m] \mid \psi(j) = |]0, c_j] \cap B|\}\ .$$

Then $s_B i_{C,G} = \psi$ if and only if $G \subseteq U_\psi$ and $[m] - G \subseteq V_\psi$, i.e.

$$[m] - V_\psi \subseteq G \subseteq U_\psi \tag{15.7}$$

Thus the coefficient of $\psi$ in $s_B i_C$ is equal to

$$\sum_{[m]-V_\psi \subseteq G \subseteq U_\psi} (-1)^{|G|} .$$

This is equal to zero if $[m]-V_\psi \not\subseteq U_\psi$ or if $[m]-V_\psi$ is a proper subset of $U_\psi$. Hence if $\psi$ appears in $s_B i_C$, then $[m] = U_\psi \sqcup V_\psi$, i.e.

$$[m] = \big\{ j \in [m] \mid \psi(j) = |]0, c_j - 1] \cap B| \big\} \sqcup \big\{ j \in [m] \mid \psi(j) = |]0, c_j] \cap B| \big\} ,$$

and in this case, the coefficient of $\psi$ in $s_B i_C$ is equal to $(-1)^{|U_\psi|}$. In particular, for any $j \in [m]$, we have $|]0, c_j - 1] \cap B| \neq |]0, c_j] \cap B|$, hence $c_j \in B$, and $C \subseteq B$. Thus $s_B i_C = 0$ if $C \not\subseteq B$.

(b) In particular, if $s_B i_C \neq 0$, then $C \subseteq B$, and $m \leq l$. Now $s_B i_C$ is a $k$-linear combination of morphisms $\psi \in \mathrm{Hom}_{\mathcal{L}}(\underline{l}, \underline{m})$. By Lemma 15.2, the product $i_A \psi$ (which is $F_{i_A}(\psi)$ by definition) is equal to zero, unless the image of $\psi$ contains $[l]$. In the latter case, $\psi$ is surjective because $\psi(0) = 0$. Hence $m \geq l$, thus $l = m$ and $\psi$ is bijective. As $\psi$ is order-preserving, it is the identity map.

Moreover $B = C$ because $C \subseteq B$ and $l = m$. Then, with the notation of part (a), we have

$$|]0, c_j] \cap B| = |]0, c_j] \cap C| = j = \psi(j) ,$$

for any $j \in [m]$. Therefore $U_\psi = \emptyset$, hence $(-1)^{|U_\psi|} = 1$ and the coefficient of $\psi = \mathrm{id}$ in $s_B i_C$ is equal to 1. It follows that $i_A s_B i_C = i_A \,\mathrm{id} = i_A$ in this case.

(c) This follows from (a) and (b). □

Our next result describes completely the structure of the $k$-algebra $\mathrm{End}_{k\mathcal{L}}(\underline{n})$ of all endomorphisms of the lattice $\underline{n}$.

**15.8. Theorem.** *Let $n \in \mathbb{N}$. For $l \in \underline{n}$, let $M_{\binom{n}{l}}(k)$ denote the algebra of square matrices indexed by the set of subsets of $[n]$ of cardinality $l$, with coefficients in $k$. For subsets $A$ and $B$ of $[n]$ of cardinality $l$, let $m_{A,B}$ denote the matrix with coefficient 1 in position $(A, B)$, and 0 elsewhere.*

*(a) There is an isomorphism of $k$-algebras*

$$\mathcal{I} : \bigoplus_{l=0}^{n} M_{\binom{n}{l}}(k) \longrightarrow \mathrm{End}_{k\mathcal{L}}(\underline{n})$$

*sending $m_{A,B} \in M_{\binom{n}{l}}(k)$ to $f_{A,B}$, for any $l \in \underline{n}$ and any pair $(A, B)$ of subsets of $[n]$ with $|A| = |B| = l$.*

*(b) In particular, if $k$ is a field, then $\mathrm{End}_{k\mathcal{L}}(\underline{n})$ is semi-simple.*

**Proof :** Throughout this proof, $\mathcal{X}_n$ denotes the set of pairs $(A, B)$ of subsets of $[n]$ such that $|A| = |B|$. The matrices $m_{A,B}$, for $(A, B) \in \mathcal{X}_n$, form a $k$-basis of $\mathcal{M} = \bigoplus_{l=0}^{n} M_{\binom{n}{l}}(k)$. They satisfy the relations

$$m_{A,B}\, m_{C,D} = \begin{cases} 0 & \text{if } B \neq C \\ m_{A,D} & \text{if } B = C \end{cases} .$$

Indeed, if $|B| \neq |C|$, then $m_{A,B}$ and $m_{C,D}$ are not in the same block of $\mathcal{M}$, so their product is 0. And if $|B| = |C| = l$, then the corresponding relations are the standard relations within the matrix algebra $M_{\binom{n}{l}}(k)$.

By Lemma 15.6, it follows that $\mathcal{I}$ is an algebra homomorphism. Proving that $\mathcal{I}$ is an isomorphism is equivalent to proving that the elements $f_{A,B}$, for $(A,B) \in \mathcal{X}_n$, form a $k$-basis of $\mathrm{End}_{k\mathcal{L}}(\underline{n})$. Let $\mathcal{B}_n$ denote the standard $k$-basis of the latter algebra, i.e. the set of order-preserving functions $\underline{n} \to \underline{n}$ which map 0 to 0. For such a function $\varphi$, let

$$A_\varphi = \varphi(\underline{n}) \cap [n] \;\text{ and }\; B_\varphi = \big\{ j \in [n] \mid \varphi(j) \neq \varphi(j-1) \big\} .$$

Clearly $|A_\varphi| = |\varphi(\underline{n})| - 1 = |B_\varphi|$, thus $(A_\varphi, B_\varphi) \in \mathcal{X}_n$. Conversely, if $(A,B) \in \mathcal{X}_n$, let $\varphi_{A,B}$ denote the element $i_{A,\emptyset}\, s_B$ of $\mathrm{End}_{\mathcal{L}}(\underline{n})$. We claim that the maps

$$\sigma : \varphi \mapsto (A_\varphi, B_\varphi) \quad \text{ and } \quad \tau : (A,B) \mapsto \varphi_{A,B}$$

are inverse to each other, hence bijections between $\mathcal{B}_n$ and $\mathcal{X}_n$.

To see this, let $(A,B) \in \mathcal{X}_n$, and set $\varphi = \varphi_{A,B}$. Let $A = \{a_1, \ldots, a_r\}$ in increasing order, and set moreover $a_0 = 0$. Then by definition

$$\forall j \in \underline{n}, \;\; \varphi(j) = a_{|]0,j] \cap B|} \; .$$

In particular, for $j \in [n]$, we have $\varphi(j) \neq \varphi(j-1)$ if and only if $|]0,j] \cap B| \neq |]0,j-1] \cap B|$, i.e. if and only if $j \in B$. In other words $B_\varphi = B$. Moreover $\varphi(\underline{n}) = \{a_0, a_1, \ldots, a_r\}$, hence $A_\varphi = \varphi(\underline{n}) \cap [n] = \{a_1, \ldots, a_r\} = A$. This proves that $\sigma \circ \tau$ is the identity map of $\mathcal{X}_n$. In particular $\tau$ is injective, and $\sigma$ is surjective.

Conversely, let $\varphi \in \mathcal{B}_n$, and set $A = A_\varphi$ and $B = B_\varphi$. Let $A = \{a_1, \ldots, a_r\}$ in increasing order, and set moreover $a_0 = 0$. Similarly, let $B = \{b_1, \ldots, b_r\}$ in increasing order, and set moreover $b_0 = 0$ and $b_{r+1} = n+1$. Then the intervals $[b_i, \ldots, b_{i+1} - 1]$, for $i \in \{0, \ldots, r\}$ form a partition of $\underline{n}$, and $\varphi$ is constant on each of these intervals, and takes two distinct values on any two of these distinct intervals, by definition of $B$. More precisely $\varphi([b_i, \ldots, b_{i+1} - 1]) = \{a_i\}$, for any $i \in \underline{n}$. It follows that the map $\varphi$ can be recovered from the knowledge of the pair $(A,B)$. Therefore $\sigma$ is injective, hence bijective. Since $\sigma \circ \tau \circ \sigma = \sigma$, it follows that $\tau \circ \sigma$ is the identity map of $\mathcal{B}_n$, which completes the proof of the claim.

Now for $(A,B) \in \mathcal{X}_n$

$$f_{A,B} = \sum_{C \subseteq [l]} (-1)^{|C|}\, i_{A,C}\, s_B \; ,$$

and it is clear form the definition of $i_{A,C}$ that $i_{A,C}\, s_B(j) \leq i_{A,\emptyset} s_B(j)$, for all $j \in \underline{n}$. It follows that $f_{A,B}$ is a linear combination of functions all smaller than $\varphi_{A,B}$, for the standard ordering of functions $\underline{n} \to \underline{n}$. Moreover, the coefficient of $\varphi_{A,B}$ in the expansion of $f_{A,B}$ in the basis $\mathcal{B}_n$ is equal to 1. It follows that the transition matrix from $\mathcal{B}_n$ to the elements $f_{A,B}$ is triangular, with ones on the diagonal. Hence the elements $f_{A,B}$, for $(A,B) \in \mathcal{X}_n$, form a basis of $\mathrm{End}_{k\mathcal{L}}(\underline{n})$, and this completes the proof Theorem 15.8. ∎

Theorem 15.8 is similar to the result proved in [**FHH**] about the planar rook algebra. Over the field $\mathbb{C}$ of complex numbers, this algebra is actually isomorphic to $\mathrm{End}_{\mathbb{C}\mathcal{L}}(\underline{n})$. However, the planar rook monoid is not isomorphic to the monoid of endomorphisms of $\underline{n}$ in $\mathcal{L}$. Only the corresponding monoid algebras become isomorphic (over $\mathbb{C}$).

Now we want to use the functor $F_? : k\mathcal{L} \to \mathcal{F}_k$ to deduce information on the correspondence functor $F_{\underline{n}}$. By Theorem 11.11, we already know that $F_{\underline{n}}$ is projective, because the total order $\underline{n}$ is a distributive lattice.

**15.9. Notation.** *For $n \in \mathbb{N}$, denote by $\varepsilon_n$ the element $i_{[n]}$ of $\mathrm{End}_{k\mathcal{L}}(\underline{n})$. In other words*

$$\varepsilon_n = \sum_{C \subseteq [n]} (-1)^{|C|} i_{[n],C} \; ,$$

*where, as before,* $i_{[n],C} \in \mathrm{End}_{\mathcal{L}}(\underline{n})$ *is defined by* $i_{[n],C}(j) = j-1$ *if* $j \in C$ *and* $i_{[n],C}(j) = j$ *otherwise, for any* $j \in \underline{n}$.

**15.10. Proposition.**

(a) *The element* $\varepsilon_n$ *is a central idempotent of* $\mathrm{End}_{k\mathcal{L}}(\underline{n})$. *Moreover*

$$\varepsilon_n \,\mathrm{End}_{k\mathcal{L}}(\underline{n}) = k\varepsilon_n \ .$$

(b) *The kernel of the idempotent endomorphism* $F_{\varepsilon_n}$ *of* $F_{\underline{n}}$ *is equal to the subfunctor* $H_{\underline{n}}$, *defined in Notation 12.1.*

**Proof :** (a) If $A = [n]$, then the map $s_A$ is the identity map of $\underline{n}$. Thus $\varepsilon_n = i_{[n]} = i_{[n]}s_{[n]} = f_{[n],[n]}$, which is an idempotent by Lemma 15.6. Moreover the inverse image of $\varepsilon_n$ under the algebra isomorphism

$$\mathcal{M} = \bigoplus_{l=0}^{n} M_{\binom{n}{l}}(k) \longrightarrow \mathrm{End}_{k\mathcal{L}}(\underline{n})$$

of Theorem 15.8 is the matrix $e_n = m_{[n],[n]}$ of the component $M_{\binom{n}{n}}(k) = k$ of $\mathcal{M}$. But $e_n$ is central in $\mathcal{M}$ and $e_n\mathcal{M} = ke_n$.

(b) Let $\varphi : X \to \underline{n}$ be any map. Then $F_{\varepsilon_n}(\varphi) = i_{[n]}\varphi = 0$ by Lemma 15.2, unless $\varphi(X)$ contains $[n]$. It follows that $H_{\underline{n}} \subseteq \mathrm{Ker}\ F_{\varepsilon_n}$.

To show that $\mathrm{Ker}\ F_{\varepsilon_n} \subseteq H_{\underline{n}}$, let $X$ be a finite set and denote by $Z_n(X)$ the set of functions $\varphi : X \to \underline{n}$ such that $\varphi(X) \supseteq [n]$. Then by definition

$$F_{\underline{n}}(X) = H_{\underline{n}}(X) \oplus kZ_n(X) \ .$$

Let $\pi = \pi_X$ denote the idempotent endomorphism of the $k$-module $F_{\underline{n}}(X)$ with image $kZ_n(X)$ and kernel $H_{\underline{n}}(X)$. For any $\varphi : X \to \underline{n}$, we have that

$$\pi F_{\varepsilon_n}(\varphi) = \pi\Big(\sum_{C\subseteq[n]} (-1)^{|C|}(i_{[n],C} \circ \varphi)\Big) \ .$$

But $\pi(i_{[n],C} \circ \varphi) = 0$, unless the image of $i_{[n],C} \circ \varphi$ contains $[n]$. In particular $i_{[n],C}(\underline{n}) \supseteq [n]$, hence $C = \emptyset$ and $i_{[n],C} = \mathrm{id}_{\underline{n}}$. It follows that $\pi F_{\varepsilon_n} = \pi$. Hence $\mathrm{Ker}\, F_{\varepsilon_n} \subseteq \mathrm{Ker}\, \pi = H_{\underline{n}}(X)$, and $\mathrm{Ker}\, F_{\varepsilon_n} = H_{\underline{n}}$. This completes the proof of Proposition 15.10. ∎

**15.11. Lemma.** *Let* $n \in \mathbb{N}$, *let* $A \subseteq [n]$, *and let* $l = |A|$. *Then* $s_Ai_A = \varepsilon_l$ *in* $\mathrm{End}_{k\mathcal{L}}(\underline{l})$.

**Proof :** By definition

$$s_Ai_A = \sum_{C\subseteq[l]} (-1)^{|C|} s_Ai_{A,C} \ .$$

Moreover, if $A = \{a_1, \ldots, a_l\}$ in increasing order, then for $j \in \underline{l}$

$$s_Ai_{A,C}(j) = \begin{cases} 0 & \text{if } j = 0 \\ |]0, a_j] \cap A| & \text{if } j \notin C \\ |]0, a_j - 1] \cap A| & \text{if } j \in C \end{cases} \ .$$

But $]0, a_j] \cap A = \{a_1, \ldots, a_j\}$ and $]0, a_j - 1] \cap A = \{a_1, \ldots, a_{j-1}\}$. It follows that $|]0, a_j] \cap A| = j$, for any $j \in \underline{l}$, and $|]0, a_j - 1] \cap A| = j - 1$. Hence $s_Ai_{A,C} = i_{[l],C}$. Now Lemma 15.11 follows from Notation 15.9. ∎

**15.12. Theorem.** *Let $n \in \mathbb{N}$ and let $\mathbb{S}_n$ denote the image of the idempotent endomorphism $F_{\varepsilon_n}$ of $F_{\underline{n}}$.*

*(a) For an integer $l$ with $0 \leq l \leq n$, set*

$$\beta_l = \sum_{\substack{A\subseteq[n]\\|A|=l}} f_{A,A} \ .$$

*Then the elements $\beta_l$, for $0 \leq l \leq n$, are orthogonal central idempotents of $\mathrm{End}_{k\mathcal{L}}(\underline{n})$, and their sum is equal to the identity.*

*(b) There are isomorphisms of correspondence functors*

$$\begin{aligned} F_{\beta_l}F_{\underline{n}} &\cong \mathbb{S}_l^{\oplus\binom{n}{l}} \ , \ \textit{for}\ 0 \leq l \leq n \ ,\\ F_{\underline{n}} &\cong \bigoplus_{A\subseteq[n]} \mathbb{S}_{|A|} \ . \end{aligned}$$

**Proof :** (a) For $A \leq [n]$ with $|A| = l$, the inverse image of $f_{A,A}$ under the algebra isomorphism

$$\mathcal{I} : \mathcal{M} = \bigoplus_{l=0}^{n} M_{\binom{n}{l}}(k) \longrightarrow \mathrm{End}_{k\mathcal{L}}(\underline{n})$$

of Theorem 15.8 is the matrix $m_{A,A}$ of the component indexed by $l$ of $\mathcal{M}$. The coefficients of this matrix are all 0, except one coefficient equal to 1 on the diagonal, at place indexed by the subset $A$ of $[n]$. It follows that the inverse image of $\beta_l$ under $\mathcal{I}$ is the identity element of the component indexed by $l$ of $\mathcal{M}$. Assertion (a) follows.

(b) By Theorem 11.7, the functor $F_?$ induces an isomorphism of $k$-algebras

$$\mathrm{End}_{k\mathcal{L}}(\underline{n}) \cong \mathrm{End}_{\mathcal{F}_k}(F_{\underline{n}}) \ .$$

Now the idempotents $f_{A,A}$ of $\mathrm{End}_{k\mathcal{L}}(\underline{n})$, for $A \subseteq [n]$, are orthogonal, and their sum is equal to the identity. It follows that the endomorphisms $F_{f_{A,A}}$ of $F_{\underline{n}}$ are orthogonal idempotents, and their sum is the identity. Hence we obtain a decomposition of correspondence functors

$$F_{\underline{n}} = \bigoplus_{A\subseteq[n]} F_{f_{A,A}}(F_{\underline{n}}) \ .$$

Moreover for any $A \subseteq [n]$ with $|A| = l$, we have $s_A f_{A,A} = s_A i_A s_A = \varepsilon_l s_A$ by Lemma 15.11. It follows that the morphism $F_{s_A} : F_{\underline{n}} \to F_{\underline{l}}$ restricts to a morphism $\pi_A : F_{f_{A,A}}(F_{\underline{n}}) \to F_{\varepsilon_l}(F_{\underline{l}})$. Conversely, since $f_{A,A} i_A = i_A s_A i_A = i_A$ by Lemma 15.6, the morphism $F_{i_A} : F_{\underline{l}} \to F_{\underline{n}}$ restricts to $\rho_A : F_{\varepsilon_l}(F_{\underline{l}}) \to F_{f_{A,A}}(F_{\underline{n}})$. Since moreover $i_A s_A f_{A,A} = f_{A,A}^2 = f_{A,A}$ and $s_A i_A = \varepsilon_l$, the morphisms $\pi_A$ and $\rho_A$ are inverse to each other. They are therefore isomorphisms between $F_{f_{A,A}}(F_{\underline{n}})$ and $F_{\varepsilon_l}(F_{\underline{l}}) \cong \mathbb{S}_l$. It follows in particular that $F_{\beta_l}F_{\underline{n}} \cong \mathbb{S}_l^{\oplus\binom{n}{l}}$, which completes the proof. ∎

**15.13. Corollary.** *Let $m, n \in \mathbb{N}$. Then*

$$\mathrm{Hom}_{\mathcal{F}_k}(\mathbb{S}_n, \mathbb{S}_m) = \begin{cases} 0 & \textit{if}\ n \neq m \ ,\\ k \cdot \mathrm{id}_{\mathbb{S}_n} & \textit{if}\ n = m \ . \end{cases}$$

**Proof :** Since $\mathbb{S}_n = F_{\varepsilon_n}F_{\underline{n}}$, the case $n = m$ follows from Assertion (a) of Proposition 15.10. Now for integers $l, m \in \{0, \dots, n\}$, we have that

$$\mathrm{Hom}_{\mathcal{F}_k}(F_{\beta_l}F_{\underline{n}}, F_{\beta_m}F_{\underline{n}}) \cong \mathrm{Hom}_{\mathcal{F}_k}(\mathbb{S}_l, \mathbb{S}_m)^{\oplus\binom{n}{l}\binom{n}{m}} \ .$$

Since $F_{\beta_l}$ and $F_{\beta_m}$ are central idempotents of $\mathrm{End}_{\mathcal{F}_k}(F_{\underline{n}})$, and since they are orthogonal if $l \neq m$. It follows that $\mathrm{Hom}_{\mathcal{F}_k}(F_{\beta_l}F_{\underline{n}}, F_{\beta_m}F_{\underline{n}}) = 0$ if $l \neq m$, hence $\mathrm{Hom}_{\mathcal{F}_k}(\mathbb{S}_l, \mathbb{S}_m) = 0$. □

Now we prove that the functor $\mathbb{S}_n$ is actually isomorphic to a fundamental functor and we compute the dimensions of all its evaluations.

**15.14. Theorem.** *Let $\mathbb{S}_n$ denote the image of the idempotent endomorphism $F_{\varepsilon_n}$ of $F_{\underline{n}}$.*

*(a) $\mathbb{S}_n \cong F_{\underline{n}}/H_{\underline{n}}$.*

*(b) $\mathbb{S}_n$ is isomorphic to the fundamental functor $\mathbb{S}_{[n],\mathrm{tot}}$, where* tot *denotes the total order on $[n]$.*

*(c) For any finite set $X$, the $k$-module $\mathbb{S}_n(X)$ is free of rank*

$$\mathrm{rank}\big(\mathbb{S}_n(X)\big) = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}(i+1)^{|X|} \ .$$

**Proof :** (a) This follows from the second statement in Proposition 15.10.

(b) We are going to use the results of Section 14 applied to the lattice $T = \underline{n}^{op}$. The set of its irreducible elements is

$$E = \{0, 1, \ldots, n{-}1\} \ ,$$

with a total order $R$ being the opposite of the usual order. Now we have

$$F_{T^{op}} = F_{(\underline{n}^{op})^{op}} = F_{\underline{n}}$$

and its evaluation at $E$ contains an element

$$\gamma_T = \gamma_{\underline{n}^{op}} = \sum_{A\subseteq E}(-1)^{|A|}\eta_A^\circ \ .$$

Recall from Notation 14.13 that $\eta_A^\circ : E \to T^{op} = \underline{n}$ denotes the same map as $\eta : E \to T = \underline{n}^{op}$ and that $\eta$ is defined by

$$\forall e \in E,\ \ \eta_A(e) = \begin{cases} r(e) & \text{if } e \in A \ , \\ e & \text{if } e \notin A \ , \end{cases} \qquad \text{that is,} \qquad \eta_A(e) = \begin{cases} e+1 & \text{if } e \in A \ , \\ e & \text{if } e \notin A \ , \end{cases}$$

because $r(e) = e + 1$ in the lattice $\underline{n}^{op}$.

Now we define $\omega : E \to \underline{n}$ by $\omega(e) = e + 1$. Then $\omega \in F_{\underline{n}}(E)$ and when we apply the idempotent $F_{\varepsilon_n}$ we claim that we obtain

$$F_{\varepsilon_n}(\omega) = (-1)^n\gamma_{\underline{n}^{op}} \ . \qquad (\mathbf{15.15})$$

The definition of $\varepsilon_n$ yields

$$F_{\varepsilon_n}(\omega) = \sum_{C\subseteq[n]}(-1)^{|C|}i_{[n],C}\circ\omega$$

and we have

$$(i_{[n],C}\circ\omega)(e) = i_{[n],C}(e+1) = \begin{cases} e+1 & \text{if } e+1 \notin C \\ e & \text{if } e+1 \in C \end{cases}$$

For each $C \subseteq [n] = \{1, \ldots, n\}$, define $A$ to be the complement of $C - 1$ in $E = \{0, \ldots, n-1\}$, so that $C$ is the complement of $A+1$ in $[n]$. In other words

$$j \notin A \iff j+1 \notin A+1 \iff j+1 \in C \ .$$

Then we see that $i_{[n],C} \circ \omega = \eta_A^\circ$ and it follows that

$$\begin{aligned} F_{\varepsilon_n}(\omega) &= \sum_{C \subseteq [n]} (-1)^{|C|} i_{[n],C} \circ \omega \\ &= \sum_{A \subseteq E} (-1)^{n-|A|} \eta_A^\circ \\ &= (-1)^n \sum_{A \subseteq E} (-1)^{|A|} \eta_A^\circ \\ &= (-1)^n \gamma_{\underline{n}^{op}} \ . \end{aligned}$$

This proves Claim 15.15 above.

Now $F_{\underline{n}}$ is generated by $\omega \in F_{\underline{n}}(E)$, because it is generated by $\iota \in F_{\underline{n}}([n])$ (where $\iota : [n] \to \underline{n}$ is the inclusion), hence also by any injection from a set $E$ of cardinality $n$ to $\underline{n}$ (by composing with a bijection between $E$ and $[n]$). Since $F_{\varepsilon_n}$ is an idempotent endomorphism of the correspondence functor $F_{\underline{n}}$, we see that $F_{\varepsilon_n} F_{\underline{n}}$ is generated by $F_{\varepsilon_n}(\omega)$. In other words, in view of Claim 15.15 above, $\mathbb{S}_n$ is generated by $\gamma_{\underline{n}^{op}} \in F_{\underline{n}}(E)$. Now Theorem 14.16 asserts that the subfunctor generated by $\gamma_T = \gamma_{\underline{n}^{op}}$ is isomorphic to $\mathbb{S}_{E,R}$. But $(E,R) \cong ([n], \mathrm{tot})$ via the map $e \mapsto n - e$, so we obtain

$$\mathbb{S}_n = F_{\varepsilon_n} F_{\underline{n}} = \langle \gamma_{\underline{n}^{op}} \rangle \cong \mathbb{S}_{E,R} \cong \mathbb{S}_{[n],\mathrm{tot}} \ .$$

(c) The $k$-module $\mathbb{S}_n(X)$ is isomorphic to $kZ_n(X)$, where $Z_n(X)$ is as in the proof of Proposition 15.10 above, namely the set of all maps $\varphi : X \to \underline{n}$ such that $[n] \subseteq \varphi(X) \subseteq \underline{n}$. Therefore $\mathbb{S}_n(X)$ is free of rank $|Z_n(X)|$. Lemma 8.1 shows that this rank is equal to

$$|Z_n(X)| = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n+1-i)^{|X|} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (j+1)^{|X|}$$

as required. □

**15.16. Remark.** We shall see in Section 17 that a similar formula holds for the rank of the evaluation of any fundamental functor, but the proof in the general case is much more elaborate.

Also, Corollary 15.13 holds more generally for fundamental functors and the general case will be proved in Section 18.

**15.17. Corollary.** *Let $k$ be a field.*

*(a) The functor $\mathbb{S}_n$ is simple, isomorphic to $S_{[n],\mathrm{tot},k}$, where $k$ is the trivial module for the trivial group* $\mathrm{Aut}([n], \mathrm{tot}) = \{\mathrm{id}\}$.

*(b) $\mathbb{S}_n$ is simple, projective, and injective in $\mathcal{F}_k$.*

**Proof :** (a) It is clear that $\mathrm{Aut}([n], \mathrm{tot})$ is the trivial group, with a single simple module $k$. Since $\mathbb{S}_n \cong \mathbb{S}_{[n],\mathrm{tot}}$ by Theorem 15.14, we obtain (using Remark 4.8) the simple functor

$$S_{[n],\mathrm{tot},k} = \mathbb{S}_{[n],\mathrm{tot}} \otimes_k k \cong \mathbb{S}_{[n],\mathrm{tot}} \cong \mathbb{S}_n \ .$$

(b) Since $\underline{n}$ is a distributive lattice, $F_{\underline{n}}$ is projective and injective by Corollary 14.11. Therefore so is its direct summand $\mathbb{S}_n$. Hence $\mathbb{S}_n$ is simple, projective, and injective. □

Our last purpose in this section is to find, for any finite lattice $T$, direct summands of $F_T$ isomorphic to $\mathbb{S}_n$. We start with a lemma, which is probably well known.

**15.18. Lemma.** *Let $n \in \mathbb{N}$, and $T$ be a finite lattice. Let $\mathcal{U}_n$ denote the set of non decreasing sequences $u_0 \leq u_1 \leq \ldots \leq u_{n-1} \leq u_n = \hat{1}$ of elements of $T$, and let $\mathcal{V}_n$ be the subset of $\mathcal{U}_n$ consisting of (strictly) increasing sequences.*

*For $u \in \mathcal{U}_n$, let $\check{u} : T \to \underline{n}$ be the map defined by*

$$\forall t \in T, \;\; \check{u}(t) = \mathrm{Min}\{j \in \underline{n} \mid t \leq u_j\} \; .$$

*Conversely, if $\varphi$ is a map from $T$ to $\underline{n}$, let $\hat{\varphi}$ be the sequence of elements of $T$ defined by*

$$\forall j \in \underline{n}, \;\; \hat{\varphi}_j = \bigvee_{\substack{t \in T \\ \varphi(t) \leq j}} t \; .$$

*(a) The assigments $u \mapsto \check{u}$ and $\varphi \mapsto \hat{\varphi}$ are inverse bijections between $\mathcal{U}_n$ and $\mathrm{Hom}_{\mathcal{L}}(T, \underline{n})$.*

*(b) Let $u \in \mathcal{U}_n$. The following assertions are equivalent:*

*(i) The map $\check{u}$ is surjective.*

*(ii) $\varepsilon_n \check{u} \neq 0$.*

*(iii) $u \in \mathcal{V}_n$.*

**Proof :** (a) Let $u \in \mathcal{U}_n$. Then $\check{u} \in \mathrm{Hom}_{\mathcal{L}}(T, \underline{n})$: indeed for $A \subseteq T$

$$\begin{aligned}
\check{u}(\bigvee_{t \in A} t) &= \mathrm{Min}\{j \in \underline{n} \mid \bigvee_{t \in A} t \leq u_j\} \\
&= \mathrm{Min}\Big(\bigcap_{t \in A} \{j \in \underline{n} \mid t \leq u_j\}\Big) \\
&= \mathop{\mathrm{Max}}_{t \in A} \Big(\mathrm{Min}\{j \in \underline{n} \mid t \leq u_j\}\Big) \\
&= \mathop{\mathrm{Max}}_{t \in A} \check{u}(t) = \bigvee_{t \in A} \check{u}(t) \; .
\end{aligned}$$

Conversely, if $\varphi \in \mathrm{Hom}_{\mathcal{L}}(T, \underline{n})$, then the sequence $\hat{\varphi}_j = \bigvee\limits_{\substack{t \in T \\ \varphi(t) \leq j}} t$, for $j \in \underline{n}$, is a nondecreasing sequence of elements of $T$, and $\hat{\varphi}_n = \hat{1}$. Hence $\hat{\varphi} \in \mathcal{U}_n$.

Now let $u \in \mathcal{U}_n$, and set $\varphi = \check{u} \in \mathrm{Hom}_{\mathcal{L}}(T, \underline{n})$. Then for $j \in \underline{n}$ and any $t \in T$

$$\hat{\varphi}_j = \bigvee_{\substack{t \in T \\ \varphi(t) \leq j}} t = \bigvee_{\substack{t \in T \\ \mathrm{Min}\{l \in \underline{n} \mid t \leq u_l\} \leq j}} t \; = \bigvee_{\substack{t \in T \\ t \leq u_j}} t = u_j \; ,$$

thus $\hat{\varphi} = u$.

Conversely, let $\varphi \in \mathrm{Hom}_{\mathcal{L}}(T, \underline{n})$, and set $u = \hat{\varphi}$. Then for $t \in T$

$$\check{u}(t) = \mathrm{Min}\{j \in \underline{n} \mid t \leq u_j\} = \mathrm{Min}\{j \in \underline{n} \mid t \leq \bigvee_{\substack{t' \in T \\ \varphi(t') \leq j}} t'\}$$

But if $t \leq \bigvee\limits_{\substack{t' \in T \\ \varphi(t') \leq j}} t'$, then

$$\varphi(t) \leq \varphi\Big(\bigvee_{\substack{t' \in T \\ \varphi(t') \leq j}} t'\Big) = \mathop{\mathrm{Max}}_{t' \in T, \; \varphi(t') \leq j} \varphi(t') \leq j \; ,$$

so we obtain

$$\check{u}(t) = \mathrm{Min}\{j \in \underline{n} \mid t \leq \bigvee_{\substack{t' \in T \\ \varphi(t') \leq j}} t'\} \geq \mathrm{Min}\{j \in \underline{n} \mid \varphi(t) \leq j\} = \varphi(t) \; .$$

Now if $l = \varphi(t)$, then in particular $\varphi(t) \leq l$, thus

$$u_l = \varphi_l = \bigvee_{\substack{t' \in T \\ \varphi(t') \leq l}} t' \geq t \; .$$

Hence $\varphi(t) = l \geq \mathrm{Min}\{j \in \underline{n} \mid t \leq u_j\} = \check{u}(t)$. It follows that $\varphi(t) = \check{u}(t)$, i.e. $\varphi = \check{u}$, as was to be shown.

(b) Let $u \in \mathcal{U}_n$. Clearly, if $u_j = u_{j-1}$ for some $j \in [n]$, then $j \notin \mathrm{Im}(\check{u})$, so $\check{u}$ is not surjective. Conversely, if $u$ is strictly increasing, then $\check{u}(u_j) = j$ for any $j \in \underline{n}$, so $\check{u}$ is surjective. This proves that (i) is equivalent to (iii).

Now it follows from Proposition 15.10 that $\varepsilon_n \check{u} \neq 0$ if and only if $\check{u} \notin H_n(T)$, i.e. the image of $\check{u}$ contains $[n]$. But this holds if and only if $\check{u}$ is surjective, since $\check{u}(\hat{0}) = 0$. Hence (ii) is equivalent to (i). □

**15.19. Theorem.** *Let $T$ be a finite lattice. For $n \in \mathbb{N}$, and $u \in \mathcal{V}_n$, let $\tau_u$ denote the composition of morphisms*

$$F_T \xrightarrow{F_{\check{u}}} F_{\underline{n}} \xrightarrow{F_{\varepsilon_n}} \mathbb{S}_n \; .$$

*(a) The maps $\tau_u$, for $u \in \mathcal{V}_n$, form a $k$-basis of* $\mathrm{Hom}_{\mathcal{F}_k}(F_T, \mathbb{S}_n)$.

*(b) The map*

$$\tau = \bigoplus_{\substack{n \in \mathbb{N} \\ u \in \mathcal{V}_n}} \tau_u : F_T \to \bigoplus_{\substack{n \in \mathbb{N} \\ u \in \mathcal{V}_n}} \mathbb{S}_n$$

*is split surjective.*

**Proof :** (a) Since the functor $k\mathcal{L} \to \mathcal{F}_k$ defined by $T \mapsto F_T$ is fully faithful by Theorem 11.7, $\mathrm{Hom}_{\mathcal{F}_k}(F_T, F_{\underline{n}})$ is isomorphic to $\mathrm{Hom}_{k\mathcal{L}}(T, \underline{n})$. By Lemma 15.18, this is a free $k$-module with basis the set of maps $\check{u} : T \to \underline{n}$, for $u \in \mathcal{U}_n$. Therefore $\mathrm{Hom}_{\mathcal{F}_k}(F_T, F_{\underline{n}})$ is a free $k$-module with basis the set of maps $F_{\check{u}} : F_T \to F_{\underline{n}}$, for $u \in \mathcal{U}_n$. Its submodule $\mathrm{Hom}_{\mathcal{F}_k}(F_T, \mathbb{S}_n)$ is the image under composition with the idempotent $F_{\varepsilon_n}$. If $u \in \mathcal{U}_n - \mathcal{V}_n$, then $\varepsilon_n \check{u} = 0$ by Lemma 15.18. Therefore $\tau_u = F_{\varepsilon_n \check{u}} = 0$ if $u \in \mathcal{U}_n - \mathcal{V}_n$ and it follows that the elements $\tau_u$, for $u \in \mathcal{V}_n$, generate $\mathrm{Hom}_{\mathcal{F}_k}(F_T, \mathbb{S}_n)$. These elements are also linearly independent, since by Proposition 15.10, the kernel of $F_{\varepsilon_n}$ is equal to $H_{\underline{n}}$, and since (the images of) the maps $\check{u}$, for $u \in \mathcal{V}_n$, form part of the $k$-basis of $(F_{\underline{n}}/H_{\underline{n}})(T)$.

(b) It suffices to show that the map $\tau$ is surjective. It will be split because the functors $\mathbb{S}_n$ are projective in $\mathcal{F}_k$.

We first claim that, if $M$ and $N$ are objects in an abelian category $\mathcal{A}$, if $M$ is projective, and if $\mathrm{Hom}_{\mathcal{A}}(M, N) = 0$, then any subobject of $M \oplus N$ which maps onto both $M$ and $N$ is equal to $M \oplus N$. Indeed for such an object $L$, there is an isomorphism $\theta : M/(L \cap M) \stackrel{\cong}{\to} N/(L \cap N)$ sending the class $m + (L \cap M)$ for $m \in M$, to the class $n + (L \cap N)$, where $n \in N$ is such that $m + n \in L$. (This is a special case of the Goursat lemma on subgroups in a direct product of groups). As $M$ is projective, this isomorphism can be lifted to a morphism $\psi : M \to N$ making the following diagram commutative

$$\begin{array}{ccc} M & \xrightarrow{\quad\psi\quad} & N \\ \downarrow & & \downarrow \\ M/(L \cap M) & \xrightarrow[\cong]{\theta} & N/(L \cap N) \end{array} ,$$

where the vertical arrows are the canonical surjections. But $\psi = 0$ by assumption, hence $\theta = 0$, thus $M = L \cap M$ and $N = L \cap N$, i.e. $L = M \oplus N$.

In order to prove the surjectivity of $\tau$, it is enough to show that for any $n \in \mathbb{N}$, the map

$$\tau_n = \bigoplus_{u \in \mathcal{V}_n} \tau_u : F_T \to \bigoplus_{u \in \mathcal{V}_n} \mathbb{S}_n$$

is surjective. This reduction follows from the claim above, using the fact that, for $n \neq m$, we have $\mathrm{Hom}_{\mathcal{F}_k}(\mathbb{S}_n, \mathbb{S}_m) = 0$ by Corollary 15.13.

We now fix an $n \in \mathbb{N}$ and show that the map $\tau_n$ is surjective. We consider the special case where the ground ring is $\mathbb{Z}$, and we use the notation $F_T^{\mathbb{Z}}$, $\mathbb{S}_n^{\mathbb{Z}}$, $\tau_n^{\mathbb{Z}}$ to denote $F_T$, $\mathbb{S}_n$, and $\tau_n$ in this case. Let $C$ be the cokernel of $\tau_n^{\mathbb{Z}}$. We have an exact sequence

$$F_T^{\mathbb{Z}} \xrightarrow{\tau_n^{\mathbb{Z}}} \bigoplus_{u \in \mathcal{V}_n} \mathbb{S}_n^{\mathbb{Z}} \longrightarrow C \longrightarrow 0 \ .$$

Tensoring this sequence with an arbitrary commutative ring $k$ gives the top row of the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
k \otimes_{\mathbb{Z}} F_T^{\mathbb{Z}} & \xrightarrow{k \otimes_{\mathbb{Z}} \tau_n^{\mathbb{Z}}} & \bigoplus_{u \in \mathcal{V}_n} k \otimes_{\mathbb{Z}} \mathbb{S}_n^{\mathbb{Z}} & \longrightarrow & k \otimes_{\mathbb{Z}} C & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \| & & \\
F_T & \xrightarrow{\tau_n} & \bigoplus_{u \in \mathcal{V}_n} \mathbb{S}_n & \longrightarrow & k \otimes_{\mathbb{Z}} C & \longrightarrow & 0
\end{array}$$

Hence if we show that $\tau_n$ is surjective when $k$ is a field, we get that $k \otimes_{\mathbb{Z}} C = 0$ for any field $k$. In particular, $\mathbb{F}_p \otimes_{\mathbb{Z}} C = 0$ for any prime $p$, hence $C = 0$, and then $\tau_n$ is surjective for any commutative ring $k$.

Thus we can assume that $k$ is a field. In this case $\mathbb{S}_n$ is a simple functor by Corollary 15.17 and $\mathrm{End}_{\mathcal{F}_k}(\mathbb{S}_n) \cong k$ by Corollary 15.13. By (a), we know that the maps $\tau_u$, for $u \in \mathcal{V}_n$, form a $k$-basis of $\mathrm{Hom}_{\mathcal{F}_k}(F_T, \mathbb{S}_n)$. By a standard argument of linear algebra, which we sketch below, it follows that the map $\tau_n$ is surjective, which completes the proof of Theorem 15.19.

The linear algebra argument goes as follows. Let $M = \bigcap_{u \in \mathcal{V}_n} \mathrm{Ker}\, \tau_u \subseteq F_T$. Then $F_T/M$ embeds via $\tau_n$ into the semi-simple functor $\bigoplus_{u \in \mathcal{V}_n} \mathbb{S}_n$, hence it is semi-simple, a direct sum of $m$ copies of $\mathbb{S}_n$ for some $m$. Moreover $m \leq v$, where $v = |\mathcal{V}_n|$. The matrix of the map $\tau_n : F_T/M \to \bigoplus_{u \in \mathcal{V}_n} \mathbb{S}_n$ has coefficients in $\mathrm{End}_{\mathcal{F}_k}(\mathbb{S}_n) \cong k$ and has $v$ rows and $m$ columns. Since the maps $\tau_u$, for $u \in \mathcal{V}_n$, are linearly independent by (a), the rows of this matrix are linearly independent. Therefore the rank of the matrix is $v$ and this forces $m \geq v$. Hence $m = v$ and consequently the $v$ columns of the matrix generate the image. This means that $\tau_n$ is surjective. □

## 16. On the structure of lattices

The purpose of this section is to define, in any finite lattice, two operations $r^\infty$ and $s^\infty$, as well as some special elements which we call *bulbs*, lying at the bottom of a totally ordered subset with strong properties. Such bulbs play a crucial role in the description of the evaluation of fundamental functors and simple functors, which will be carried out in Section 17.

Let $T$ be a finite lattice and let $(E, R)$ be the full subposet of irreducible elements in $T$. For simplicity, we write $\leq$ for $\leq_T$ and $[t_1, t_2]$ for $[t_1, t_2]_T$. Recall that $T$ is *generated* by $E$ in the sense that any element $x \in T$ is a join of elements of $E$ (see Section 5).

If $t \in T$, recall from Notation 14.12 that $r(t)$ denotes the join of all elements in $[\hat{0}, t[$. If $t \notin E$, then $t$ can be written as the join of two smaller elements, so $r(t) = t$. If $e \in E$, then $r(e)$ is the unique maximal element of $[\hat{0}, e[$. We put $r^k(t) = r(r^{k-1}(t))$ and $r^\infty(t) = r^n(t)$ if $n$ is such that $r^n(t) = r^{n+1}(t)$. Note that the map $r : T \to T$ is order-preserving.

**16.1. Lemma.** *Let $t \in T$.*

*(a) $r^\infty(t) \notin E$.*
*(b) $r^\infty(t) = t$ if and only if $t \in T - E$.*
*(c) If $e \in E$, $r^\infty(e)$ is the unique greatest element of $T - E$ smaller than $e$.*
*(d) If $t' \in T$ with $t \leq t'$, then $r^\infty(t) \leq r^\infty(t')$.*

**Proof :** The proof is a straightforward consequence of the definitions. □

**16.2. Lemma.** *Let $e \in E$.*

*(a) $[\hat{0}, e] = [\hat{0}, r^\infty(e)] \sqcup\, ]r^\infty(e), e]$*
*(b) $]r^\infty(e), e]$ is contained in $E$.*
*(c) $[r^\infty(e), e]$ is totally ordered.*
*(d) More precisely, $[r^\infty(e), e] = \{r^n(e), \ldots, r^1(e), e\}$ if $r^\infty(e) = r^n(e)$.*
*(e) $r^\infty(r^i(e)) = r^\infty(e)$ for all $0 \leq i \leq n-1$.*

**Proof :** (b) This is an immediate consequence of the definition of $r^\infty(e)$.

(a) Let $f \in [\hat{0}, e]$. Then $f \vee r^\infty(e) \in [r^\infty(e), e]$. If $f \vee r^\infty(e) = r^\infty(e)$, then $f \in [\hat{0}, r^\infty(e)]$. Otherwise, $f \vee r^\infty(e) \in\, ]r^\infty(e), e]$, hence $f \vee r^\infty(e) \in E$ by (b), that is, $f \vee r^\infty(e)$ is irreducible. It follows that $f \vee r^\infty(e) = f$ or $f \vee r^\infty(e) = r^\infty(e)$. But the second case is impossible since $r^\infty(e) \notin E$. Therefore $f \vee r^\infty(e) = f$, that is, $f \in\, ]r^\infty(e), e]$.

(c) Let $f, g \in [r^\infty(e), e]$. We may suppose that they are not both equal to $r^\infty(e)$. Then $f \vee g \in\, ]r^\infty(e), e]$ and again $f \vee g$ is irreducible by (b). Thus $f \vee g$ is equal to either $f$ or $g$. Consequently $g \leq f$ or $f \leq g$.

(d) This is an immediate consequence of (b) and (c).

(e) This follows from the definition of $r^\infty(e)$. □

We write $\wedge E$ for the subset of $T$ consisting of all meets of elements of $E$, that is, elements of the form $\bigwedge_{i \in I} e_i$ where $I$ is a finite set of indices and $e_i \in E$ for every $i \in I$. Note that we include the possibility that $I$ be the empty set, in which case one gets the unique greatest element $\hat{1}$.

**16.3. Notation.** *If $t \in T$, define $s(t)$ to be the meet of all the irreducible elements which are strictly larger than $t$.*

Notice that this definition of $s$ is 'dual' to the definition of $r$, because $r(t)$ is the join of all the irreducible elements which are strictly smaller than $t$. It is clear that the map $s : T \to T$ is order-preserving.

In order to describe the effect of $s$, note first that $s(t) = t$ if $t \in \wedge E - E$ and $s(t) > t$ if $t \notin \wedge E$. Now if $e \in E$, there are 3 cases:

(1) If $]e,\hat{1}] \cap E$ has at least two minimal elements, then $s(e) = e$.
(2) If $]e,\hat{1}] \cap E$ has a unique minimal element $e^+$, then $s(e) = e^+$.
(3) If $]e,\hat{1}] \cap E$ is empty (that is, $e$ is maximal in $E$), then $s(e) = \hat{1}$.

Note that the equality $s(e) = e$ also occurs in the third case for $e = \hat{1}$, provided $\hat{1}$ is irreducible. We define inductively $s^k(t) = s(s^{k-1}(t))$ and $s^\infty(t) = s^n(t)$ where $n$ is such that $s^n(t) = s^{n+1}(t)$.

**16.4. Definition.** *An element $t \in T$ is called a* bulb *if $t \notin \wedge E$ and if there exists $e \in E$ such that $t = r^\infty(e)$ and $s(e) = e$.*

Since the poset $]r^\infty(e), e]$ is totally ordered and consists of elements of $E$, it is clear that $r^\infty(f) = r^\infty(e)$ for every $f \in \, ]r^\infty(e), e]$ (see Lemma 16.2). Since it may happen that $s(f) = f$ for certain elements $f \in \, ]r^\infty(e), e]$, the element $e$ in the definition above is not necessarily unique. The following lemma shows that $e$ becomes unique if it is chosen minimal among all elements $f \in E$ such that $r^\infty(f) = t$ and $s(f) = f$.

**16.5. Lemma.** *Let $t \in T$ be a bulb and let $e \in E$ be a minimal element such that $t = r^\infty(e)$ and $s(e) = e$. Let $k \geq 1$ be the smallest positive integer such that $r^k(e) = r^\infty(e)$, so that $[t, e] = \{r^k(e), r^{k-1}(e), \ldots, r^1(e), e\}$.*

*(a) $s^i(t) = r^{k-i}(e)$ for $1 \leq i \leq k$ and $s^k(t) = s^\infty(t) = e$.*
*(b) $[t, s^\infty(t)] = \{r^k(e), \ldots, e\} = \{t, s(t), \ldots, s^k(t)\}$.*
*(c) $e = s^\infty(t)$, in other words $e$ is unique. Moreover, $s^\infty(t) \in E$.*
*(d) $r^\infty(s^i(t)) = t$ for $1 \leq i \leq k$.*

**Proof :** By Lemma 16.2, $r^\infty(r^i(e)) = r^\infty(e) = t$ for all $0 \leq i \leq k-1$. If $1 \leq i \leq k-1$, then $r^i(e) \in E$, $r^\infty(r^i(e)) = t$ and $r^i(e) < e$. By minimality of $e$, it follows that $s(r^i(e)) \neq r^i(e)$, hence $s(r^i(e)) > r^i(e)$. Moreover, $s(r^i(e)) \leq r^{i-1}(e)$ by definition of $s(r^i(e))$ and the fact that $r^{i-1}(e) \in E$. Since $[t, e]$ is totally ordered, this forces the equality $s(r^i(e)) = r^{i-1}(e)$. This equality also holds if $i = k$ because $r^k(e) = t$ and $t \notin \wedge E$, so $s(t) > t$, and again $s(t) \leq r^{k-1}(e)$ so that $s(t) = r^{k-1}(e)$.

Then one obtains $s^i(t) = r^{k-i}(e)$ for $1 \leq i \leq k$ and in particular $s^k(t) = s^\infty(t) = r^0(e) = e$. The first three statements follow. The fourth is a consequence of Lemma 16.2. □

**16.6. Example.** If $T = \underline{n} = \{0, 1, \ldots, n\}$ is totally ordered, then $E = \mathrm{Irr}(T) = \{1, \ldots, n\}$ and 0 is a bulb in $T$.

**16.7. Notation.** *We let $G_1 = G_1(T)$ be the set of all bulbs in $T$ and we define*

$$G = \wedge E \sqcup G_1 \; .$$

We first show that $G$ has another characterization.

**16.8. Lemma.** *Let $G$ be as above. Then*

$$G = E \sqcup G^\sharp \; ,$$

*where $G^\sharp = \{a \in T \mid a = r^\infty s^\infty(a)\}$.*

**Proof :** First observe that $E \sqcup G^\sharp$ is a disjoint union because an element of the form $a = r^\infty s^\infty(a)$ satisfies $r(a) = a$, so it cannot belong to $E$.

In order to prove that $G \subseteq E \sqcup G^\sharp$, let $a \in G$. If $a \in E$, then obviously $a \in E \sqcup G^\sharp$. If $a \in \wedge E - E$, then $a = s(a)$, hence $a = s^\infty(a)$. Moreover $a = r^\infty(a)$

since $a \notin E$. Hence $a = r^\infty s^\infty(a)$, that is $a \in G^\sharp$. Finally if $a \in G_1$, then $a \notin E$ and $a = r^\infty s^\infty(a)$, by definition of a bulb. This proves that $G \subseteq E \sqcup G^\sharp$.

For the reverse inclusion, first note that $E \subseteq G$ because $E \subseteq \wedge E$. Now let $a \in G^\sharp$ and set $b = s^\infty(a)$. If $b \notin E$, then $b = r^\infty(b)$, hence $b = a$ and $a = s(a)$. It follows that $a \in \wedge E$, hence $a \in G$. If now $b \in E$, there are two cases. Either $a \in \wedge E$, hence $a \in G$ and we are done, or $a \notin \wedge E$. But then we have $b = s(b)$ and $a = r^\infty(b)$, so $a$ is a bulb by definition, so that $a \in G$. This proves the inclusion $E \sqcup G^\sharp \subseteq G$. □

The following two propositions will be crucial for our results on evaluations of fundamental functors in Section 17.

**16.9. Proposition.** *Let $a \in T - \wedge E$ and suppose that $s^\infty(a) \in E$. Write $s^\infty(a) = s^m(a)$ where $m$ is such that $s^{m-1}(a) < s^m(a)$. Let $b := r^\infty(s^\infty(a))$.*

*(a) There exists $0 \leq r \leq m-1$ such that $s^r(a) < b < s^{r+1}(a)$.*
*(b) $b \in G$.*
*(c) $]s^r(a), \hat{1}] \cap E = [b, \hat{1}] \cap E = [s^{r+1}(a), \hat{1}] \cap E$.*

**Proof :** (a) Define $e_i = s^i(a)$ for all $0 \leq i \leq m$. Note that $e_1, \ldots, e_{m-1}$ all belong to $E$ because they belong to $\wedge E$ (since they are in the image of the operator $s$) and moreover $s(e_i) > e_i$. Also $e_m = s^\infty(a) \in E$ by assumption.

We have $a \leq r^\infty(s^\infty(a)) < s^\infty(a)$, because $s^\infty(a) \in E$ by assumption. Therefore, there is an integer $r \leq m-1$ such that $b \leq e_{r+1}$ but $b \not\leq e_r$. The inequality $b < e_{r+1}$ is strict because $b \notin E$ while $e_{r+1} \in E$. (The case $r = 0$ occurs when $a \leq b < e_1$.)

In particular $b \leq r^\infty(e_{r+1}) \leq r^\infty(s^\infty(a)) = b$, hence $b = r^\infty(e_{r+1})$. We want to show that the element $e_r \vee b$ cannot be irreducible. Otherwise $e_r \vee b = b$ or $e_r \vee b = e_r$. The first case is impossible because $b$ is not irreducible since $b = r^\infty(e_{r+1}) \notin E$. The second case is impossible because it would imply $b \leq e_r$, contrary the the definition of $r$. Therefore $e_r \vee b \notin E$ and $e_r \vee b \leq e_{r+1}$. By definition of $r^\infty(e_{r+1})$, we obtain $e_r \vee b \leq r^\infty(e_{r+1}) = b$. It follows that $e_r < b < e_{r+1}$, as required.

(b) To prove that $b \in G$, we first note that if $b \in \wedge E$, then $b \in G$. Otherwise, $b \notin \wedge E$, $b = r^\infty(s^\infty(a))$ and $s(s^\infty(a)) = s^\infty(a)$, proving that $b$ is a bulb, that is, $b \in G_1$.

(c) By the definition of $s(e_r)$, there is a unique minimal element in $]e_r, \hat{1}] \cap E$, namely $s(e_r) = e_{r+1}$. Therefore, $]e_r, \hat{1}] \cap E = [e_{r+1}, \hat{1}] \cap E$. In particular $]e_r, \hat{1}] \cap E = [b, \hat{1}] \cap E$. □

**16.10. Notation.** *Let $\sigma : G \to I^\uparrow(E, R)$ be the map defined by*

$$\sigma(t) = \begin{cases} [t, \hat{1}] \cap E & \text{if } t \in E\,, \\ ]s^\infty(t), \hat{1}] \cap E & \text{if } t \notin E\,. \end{cases}$$

*For any $B \in I^\uparrow(E, R)$, define $\wedge B = \bigwedge_{e \in B} e$. By definition of $s^\infty(t)$, we obtain*

$$\wedge\sigma(t) = \begin{cases} t & \text{if } t \in E\,, \\ s^\infty(t) & \text{if } t \notin E\,. \end{cases}$$

**16.11. Proposition.** *Let $t \in G$ and $t' \in T$ such that $t' \leq \wedge\sigma(t)$.*

*(a) $s^\infty(t') \leq s^\infty(t)$ and $r^\infty(t') \leq r^\infty(t)$.*
*(b) $t' \leq t$, except possibly if $t$ is a bulb (i.e. $t \in G_1$).*
*(c) If $t' \not\leq t$, then $t \in G_1$ and $t = r^\infty(t')$.*

**Proof :** We have $G = (\wedge E - E) \sqcup E \sqcup G_1$ and one considers the three cases for $t$ successively.

If $t \in \wedge E - E$, then $\wedge\sigma(t) = s^\infty(t) = t$, hence $t' \leq t$ and consequently $s^\infty(t') \leq s^\infty(t)$ and $r^\infty(t') \leq r^\infty(t)$.

If $t \in E$, then $\wedge\sigma(t) = t$, hence $t' \leq t$ and consequently $s^\infty(t') \leq s^\infty(t)$ and $r^\infty(t') \leq r^\infty(t)$.

Finally, if $t \in G_1$, then $\wedge\sigma(t) = s^\infty(t)$, thus $s^\infty(t') \leq s^\infty(t)$. Moreover, using part (d) of Lemma 16.1 and part (d) of Lemma 16.5, we obtain

$$r^\infty(t') \leq r^\infty(\wedge\sigma(t)) = r^\infty(s^\infty(t)) = t = r^\infty(t) \ .$$

This proves (a), and also (b) because the relation $t' \not\leq t$ can appear only if $t \in G_1$. In that case case, by Lemma 16.5, we can write $t = r^\infty(e)$ where $e = s^\infty(t)$. Since $t' \leq s^\infty(t)$, we get $t' \in [\hat{0}, t] \sqcup ]t, s^\infty(t)]$ by Lemma 16.2, hence $t' \in ]t, s^\infty(t)]$ because $t' \not\leq t$. In other words, $t' \in \{s(t), \ldots, s^k(t)\}$ by Lemma 16.5. Therefore $t' = s^i(t)$ for some $i \geq 1$ and so $r^\infty(t') = t$ by Lemma 16.5. This proves (c) and completes the proof. □

## 17. Evaluations of fundamental functors and simple functors

As usual, $E$ denotes a fixed finite set and $R$ an order relation on $E$. Our purpose is to prove that, for an arbitrary commutative ring $k$ and for any finite set $X$, the evaluation $\mathbb{S}_{E,R}(X)$ of the fundamental correspondence functor $\mathbb{S}_{E,R}$ is a free $k$-module, by finding an explicit $k$-basis. In case $k$ is a field, we will then find the dimension of $S_{E,R,V}(X)$, where $S_{E,R,V}$ is a simple correspondence functor.

Let $T$ be any lattice such that $(E, R)$ is the full subposet of irreducible elements in $T$. Note that, by Theorem 21.16 in the first appendix, $I_\downarrow(E, R)$ is the largest such lattice and all the others are described as quotients of $I_\downarrow(E, R)$. By Theorem 13.1, the fundamental functor $\mathbb{S}_{E,R^{op}}$ is isomorphic to a quotient of $F_T$ via a morphism $\Theta_T : F_T \to \mathbb{S}_{E,R^{op}}$. For this reason, we work with $\mathbb{S}_{E,R^{op}}$ rather than $\mathbb{S}_{E,R}$.

**17.1. Notation.** *Let $G = G(T)$ be the subset defined in Notation 16.7 and let $X$ be a finite set. We define $\mathcal{B}_X$ to be the set of all maps $\varphi : X \to T$ such that $E \subseteq \varphi(X) \subseteq G$.*

Our main purpose is to prove that the set $\Theta_{T,X}(\mathcal{B}_X)$ is a $k$-basis of $\mathbb{S}_{E,R^{op}}(X)$. We shall first prove that $\Theta_{T,X}(\mathcal{B}_X)$ generates $k$-linearly $\mathbb{S}_{E,R^{op}}(X)$. Then, in the second part of this section, we shall show that $\Theta_{T,X}(\mathcal{B}_X)$ is $k$-linearly independent.

By Lemma 16.8, we have $G = E \sqcup G^\sharp$ and we denote by $\Gamma$ the complement of $G$ in $T$, namely

$$\Gamma = \{a \in T \mid a \notin E,\ a < r^\infty s^\infty(a)\} \ ,$$

where $r^\infty$ and $s^\infty$ are defined as in Section 16.

**17.2. Lemma.** *Let $a \in \Gamma$, and let $b = r^\infty s^\infty(a)$. There exists an integer $r \geq 0$ such that*

$$a < s(a) < \ldots < s^r(a) < b = r^\infty s^\infty(a) \leq s^{r+1}(a) \ ,$$

*and $s^j(a) \in E$ for $j \in \{1, \ldots, r\}$. Moreover $b = r^\infty s^\infty\big(s^j(a)\big)$ for $j \in \{1, \ldots, r\}$, and $b \in G^\sharp$.*

**Proof :** We know that $a = s^0(a) \notin E$ because $a \in \Gamma$. Suppose first that there exists an integer $r \geq 0$ such that $c = s^{r+1}(a) \notin E$. In this case, we choose $r$ minimal with this property, so that $s^j(a) \in E$ for $1 \leq j \leq r$. We have $c = s\big(s^r(a)\big) \in \wedge E - E$, hence $c = s(c) = s^\infty(c) = s^\infty(a)$. Moreover $b = r^\infty(c) = c$, because $c \notin E$, and $b = r^\infty s^\infty(b)$, so $b \in G^\sharp$.

Suppose now that $s^r(a) \in E$ for all $r \in \mathbb{Z}_{>0}$. Then Proposition 16.9 applies and there exists an integer $r \geq 0$ such that $s^r(a) < b < s^{r+1}(a)$. Moreover $b \in G$ and $b \notin E$, because $b = r^\infty(b)$, so $b \in G^\sharp$. □

**17.3. Definition.** *For $a \in \Gamma$, the sequence $a < s(a) < \ldots < s^r(a) < b$ defined in Lemma 17.2 is called the* reduction sequence *associated to $a$.*

**17.4. Notation.** *Let $n \geq 1$ and let $\underline{a} = (a_0, a_1, \ldots, a_n)$ be a sequence of distinct elements of $T$. We denote by $[a_0, \ldots, a_n] : T \to T$ the map defined by*

$$\forall t \in T, \ \ [a_0, \ldots, a_n](t) = \begin{cases} a_{j+1} & \textit{if } t = a_j, \ j \in \{0, \ldots, n-1\} \\ t & \textit{otherwise.} \end{cases}$$

*We denote by $\kappa_{\underline{a}}$ the element of $k(T^T) = F_T(T)$ defined by*

$$\kappa_{\underline{a}} = [a_0, a_1] - [a_0, a_1, a_2] + \ldots + (-1)^{n-1}[a_0, a_1, \ldots, a_n] \ .$$

*For $a \in \Gamma$, we set $u_a = \kappa_{\underline{a}}$, where $\underline{a}$ is the reduction sequence associated to $a$.*

**17.5. Theorem.** *Let $T$ be a finite lattice, let $(E, R)$ be the full subposet of its irreducible elements, let $\Gamma = \{a \in T \mid a \notin E, \ a < r^\infty s^\infty(a)\}$, and let $X$ be a finite set. Then*

$$\forall a \in \Gamma, \forall \varphi : X \to T, \ \ \varphi - u_a \circ \varphi \in \operatorname{Ker} \Theta_{T,X} \ ,$$

*where $u_a \circ \varphi$ is defined by bilinearity from the composition of maps $T^T \times T^X \to T^X$.*

**Proof :** The kernel of the map $\Theta_{T,X} : F_T(X) \to \mathbb{S}_{E,R^{op}}(X)$ was given in Theorem 13.4. Let $\sum\limits_{\varphi:T\to X} \lambda_\varphi \varphi \in F_T(X)$, where $\lambda_\varphi \in k$. Then $\sum\limits_{\varphi:T\to X} \lambda_\varphi \varphi \in \operatorname{Ker} \Theta_{T,X}$ if and only if the coefficients $\lambda_\varphi$ satisfy a system of linear equations indexed by maps $\psi : X \to I^\uparrow(E, R)$. The equation $(E_\psi)$ indexed by such a map $\psi$ is the following :

$$(E_\psi) : \qquad \sum_{\varphi \underset{E,R}{\vdash} \psi} \lambda_\varphi = 0 \ ,$$

where $\varphi \underset{E,R}{\vdash} \psi$ means that $\varphi : X \to T$ and $\psi : X \to I^\uparrow(E, R)$ satisfy the equivalent conditions of Lemma 13.9. We shall use condition (f) of Lemma 13.9, namely

$$\varphi \underset{E,R}{\vdash} \psi \iff \begin{cases} \forall t \in T, \ \ \psi\big(\varphi^{-1}(t)\big) \subseteq [t, \cdot[_T \cap E \ , \\ \forall e \in E, \ \ \psi\big(\varphi^{-1}(e)\big) = [e, \cdot[_E \ . \end{cases}$$

Let $a \in \Gamma$, and let $(a, e_1, e_2, \ldots, e_r, b)$ be the associated reduction sequence. For $i \in \{1, \ldots, r\}$, set $\varphi_i = [a, e_1, \ldots, e_i] \circ \varphi$. Also define $\varphi_{r+1} = [a, e_1, \ldots, e_r, b] \circ \varphi$. In particular, for any $i \in \{1, \ldots, r+1\}$,

$$\text{if } \varphi(x) \in T - \{a, e_1, \ldots, e_r\} \ , \ \ \text{then } \varphi_i(x) = \varphi(x) \ .$$

The other values of the maps $\varphi_i$ are given in the following table:

(**17.6**)

| $x \in$ | $\varphi^{-1}(a)$ | $\varphi^{-1}(e_1)$ | $\varphi^{-1}(e_2)$ | $\dots$ | $\varphi^{-1}(e_{r-1})$ | $\varphi^{-1}(e_r)$ |
|---|---|---|---|---|---|---|
| $\varphi(x)$ | $a$ | $e_1$ | $e_2$ | $\dots$ | $e_{r-1}$ | $e_r$ |
| $\varphi_1(x)$ | $e_1$ | $e_1$ | $e_2$ | $\dots$ | $e_{r-1}$ | $e_r$ |
| $\varphi_2(x)$ | $e_1$ | $e_2$ | $e_2$ | $\dots$ | $e_{r-1}$ | $e_r$ |
| $\varphi_3(x)$ | $e_1$ | $e_2$ | $e_3$ | $\dots$ | $e_{r-1}$ | $e_r$ |
| $\dots$ | $\dots$ | $\dots$ | $\dots$ | $\dots$ | $\dots$ | $\dots$ |
| $\varphi_r(x)$ | $e_1$ | $e_2$ | $e_3$ | $\dots$ | $e_r$ | $e_r$ |
| $\varphi_{r+1}(x)$ | $e_1$ | $e_2$ | $e_3$ | $\dots$ | $e_r$ | $b$ |

We want to prove that the element

$$\varphi - u_a \circ \varphi = \varphi - \varphi_1 + \varphi_2 - \ldots + (-1)^{r-1}\varphi_{r+1}$$

belongs to $\operatorname{Ker}\Theta_{T,X}$. We must prove that it satisfies the equation $(E_\psi)$ for every $\psi$, so we must find which of the functions $\varphi, \varphi_1, \varphi_2, \ldots, \varphi_{r+1}$ are linked with $\psi$ under the relation $\underset{E,R}{\vdash}$. We are going to prove that only two consecutive functions can be linked with a given $\psi$, from which it follows that the corresponding equation $(E_\psi)$ reduces to either $1-1=0$, or $-1+1=0$. It follows from this that $\varphi - u_a \circ \varphi$ satisfies all equations $(E_\psi)$, hence belongs to $\operatorname{Ker}\Theta_{T,X}$, as required.

The linking with $\psi$ is controlled by the following conditions:

$$\varphi \underset{E,R}{\vdash} \psi \iff \begin{cases} \forall t \in U, \;\; \psi\big(\varphi^{-1}(t)\big) \subseteq [t,\cdot[_T \cap E \\ \forall e \in V, \;\; \psi\big(\varphi^{-1}(e)\big) = [e,\cdot[_E \\ \psi\big(\varphi^{-1}(a)\big) \subseteq [a,\cdot[_T \cap E = [e_1,\cdot[_E \\ \psi\big(\varphi^{-1}(e_i)\big) = [e_i,\cdot[_E \;\; \forall i \in \{1,\ldots,r\}\,. \end{cases}$$

Similarly, since $\varphi_1^{-1}(e_1) = \varphi^{-1}(a) \sqcup \varphi^{-1}(e_1)$,

$$\varphi_1 \underset{E,R}{\vdash} \psi \iff \begin{cases} \forall t \in U, \;\; \psi\big(\varphi^{-1}(t)\big) \subseteq [t,\cdot[_T \cap E \\ \forall e \in V, \;\; \psi\big(\varphi^{-1}(e)\big) = [e,\cdot[_E \\ \psi\big(\varphi^{-1}(a) \sqcup \varphi^{-1}(e_1)\big) = [e_1,\cdot[_E \\ \psi\big(\varphi^{-1}(e_i)\big) = [e_i,\cdot[_E \;\; \forall i \in \{2,\ldots,r\}\,. \end{cases}$$

Similarly,

$$\varphi_2 \underset{E,R}{\vdash} \psi \iff \begin{cases} \forall t \in U, \;\; \psi\big(\varphi^{-1}(t)\big) \subseteq [t,\cdot[_T \cap E \\ \forall e \in V, \;\; \psi\big(\varphi^{-1}(e)\big) = [e,\cdot[_E \\ \psi\big(\varphi^{-1}(a)\big) = [e_1,\cdot[_E \\ \psi\big(\varphi^{-1}(e_1) \sqcup \varphi^{-1}(e_2)\big) = [e_2,\cdot[_E \\ \psi\big(\varphi^{-1}(e_i)\big) = [e_i,\cdot[_E \;\; \forall i \in \{3,\ldots,r\}\,. \end{cases}$$

$$\varphi_3 \underset{E,R}{\vdash} \psi \iff \begin{cases} \forall t \in U, \;\; \psi\big(\varphi^{-1}(t)\big) \subseteq [t,\cdot[_T \cap E \\ \forall e \in V, \;\; \psi\big(\varphi^{-1}(e)\big) = [e,\cdot[_E \\ \psi\big(\varphi^{-1}(a)\big) = [e_1,\cdot[_E \\ \psi\big(\varphi^{-1}(e_1)\big) = [e_2,\cdot[_E \\ \psi\big(\varphi^{-1}(e_2) \sqcup \varphi^{-1}(e_3)\big) = [e_3,\cdot[_E \\ \psi\big(\varphi^{-1}(e_i)\big) = [e_i,\cdot[_E \;\; \forall i \in \{4,\ldots,r\}\,. \end{cases}$$

$\dots$

$$\varphi_{r-1} \underset{E,R}{\vdash} \psi \iff \begin{cases} \forall t \in U, \ \ \psi\big(\varphi^{-1}(t)\big) \subseteq [t, \cdot[_T \cap E \\ \forall e \in V, \ \ \psi\big(\varphi^{-1}(e)\big) = [e, \cdot[_E \\ \psi\big(\varphi^{-1}(a)\big) = [e_1, \cdot[_E \\ \psi\big(\varphi^{-1}(e_i)\big) = [e_{i+1}, \cdot[_E \ \forall i \in \{1, \dots r-3\} \\ \psi\big(\varphi^{-1}(e_{r-2}) \sqcup \varphi^{-1}(e_{r-1})\big) = [e_{r-1}, \cdot[_E \\ \psi\big(\varphi^{-1}(e_r)\big) = [e_r, \cdot[_E \ . \end{cases}$$

$$\varphi_{r} \underset{E,R}{\vdash} \psi \iff \begin{cases} \forall t \in U, \ \ \psi\big(\varphi^{-1}(t)\big) \subseteq [t, \cdot[_T \cap E \\ \forall e \in V, \ \ \psi\big(\varphi^{-1}(e)\big) = [e, \cdot[_E \\ \psi\big(\varphi^{-1}(a)\big) = [e_1, \cdot[_E \\ \psi\big(\varphi^{-1}(e_i)\big) = [e_{i+1}, \cdot[_E \ \ \forall i \in \{1, \dots, r-2\} \\ \psi\big(\varphi^{-1}(e_{r-1}) \sqcup \varphi^{-1}(e_r)\big) = [e_r, \cdot[_E \ . \end{cases}$$

$$\varphi_{r+1} \underset{E,R}{\vdash} \psi \iff \begin{cases} \forall t \in U, \ \ \psi\big(\varphi^{-1}(t)\big) \subseteq [t, \cdot[_T \cap E \\ \forall e \in V, \ \ \psi\big(\varphi^{-1}(e)\big) = [e, \cdot[_E \\ \psi\big(\varphi^{-1}(a)\big) = [e_1, \cdot[_E \\ \psi\big(\varphi^{-1}(e_i)\big) = [e_{i+1}, \cdot[_E \ \ \forall i \in \{1, \dots, r-1\} \\ \psi\big(\varphi^{-1}(e_r)\big) \subseteq [b, \cdot[_T \cap E \ . \end{cases}$$

In particular $\varphi \underset{E,R}{\vdash} \psi$ implies $\varphi_1 \underset{E,R}{\vdash} \psi$, and also $\varphi_i \underset{E,R}{\nvdash} \psi$ for $i \geq 2$, since $\varphi \underset{E,R}{\vdash} \psi$ implies $\psi\big(\varphi^{-1}(e_1)\big) = [e_1, \cdot[_E$, but $\varphi_i \underset{E,R}{\vdash} \psi$ implies $\psi\big(\varphi^{-1}(e_1)\big) \subseteq [e_2, \cdot[_E$ when $i \geq 2$. Therefore only $\varphi$ and $\varphi_1$ are involved in this case.

Now if $\varphi_1 \underset{E,R}{\vdash} \psi$ but $\varphi \underset{E,R}{\nvdash} \psi$, then $\psi\big(\varphi^{-1}(e_1)\big) \subseteq ]e_1, \cdot[_T \cap E = [e_2, \cdot[_E$ (as $e_2 = s(e_1)$) and $\psi\big(\varphi^{-1}(a)\big) = [e_1, \cdot[_E$, hence in particular $\varphi_2 \underset{E,R}{\vdash} \psi$, since $\varphi_1 \underset{E,R}{\vdash} \psi$ also implies $\psi\big(\varphi^{-1}(e_i)\big) = [e_i, \cdot[_E$ for $i \in \{2, \dots, n\}$. On the other hand $\varphi_i \underset{E,R}{\nvdash} \psi$, for $i \geq 3$, since $\varphi_1 \underset{E,R}{\vdash} \psi$ implies $\psi\big(\varphi^{-1}(e_2)\big) = [e_2, \cdot[_E$, but if $i \geq 3$, then $\varphi_i \underset{E,R}{\vdash} \psi$ implies $\psi\big(\varphi^{-1}(e_1)\big) \subseteq [e_3, \cdot[_E$. Therefore only $\varphi_1$ and $\varphi_2$ are involved in this case.

By induction, the same argument shows, for $i \in \{1, \dots, r-1\}$, that $\varphi_i \underset{E,R}{\vdash} \psi$ but $\varphi_{i-1} \underset{E,R}{\nvdash} \psi$, then $\varphi_{i+1} \underset{E,R}{\vdash} \psi$. Therefore only $\varphi_i$ and $\varphi_{i+1}$ are involved in this case.

If now $\varphi_r \underset{E,R}{\vdash} \psi$ but $\varphi_{r-1} \underset{E,R}{\nvdash} \psi$, then $\psi\big(\varphi^{-1}(e_{r-1}) \sqcup \varphi^{-1}(e_r)\big) = [e_r, \cdot[_E$ but $\psi\big(\varphi^{-1}(e_r)\big) \neq [e_r, \cdot[_E$. Hence $\psi\big(\varphi^{-1}(e_r)\big) \subseteq ]e_r, \cdot[_E \subseteq [b, \cdot[_T \cap E$, since $s(e_r) \geq b$. Moreover $e_r \in \psi\big(\varphi^{-1}(e_{r-1}) \sqcup \varphi^{-1}(e_r)\big)$ and $e_r \notin \psi\big(\varphi^{-1}(e_r)\big)$. It follows that $e_r \in \psi\big(\varphi^{-1}(e_{r-1})\big)$, hence $\psi\big(\varphi^{-1}(e_{r-1})\big) = [e_r, \cdot[_E$, and $\varphi_{r+1} \underset{E,R}{\vdash} \psi$. Therefore only $\varphi_r$ and $\varphi_{r+1}$ are involved in this case.

Finally, if $\varphi_{r+1} \underset{E,R}{\vdash} \psi$, then $\psi\big(\varphi^{-1}(e_{r-1})\big) = [e_r, \cdot[_E$ and $\psi\big(\varphi^{-1}(e_r)\big) \subseteq [b, \cdot[_T \cap E \subseteq [e_r, \cdot[_E$. Thus $\psi\big(\varphi^{-1}(e_{r-1}) \sqcup \varphi^{-1}(e_r)\big) = [e_r, \cdot[_E$, and $\varphi_r \underset{E,R}{\vdash} \psi$. Therefore we are again in the case when only $\varphi_r$ and $\varphi_{r+1}$ are involved.

Of course, if none of $\varphi, \varphi_1, \varphi_2, \dots, \varphi_{r+1}$ is linked with $\psi$, then the corresponding equation $(E_\psi)$ is just $0 = 0$. It follows that $\varphi - u_a \circ \varphi$ satisfies the equation $(E_\psi)$ for every $\psi$, as was to be shown. ∎

We have now paved the way for finding generators of $\mathbb{S}_{E,R^{op}}(X)$.

**17.7. Theorem.** *Let $T$ be a finite lattice, let $(E,R)$ be the full subposet of its irreducible elements, and let $\Gamma = \{a \in T \mid a \notin E,\ a < r^\infty s^\infty(a)\}$. For $a \in \Gamma$, let $u_a$ be the element of $k(T^T)$ introduced in Notation 17.4, and let $u_T$ denote the composition of all the elements $u_a$, for $a \in \Gamma$, in any order (they actually commute, see Theorem 17.9 below).*

(a) *Let $X$ be a finite set. Then for any $\varphi : X \to T$, the element $u_T \circ \varphi$ is a $k$-linear combination of functions $f : X \to T$ such that $f(X) \subseteq G$.*
(b) *Let $\mathcal{B}_X$ be the set of all maps $\varphi : X \to T$ such that $E \subseteq \varphi(X) \subseteq G$. Then the set $\Theta_{T,X}(\mathcal{B}_X)$ generates $\mathbb{S}_{E,R^{op}}(X)$ as a $k$-module.*

**Proof :** (a) It follows from Table 17.6 that for any $a \in \Gamma$ and any $\varphi : X \to T$, the element $u_a \circ \varphi = \varphi_1 - \varphi_2 + \ldots + (-1)^r \varphi_{r+1}$ is a $k$-linear combination of functions $\varphi_i$ such that $\varphi_i(X) \cap \Gamma \subseteq \big(\varphi(X) \cap \Gamma\big) - \{a\}$. We now remove successively all such elements $a$ by applying successively all $u_a$ for $a \in \Gamma$. It follows that $u_T \circ \varphi$ is a $k$-linear combination of functions $f : X \to T$ such that $f(X) \cap \Gamma = \emptyset$, that is, $f(X) \subseteq G$.

(b) Since $\Theta_{T,X} : F_T(X) \to \mathbb{S}_{E,R^{op}}(X)$ is surjective, $\mathbb{S}_{E,R^{op}}(X)$ is generated as a $k$-module by the images of all maps $\varphi : X \to T$. For any $a \in \Gamma$, $u_a \circ \varphi$ has the same image as $\varphi$ under $\Theta_{T,X}$, by Theorem 17.5. Therefore $u_T \circ \varphi$ has the same image as $\varphi$ under $\Theta_{T,X}$. Moreover $u_T \circ \varphi$ is a $k$-linear combination of functions $f : X \to T$ such that $f(X) \subseteq G$, by (a). Finally, if $E \nsubseteq f(X)$, then $f \in H_T(X) \subseteq \operatorname{Ker} \Theta_{T,X}$. So we are left with all maps $\varphi : X \to T$ such that $E \subseteq \varphi(X) \subseteq G$. □

We now mention that much more can be said about the elements $u_a$ appearing in Theorem 17.7.

**17.8. Definition.** *Let $T$ be a finite lattice. Recall that $\Gamma$ denotes the complement of $G$ in $T$. We define an oriented graph structure $\mathcal{G}(T)$ on $T$ in the following way: for $x, y \in T$, there is an edge $x \rightarrowtail y$ from $x$ to $y$ in $\mathcal{G}(T)$ if there exists $a \in \Gamma$ such that $(x, y)$ is a pair of consecutive elements in the reduction sequence associated to $a$.*

**17.9. Theorem.** *Let $T$ be a finite lattice, let $(E,R)$ the full subposet of its irreducible elements, let $\Gamma = \{a \in T \mid a \notin E,\ a < r^\infty s^\infty(a)\}$, and let $\mathcal{G}(T)$ be the graph structure on $T$ introduced in Definition 17.8. For $a \in \Gamma$, let $u_a$ be the element of $k(T^T)$ introduced in Notation 17.4, and let $u_T$ denote the composition of all the elements $u_a$, for $a \in \Gamma$.*

(a) *The graph $\mathcal{G}(T)$ has no (oriented or unoriented) cycles, and each vertex has at most one outgoing edge. Hence it is a forest.*
(b) *For $a \in \Gamma$, the element $u_a$ is an idempotent of $k(T^T)$.*
(c) *$u_a \circ u_b = u_b \circ u_a$ for any $a, b \in \Gamma$.*
(d) *The element $u_T$ is an idempotent of $k(T^T)$.*

Theorem 17.9 has apparently no direct implication on the structure of correspondence functors. Therefore, it will be proved in an appendix (Section 22), as a special case of a general results on forests.

We now move to our second part.

**17.10. Theorem.** *Let $T$ be a finite lattice, let $(E,R)$ be the full subposet of its irreducible elements, and let $X$ be a finite set. The set $\Theta_{T,X}(\mathcal{B}_X)$ is a $k$-linearly independent subset of $\mathbb{S}_{E,R^{op}}(X)$.*

**Proof :** We consider again the kernel of the map $\Theta_{T,X} : F_T(X) \to \mathbb{S}_{E,R^{op}}(X)$, which was described in Theorem 13.4 by a system of linear equations. This can be reformulated by introducing the $k$-linear map

$$\begin{aligned} \eta_{E,R,X} : F_T(X) &\longrightarrow F_{I^\uparrow(E,R)}(X) \\ \varphi &\longmapsto \sum_{\substack{\psi: X\to I^\uparrow(E,R)\\ \varphi \underset{E,R}{\vdash} \psi}} \psi \end{aligned}$$

where the notation $\varphi \underset{E,R}{\vdash} \psi$ means, as before, that $\varphi : X \to T$ and $\psi : X \to I^\uparrow(E,R)$ satisfy the equivalent conditions of Lemma 13.9. Theorem 13.4 asserts that

$$\mathrm{Ker}(\Theta_{T,X}) = \mathrm{Ker}(\eta_{E,R,X}) \ .$$

For handling the condition $\varphi \underset{E,R}{\vdash} \psi$, we shall use part (e) of Lemma 13.9, namely

(**17.11**)

$$\varphi \underset{E,R}{\vdash} \psi \iff \begin{cases} \varphi \leq \wedge\psi \ , \\ \forall e \in E,\ \exists x \in X \ \text{ such that } \ \varphi(x) = e \ \text{ and } \ \psi(x) = [e, \cdot[_E \ . \end{cases}$$

Let $N = N_{E,R,X}$ be the matrix of $\eta_{E,R,X}$ with respect to the standard basis of $F_T(X)$, consisting of maps $\varphi : X \to T$, and the standard basis of $F_{I^\uparrow(E,R)}(X)$, consisting of maps $\psi : X \to I^\uparrow(E,R)$. Explicitly,

(**17.12**)

$$N_{\psi,\varphi} = \begin{cases} 1 & \text{if } \varphi \underset{E,R}{\vdash} \psi \ , \\ 0 & \text{otherwise.} \end{cases}$$

Note that $N$ is a square matrix in the special case when $T = I_\downarrow(E,R)$, because complementation yields a bijection between $I_\downarrow(E,R)$ and $I^\uparrow(E,R)$. However, if $T$ is a proper quotient of $I_\downarrow(E,R)$, then $N$ has less columns.

In order to prove that the elements $\Theta_{T,X}(\varphi)$, for $\varphi \in \mathcal{B}_X$, are $k$-linearly independent, we shall prove that the elements $\eta_{E,R,X}(\varphi)$, for $\varphi \in \mathcal{B}_X$, are $k$-linearly independent. In other words, we have to show that the columns of $N$ indexed by $\varphi \in \mathcal{B}_X$ are $k$-linearly independent. Now we consider only the lines indexed by elements of the form $\psi = \sigma \circ \varphi'$, where $\varphi' : X \to G$ is any function, and $\sigma : G \to I^\uparrow(E,R)$ is the map defined in (16.10). We then define the square matrix $M$, indexed by $\mathcal{B}_X \times \mathcal{B}_X$, by

$$\forall \varphi, \varphi' \in \mathcal{B}_X,\ \ M_{\varphi',\varphi} = N_{\sigma\circ\varphi',\varphi} \ .$$

We are going to prove that $M$ is invertible and this will prove the required linear independence.

The invertibility of $M$ implies in particular that the map $\sigma$ must be injective, otherwise two lines of $M$ would be equal. Therefore $M$ turns out to be a submatrix of $N$, but this cannot be seen directly from its definition (unless an independent proof of the injectivity of $\sigma$ is provided).

The characterization of the condition $\varphi \underset{E,R}{\vdash} \psi$ given in (17.11) implies that

$$M_{\varphi',\varphi} = \begin{cases} 1 & \text{if } \varphi \leq \wedge\sigma\varphi' \text{ and } \forall e \in E, \exists x \in X,\ \varphi(x) = e = \varphi'(x) \ , \\ 0 & \text{otherwise} \ , \end{cases}$$

because the equality $\sigma\varphi'(x) = [e, \cdot[_E$ is equivalent to $\varphi'(x) = e$, by definition of $\sigma$ (see Notation 16.10).

By Proposition 16.11, if $t, t' \in G$ are such that $t \leq \wedge\sigma(t')$, then $r^\infty(t) \leq r^\infty(t')$ and $s^\infty(t) \leq s^\infty(t')$. Let $\preceq$ be the preorder on $G$ defined by these conditions, i.e.

for all $t, t' \in G$,

$$t \preceq t' \iff r^\infty(t) \leq r^\infty(t') \text{ and } s^\infty(t) \leq s^\infty(t') \ .$$

We extend this preorder to $\mathcal{B}_X$ by setting, for all $\varphi', \varphi \in \mathcal{B}_X$,

$$\varphi \preceq \varphi' \iff \forall x \in X,\ \varphi(x) \preceq \varphi'(x) \ ,$$

which makes sense because $\varphi(x), \varphi'(x) \in G$ by definition of $\mathcal{B}_X$. We denote by $\preceq\succeq$ the equivalence relation defined by this preorder.

Clearly the condition $M_{\varphi',\varphi} \neq 0$ implies $\varphi \leq \wedge\sigma\varphi'$, hence $\varphi \preceq \varphi'$ by Proposition 16.11 quoted above. In other words the matrix $M$ is block triangular, the blocks being indexed by the equivalence classes of the preorder $\preceq$ on $\mathcal{B}_X$. Showing that $M$ is invertible is equivalent to showing that all its diagonal blocks are invertible. In other words, we must prove that, for each equivalence class $C$ of $\mathcal{B}_X$ for the relation $\preceq\succeq$, the matrix $M_C = (M_{\varphi',\varphi})_{\varphi',\varphi \in C}$ is invertible. Let $C$ be such a fixed equivalence class.

Recall that $G = \wedge E \sqcup G_1$, where $G_1$ is the set of bulbs. If $t \in G_1$, then $s^\infty(t) \in E$ by Lemma 16.5. By Lemma 16.2, all elements of $[\,t, s^\infty(t)\,]_T$ belong to $E$ except $t$ itself. Moreover, if $x \in [\,t, s^\infty(t)\,]_T$ then $r^\infty(x) = t$ by Lemma 16.5. It follows that the sets $G_t = [\,t, s^\infty(t)\,]_T$, for $t \in G_1$, are disjoint, and contained in $G$. Let

$$G_* = G - \bigsqcup_{t \in G_1} G_t \ ,$$

so that we get a partition

$$G = \bigsqcup_{t \in \{*\} \sqcup G_1} G_t \ .$$

**17.13. Lemma.** *Let $\varphi', \varphi \in \mathcal{B}_X$. If $\varphi' \preceq\succeq \varphi$, then for all $t \in \{*\} \sqcup G_1$,*

$$\varphi'^{-1}(G_t) = \varphi^{-1}(G_t) \ .$$

**Proof :** Let $t \in G_1$ and $x \in \varphi^{-1}\big([\,t, s^\infty(t)\,]_T\big)$. Then $\varphi(x) \in [\,t, s^\infty(t)\,]_T$, hence $r^\infty\varphi(x) = t$ and $s^\infty\varphi(x) = s^\infty(t)$, by Lemma 16.5. But the relation $\varphi' \preceq\succeq \varphi$ implies that $r^\infty\varphi'(x) = r^\infty\varphi(x)$ and $s^\infty\varphi'(x) = s^\infty\varphi(x)$. Therefore $r^\infty\varphi'(x) = t$ and $s^\infty\varphi'(x) = s^\infty(t)$, from which it follows that $\varphi'(x) \in [\,t, s^\infty(t)\,]_T$, that is, $x \in \varphi'^{-1}\big([\,t, s^\infty(t)\,]_T\big)$. This shows that $\varphi^{-1}(G_t) \subseteq \varphi'^{-1}(G_t)$. By exchanging the roles of $\varphi$ and $\varphi'$, we obtain $\varphi'^{-1}(G_t) = \varphi^{-1}(G_t)$.

Now $G_*$ is the complement of $\bigsqcup_{t \in G_1} G_t$ in $G$ and the functions $\varphi', \varphi$ have their values in $G$ (by definition of $\mathcal{B}_X$). So we must have also $\varphi'^{-1}(G_*) = \varphi^{-1}(G_*)$. □

For every $t \in \{*\} \sqcup G_1$, we define

$$X_t = \varphi_0^{-1}(G_t)$$

where $\varphi_0$ is an arbitrary element of $C$. It follows from Lemma 17.13 that this definition does not depend on the choice of $\varphi_0$. Therefore, the equivalence class $C$ yields a partition

$$X = \bigsqcup_{t \in \{*\} \sqcup G_1} X_t \ ,$$

and every function $\varphi \in C$ decomposes as the disjoint union of the functions $\varphi_t$, where $\varphi_t : X_t \to G_t$ is the restriction of $\varphi$ to $X_t$.

For $t \in G_1$, define

$$E_t = ]\,t, s^\infty(t)\,]_T \ .$$

By Lemma 16.2, this consists of elements of $E$, so $E_t = E \cap G_t$. Then we define $E_* = E - \bigsqcup_{t\in G_1} E_t = E \cap G_*$, so that we get a partition

$$E = \bigsqcup_{t\in\{*\}\sqcup G_1} E_t \ .$$

For every $t \in \{*\} \sqcup G_1$ and for every $\varphi \in C$, the function $\varphi_t$ satisfies the condition $E_t \subseteq \varphi_t(X_t) \subseteq G_t$, by definition of $\mathcal{B}_X$. Moreover, if $\varphi', \varphi \in C$, then

$$M_{\varphi',\varphi} = 1 \iff \forall t \in \{*\} \sqcup G_1, \begin{cases} \forall x \in X_t,\ \varphi'_t(x) \leq \wedge\sigma\varphi_t(x) \\ \forall e \in E_t,\ \exists x \in X_t,\ \varphi'(x) = e = \varphi(x) \ . \end{cases}$$

It follows that the matrix $M_C$ is a tensor product of square matrices $M_{C,t}$, for $t \in \{*\} \sqcup G_1$. The matrix $M_{C,t}$ is indexed by the functions $\varphi_t : X_t \to G_t$ whose image contains $E_t$ and satisfies

$$(M_{C,t})_{\varphi',\varphi} = 1 \iff \begin{cases} \forall x \in X_t,\ \varphi'_t(x) \leq \wedge\sigma\varphi_t(x) \\ \forall e \in E_t,\ \exists x \in X_t,\ \varphi'(x) = e = \varphi(x) \ . \end{cases}$$

In order to show that $M_C$ is invertible, we shall prove that each matrix $M_{C,t}$ is invertible.

If $x \in X_*$, then $\varphi'(x)$ is not a bulb, by construction of $X_*$. Therefore, the condition $\varphi'_*(x) \leq \wedge\sigma\varphi_*(x)$ implies $\varphi'_*(x) \leq \varphi_*(x)$ by Proposition 16.11. It follows that the matrix $M_{C,*}$ is unitriangular, hence invertible, as required.

We are left with the matrices $M_{C,t}$ for $t \in G_1$ and we now discuss the special role played by the set $G_1$ of all bulbs. If $t$ is a bulb, then $G_t = [t, s^\infty(t)]_T$ is isomorphic to the totally ordered lattice $\underline{n}$, for some $n \geq 0$, and the set of irreducible elements $E_t = ]t, s^\infty(t)]_T$ is isomorphic to $[n] = \{1, \ldots, n\}$ (or $\emptyset$ if $n = 0$). Composing the maps $\varphi_t : X_t \to G_t$ with this isomorphism, we obtain maps $X_t \to \underline{n}$. Changing notation for simplicity, we write $X$ for $X_t$ and $\varphi$ for $\varphi_t$ and we consider all such maps $\varphi : X \to \underline{n}$ lying in $\mathcal{B}_X$, i.e. satisfying the condition $[n] \subseteq \varphi(X) \subseteq \underline{n}$. The matrix $M_{C,t}$, which we write $M$ for simplicity, is now indexed by all such maps and we have

$$M_{\varphi',\varphi} = 1 \iff \begin{cases} \forall x \in X,\ \varphi'(x) \leq \wedge\sigma\varphi(x) \\ \forall e \in [n],\ \exists x \in X,\ \varphi'(x) = e = \varphi(x) \ . \end{cases}$$

(Actually, the map $\sigma : \underline{n} \to I^\uparrow([n], \mathrm{tot})$, defined in (16.10), is easily seen to be a bijection (mapping 0 to $\emptyset$) and the matrix $M_{C,t}$ is equal to a square submatrix of the matrix $N_{[n],\mathrm{tot},X}$, analogous to the matrix $N_{E,R,X}$ of 17.12. Hence we are actually dealing with a reduction to the case of a total order.)

For every $a \in \underline{n}$ we have $r^\infty(a) = 0$ and $s^\infty(a) = n$. Thus for $\varphi \in \mathcal{B}_X$ and $x \in X$, we have

$$\wedge\varphi(x) = \begin{cases} \varphi(x) & \text{if } \varphi(x) \neq 0 \ , \\ n & \text{if } \varphi(x) = 0 \ . \end{cases}$$

Hence for $\varphi, \varphi' \in \mathcal{B}_X$, we have $\varphi' \leq \wedge\sigma\varphi$ if and only if $\varphi'(x) \leq \varphi(x)$ for any $x \in X - \varphi^{-1}(0)$.

Now the set $\mathcal{B}_X$ consists of those functions $\varphi : X \to \underline{n}$ such that $\varphi(X) \supseteq [n]$. For such a map $\varphi$, the sets $A_i = \varphi^{-1}(i)$, for $i \in [n]$, are nonempty disjoint subsets of $X$. Conversely, if $A = (A_1, \ldots, A_n)$ is a sequence of nonempty disjoint subsets of $X$, setting $A_0 = X - \bigsqcup_{i=1}^{n} A_i$, there is a unique map $\varphi_A : X \to \underline{n}$ defined for $x \in X$ by $\varphi_A(x) = i$ if $x \in A_i$. This gives a bijective correspondence

$$\mathcal{P}_n(X) \longrightarrow \mathcal{B}_X \ , \qquad A \mapsto \varphi_A$$

where $\mathcal{P}_n(X)$ denotes the set of all sequences $(A_1, \ldots, A_n)$ of nonempty disjoint subsets of $X$.

For $A, A' \in \mathcal{P}_n(X)$, we have $M_{\varphi_{A'},\varphi_A} = 1$ if and only if the following two conditions hold:

(1) If $x \in X$ is such that $\varphi'(x) = i$ and $\varphi(x) \neq 0$, then $\varphi(x) \geq i$. In other words $A'_i \subseteq A_0 \sqcup A_i \sqcup A_{i+1} \sqcup \ldots \sqcup A_n$, for any $i \in [n]$. Equivalently $A'_i \cap A_j = \emptyset$ for $1 \leq j < i \leq n$.

(2) For any $i \in [n]$, there exists $x \in X$ such that $\varphi_A(x) = i = \varphi_{A'}(x)$. In otherwords $A_i \cap A'_i \neq \emptyset$ for any $i \in [n]$.

We can now view $M$ as a square matrix indexed by $\mathcal{P}_n(X)$. For all $A, A' \in \mathcal{P}_n(X)$, we have

$$M_{A',A} = 1 \iff \begin{cases} \forall 1 \leq j < i \leq n,\ A'_i \cap A_j = \emptyset\ , \\ \forall i \in [n],\ A'_i \cap A_i \neq \emptyset\ . \end{cases}$$

In order to show that the matrix $M$ is invertible, we introduce a square matrix $U$ indexed by $\mathcal{P}_n(X)$, defined as follows. For all $A, B \in \mathcal{P}_n(X)$,

$$U_{A,B} \quad = \quad \begin{cases} (-1)^{\sum\limits_{i=1}^{n} |A_i - B_i|} & \text{if } \forall i \in [n],\ A_i \supseteq B_i\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

We now compute the product $V = U \cdot M \cdot {}^tU$. For $A, A' \in \mathcal{P}_n(X)$, we have

$$V_{A',A} = \sum_{B'} \sum_{B} U_{A',B'}\, M_{B',B}\, U_{A,B} = \sum_{(B',B)} (-1)^{\sum\limits_{i=1}^{n} |A'_i - B'_i| + |A_i - B_i|}\ ,$$

where $(B', B)$ runs through the set of pairs of elements of $\mathcal{P}_n(X)$ such that

$$\begin{gathered} \forall 1 \leq i \leq n\ ,\ \ B'_i \subseteq A'_i \ \text{ and } \ B_i \subseteq A_i\ , \\ \forall 1 \leq j < i \leq n\ ,\ \ B'_i \cap B_j = \emptyset\ , \\ \forall 1 \leq i \leq n\ ,\ \ B_i \cap B'_i \neq \emptyset\ . \end{gathered}$$

We compute the above sum giving $V_{A',A}$ by first choosing $B \in \mathcal{P}_n(X)$ such that $B_i \subseteq A_i$ for all $i \in [n]$, and then computing

$$(-1)^{\sum\limits_{i=1}^{n} (|A_i - B_i| + |A'_i|)} \sum_{B'} (-1)^{\sum\limits_{i=1}^{n} |B'_i|}\ ,$$

where $B'$ runs through the set of elements of $\mathcal{P}_n(X)$ such that

$$\textbf{(17.14)} \qquad \begin{cases} \forall 1 \leq i \leq n\ , & B'_i \subseteq A'_i\ , \\ \forall 1 \leq j < i \leq n\ , & B'_i \cap B_j = \emptyset\ , \\ \forall 1 \leq i \leq n\ , & B'_i \cap B_i \neq \emptyset\ . \end{cases}$$

For $i \in [n]$, set $D_i = A'_i - \bigsqcup\limits_{j=1}^{i-1} B_j$ and $E_i = A'_i \cap B_i$, and observe that $E_i \subseteq D_i$. Conditions 17.14 hold if and only if, for every $i \in [n]$,

$$\textbf{(17.15)} \qquad B'_i \subseteq D_i \ \text{ and } \ B'_i \cap E_i \neq \emptyset\ ,$$

and we have to compute

$$\sum_{\substack{B' \in \mathcal{P}_n(X) \\ \text{s.t.}(17.14)}} (-1)^{\sum\limits_{i=1}^{n} |B'_i|} = \prod_{i=1}^{n} \Big( \sum_{\substack{B'_i \subseteq X \\ \text{s.t.}(17.15)}} (-1)^{|B'_i|} \Big)\ .$$

By Condition (17.15), $B'_i = E'_i \sqcup D'_i$, where $E'_i$ is a nonempty subset of $E_i$ and $D'_i$ is a subset of $D_i - E_i$, and we have to sum over all such $E'_i$ and $D'_i$. For a given $i \in [n]$,

$$\sum_{\substack{B'_i \subseteq X \\ \text{s.t.}(17.15)}} (-1)^{|B'_i|} = \sum_{\emptyset \neq E'_i \subseteq E_i} \sum_{D'_i \subseteq D_i - E_i} (-1)^{|E'_i| + |D'_i|} \ ,$$

and the inner sum is zero unless $D_i - E_i = \emptyset$, or equivalently $A'_i - \bigsqcup_{j=1}^{i-1} B_j \subseteq B_i$. In this case $E_i = D_i = A'_i - \bigsqcup_{j=1}^{i-1} B_j$ and $D'_i = \emptyset$ is the only choice. So we are left with the sum

$$\sum_{\emptyset \neq E'_i \subseteq D_i} (-1)^{|E'_i|} \ ,$$

which is equal to $-1$ if $D_i \neq \emptyset$, and to 0 if $D_i = \emptyset$.

It follows that $\sum_{B'} (-1)^{\sum_{i=1}^{n} |B'_i|}$ is equal to 0, unless $B$ satisfies

$$\textbf{(17.16)} \qquad \forall i \in [n], \ \ \emptyset \neq A'_i - \bigsqcup_{j=1}^{i-1} B_j \subseteq B_i \subseteq A_i \ ,$$

in which case we obtain

$$\sum_{B'} (-1)^{\sum_{i=1}^{n} |B'_i|} = \prod_{i=1}^{n} \Big( \sum_{B'_i} (-1)^{|B'_i|} \Big) = (-1)^n \ .$$

Condition 17.16 imply that $A'_i \subseteq \bigsqcup_{j=1}^{i} B_j \subseteq \bigsqcup_{j=1}^{i} A_j$, for any $i \in [n]$, or equivalently $\bigsqcup_{j=1}^{i} A'_j \subseteq \bigsqcup_{j=1}^{i} A_j$, for every $i \in [n]$. Let $\preceq$ be the partial order relation on $\mathcal{P}_n(X)$ defined, for all $A', A \in \mathcal{P}_n(X)$, by the requirement

$$A' \preceq A \iff \forall i \in [n], \ \ \bigsqcup_{j=1}^{i} A'_j \subseteq \bigsqcup_{j=1}^{i} A_j \ .$$

The previous discussion shows that $V_{A',A} = 0$ unless $A' \preceq A$. Moreover if $A' = A$, we obtain

$$\begin{aligned} V_{A,A} &= \sum_{\substack{B \in \mathcal{P}_n(X) \\ B_i \subseteq A_i \ \forall i}} (-1)^{\sum_{i=1}^{n} (|A_i - B_i| + |A_i|)} \sum_{\substack{B' \in \mathcal{P}_n(X) \\ \text{s.t.}(17.14)}} (-1)^{\sum_{i=1}^{n} |B'_i|} \\ &= \sum_{\substack{B \in \mathcal{P}_n(X) \\ \text{s.t.}(17.16)}} (-1)^{\sum_{i=1}^{n} |B_i|} (-1)^n \\ &= (-1)^{n + \sum_{i=1}^{n} |A_i|} \ , \end{aligned}$$

because, when $A' = A$, Condition 17.16 is equivalent to $B_i = A_i$ for every $i \in [n]$, so the sum over $B$ reduces to a single term.

It follows now that for a total ordering of $\mathcal{P}_n(X)$ finer than $\preceq$, the matrix $V = U \cdot M \cdot {}^tU$ is triangular with diagonal coefficients equal to $\pm 1$, hence invertible. Moreover $U_{A,B} \neq 0$ implies that $A_i \supseteq B_i$ for all $i \in [n]$, and this is also a partial order on $\mathcal{P}_n(X)$. As $U_{A,A} = 1$ for any $A \in \mathcal{P}_n(X)$, the matrix $U$ is unitriangular for a suitable total order on $\mathcal{P}_n(X)$, hence it is also invertible. Therefore the matrix

$$M = U^{-1} \cdot V \cdot {}^tU^{-1}$$

is invertible, as was to be shown. This completes the proof of Theorem 17.10. ∎

We can finally prove one of our main results about fundamental correspondence functors, which generalizes the formula obtained in Theorem 15.14 in the case of a total order.

**17.17. Corollary.** *Let $(E,R)$ be a finite poset and let $T$ be any lattice such that $(E,R)$ is the full subposet of irreducible elements in $T$.*

*(a) $\Theta_{T,X}(\mathcal{B}_X)$ is a $k$-basis of $\mathbb{S}_{E,R^{op}}(X)$.*
*(b) The $k$-module $\mathbb{S}_{E,R^{op}}(X)$ is free of rank*

$$\mathrm{rk}_k\big(\mathbb{S}_{E,R^{op}}(X)\big) = \sum_{i=0}^{|E|}(-1)^i\binom{|E|}{i}(|G|-i)^{|X|} \; .$$

*In particular, $|G|$ only depends on $(E,R)$, and not on the choice of $T$.*

**Proof :** (a) follows from Theorem 17.7 and Theorem 17.10.

(b) The formula follows immediately from (a) and Lemma 8.1. Moreover, it implies that

$$\mathrm{rk}_k\big(\mathbb{S}_{E,R^{op}}(X)\big) \sim |G|^{|X|} \qquad \text{as } |X| \to \infty \; .$$

Since $\mathbb{S}_{E,R^{op}}$ only depends on $(E,R)$, it follows that $|G|$ only depends on $(E,R)$. ∎

We shall now determine the dimension of any evaluation of a simple correspondence functor. We first need a lemma.

**17.18. Lemma.** *Let $(E,R)$ be a poset.*

*(a) For any lattice $T$ such that $\mathrm{Irr}(T) \cong (E,R)$, restriction induces an injective group homomorphism $\mathrm{Aut}(T) \to \mathrm{Aut}(E,R)$.*
*(b) There exists a lattice $T$ such that $\mathrm{Irr}(T) \cong (E,R)$ and such that the restriction homomorphism $\mathrm{Aut}(T) \to \mathrm{Aut}(E,R)$ is an isomorphism.*

**Proof :** (a) For any lattice $T$ such that $\mathrm{Irr}(T) \cong (E,R)$, any lattice automorphism of $T$ induces an automorphism of the poset $(E,R)$. This gives a group homomorphism $\mathrm{Aut}(T) \to \mathrm{Aut}(E,R)$, which is injective since any element $t$ of $T$ is equal to the join of the irreducible elements $e \leq_T t$.

(b) Requiring that $\mathrm{Aut}(T) \cong \mathrm{Aut}(E,R)$ amounts to requiring that any automorphism of $(E,R)$ can be extended to an automorphism of $T$. This is clearly possible if we choose $T = I_\downarrow(E,R)$. ∎

We can finally prove one of our main results about simple correspondence functors.

**17.19. Theorem.** *Let $k$ be a field. Let $(E,R)$ be a poset and let $V$ be a simple left $k\,\mathrm{Aut}(E,R)$-module. Let $T$ be a lattice such that $\mathrm{Irr}(T) \cong (E,R)$ and such that the restriction homomorphism $\mathrm{Aut}(T) \to \mathrm{Aut}(E,R)$ is an isomorphism. Let $G = E \sqcup \{t \in T \mid t = r^\infty s^\infty(t)\} \subseteq T$. Then for any finite set $X$, the dimension of $S_{E,R,V}(X)$ is given by*

$$\dim_k S_{E,R,V}(X) = \frac{\dim_k V}{|\mathrm{Aut}(E,R)|}\sum_{i=0}^{|E|}(-1)^i\binom{|E|}{i}(|G|-i)^{|X|}$$

**Proof :** Observe first that for any poset $(E,R)$, the subgroups $\mathrm{Aut}(E,R)$ and $\mathrm{Aut}(E,R^{op})$ of the group of permutations of $E$ are actually equal. For any finite set $X$, let $\mathcal{B}_X$ be the set of all functions $\varphi : X \to T$ such that $E \subseteq \varphi(X) \subseteq G$. By

Corollary 17.17, the image of $\mathcal{B}_X$ under $\Theta_{T,X} : F_T(X) \to \mathbb{S}_{E,R^{op}}(X)$ is a basis of the $k$-module $\mathbb{S}_{E,R^{op}}(X)$. The group $\mathrm{Aut}(T) \cong \mathrm{Aut}(E,R) = \mathrm{Aut}(E,R^{op})$ acts on the right on $\mathcal{B}_X$ via $\varphi \cdot \alpha = \alpha^{-1} \circ \varphi$, for $\varphi \in \mathcal{B}_X$ and $\alpha \in \mathrm{Aut}(E,R)$. This action is moreover *free*: indeed, if $\varphi \cdot \alpha = \varphi$, then $\alpha^{-1}\big(\varphi(x)\big) = \varphi(x)$ for any $x \in X$, thus $\alpha(e) = e$ for any $e \in E$, since $E \subseteq \varphi(X)$, hence $\alpha = \mathrm{id}$.

It follows that $\mathbb{S}_{E,R^{op}}(X)$ is isomorphic to a direct sum of copies of the free right module $k\,\mathrm{Aut}(E,R)$. Applying this to $R^{op}$ instead of $R$, we obtain that $\mathbb{S}_{E,R}(X)$ is isomorphic to the direct sum of $n_X$ copies of the free right module $k\,\mathrm{Aut}(E,R)$, for some $n_X \in \mathbb{N}$. In particular

$$\dim_k \mathbb{S}_{E,R}(X) = n_X \,|\,\mathrm{Aut}(E,R)|\ .$$

Moreover, by Remark 4.8, the simple functor $S_{E,R,V}$ satisfies

$$S_{E,R,V} \cong \mathbb{S}_{E,R} \otimes_{k\,\mathrm{Aut}(E,R)} V\ ,$$

so that

$$S_{E,R,V}(X) = \mathbb{S}_{E,R}(X) \otimes_{k\,\mathrm{Aut}(E,R)} V \cong n_X\big(k\,\mathrm{Aut}(E,R)\big) \otimes_{k\,\mathrm{Aut}(E,R)} V \cong n_X V\ .$$

Hence $\dim_k \mathbb{S}_{E,R,V}(X) = n_X \dim_k V$. Therefore

$$\dim_k \mathbb{S}_{E,R,V}(X) = \frac{\dim_k V}{|\,\mathrm{Aut}(E,R)|} \dim_k \mathbb{S}_{E,R}(X)\ ,$$

and Theorem 17.19 now follows from Corollary 17.17. ∎

## 18. Properties of fundamental functors

By using the results of the previous section, we can obtain more information on fundamental functors, in particular about the dual of a fundamental functor, homomorphisms between fundamental functors, and the description of the action of correspondences on a fundamental functor.

At the beginning of the proof of Theorem 17.10, we considered the $k$-linear map

$$\begin{aligned} \eta_{E,R,X} : F_T(X) &\longrightarrow F_{I^\uparrow(E,R)}(X) \\ \varphi &\longmapsto \sum_{\substack{\psi : X \to I^\uparrow(E,R) \\ \varphi \underset{E,R}{\vdash} \psi}} \psi \end{aligned}$$

where the notation $\varphi \underset{E,R}{\vdash} \psi$ means, as before, that $\varphi : X \to T$ and $\psi : X \to I^\uparrow(E,R)$ satisfy the equivalent conditions of Lemma 13.9. Since $\eta_{E,R,X}$ has the same kernel as the surjective map $\Theta_{T,X} : F_T(X) \to \mathbb{S}_{E,R^{op}}(X)$, we see that $\mathbb{S}_{E,R^{op}}(X)$ is isomorphic, as a $k$-module, to the image of $\eta_{E,R,X}$.

**18.1. Proposition.** *Let $N$ be the matrix of the $k$-linear map*

$$\eta_{E,R,X} : F_T(X) \longrightarrow F_{I^\uparrow(E,R)}(X)$$

*with respect to the canonical bases (see 17.12).*

(a) *Let $k = \mathbb{Z}$ and let*

$$d = \mathrm{rk}_k\big(\mathbb{S}_{E,R^{op}}(X)\big) = \sum_{i=0}^{|E|} (-1)^i \binom{|E|}{i} (|G| - i)^{|X|}\ .$$

*Then the first $d$ elementary divisors of the matrix $N$ are equal to 1, and the next ones are equal to 0.*

(b) *The image of the $k$-linear map $\eta_{E,R,X}$ is a direct summand of $F_{I^{\uparrow}(E,R)}(X)$ as a $k$-module.*

**Proof :** (a) For any commutative ring $k$, let $I_k$ be the image of $\eta_{E,R,X}$. It is isomorphic to $\mathbb{S}_{E,R}(X)$, hence it is a free $k$-module of rank $d$, independently of $k$, by Corollary 17.17. Now for $k = \mathbb{Z}$, we have two exact sequences of abelian groups

$$0 \longrightarrow K_{\mathbb{Z}} \longrightarrow F_T(X) \xrightarrow{\eta_{E,R,X}} I_{\mathbb{Z}} \longrightarrow 0$$

$$0 \longrightarrow I_{\mathbb{Z}} \longrightarrow F_{I^{\uparrow}(E,R)}(X) \longrightarrow C_{\mathbb{Z}} \longrightarrow 0\ ,$$

where $K_{\mathbb{Z}}$ and $C_{\mathbb{Z}}$ are the kernel and cokernel of $\eta_{E,R,X}$, respectively. The first exact sequence is actually split, because $I_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module (either because it is a subgroup of the finitely generated free $\mathbb{Z}$-module $F_{I^{\uparrow}(E,R)}(X)$, or by Corollary 17.17). Hence tensoring these exact sequences with $k$ gives the exact sequences

$$0 \longrightarrow k \otimes_{\mathbb{Z}} K_{\mathbb{Z}} \longrightarrow k \otimes_{\mathbb{Z}} F_T(X) \xrightarrow{k \otimes_{\mathbb{Z}} \eta_{E,R,X}} k \otimes_{\mathbb{Z}} I_{\mathbb{Z}} \longrightarrow 0$$

$$0 \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(k, C_{\mathbb{Z}}) \longrightarrow k \otimes_{\mathbb{Z}} I_{\mathbb{Z}} \longrightarrow k \otimes_{\mathbb{Z}} F_{I^{\uparrow}(E,R)}(X) \longrightarrow k \otimes_{\mathbb{Z}} C_{\mathbb{Z}} \longrightarrow 0\ .$$

The image of the middle map

$$k \otimes_{\mathbb{Z}} I_{\mathbb{Z}} \longrightarrow k \otimes_{\mathbb{Z}} F_{I^{\uparrow}(E,R)}(X) = k\big(I^{\uparrow}(E,R)^X\big)$$

is precisely $I_k$. It follows that $I_k \cong \big(k \otimes_{\mathbb{Z}} I_{\mathbb{Z}}\big)/\operatorname{Tor}_1^{\mathbb{Z}}(k, C_{\mathbb{Z}})$, i.e. we have the following exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(k, C_{\mathbb{Z}}) \longrightarrow k \otimes_{\mathbb{Z}} I_{\mathbb{Z}} \longrightarrow I_k \longrightarrow 0\ .$$

But $k \otimes_{\mathbb{Z}} I_{\mathbb{Z}} \cong I_k$ as $k$-modules, because both are free $k$-modules of rank $d$. Therefore, if $k$ is a field, then $\operatorname{Tor}_1^{\mathbb{Z}}(k, C_{\mathbb{Z}}) = 0$. Taking $k = \mathbb{F}_p$ for some prime number $p$, it follows that $\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{F}_p, C_{\mathbb{Z}}) = 0$, and since $p$ is an arbitrary prime, the group $C_{\mathbb{Z}}$ is torsion free, hence it is a free $\mathbb{Z}$-module. Thus $I_{\mathbb{Z}}$ is a pure submodule of $F_{I^{\uparrow}(E,R)}(X)$, that is, the nonzero elementary divisors of the matrix $N$ are all equal to 1.

(b) The proof of part (a) shows that

$$0 \longrightarrow I_{\mathbb{Z}} \longrightarrow F_{I^{\uparrow}(E,R)}(X) \longrightarrow C_{\mathbb{Z}} \longrightarrow 0$$

is an exact sequence of free $\mathbb{Z}$-modules, hence split, and that $k \otimes_{\mathbb{Z}} C_{\mathbb{Z}}$ is isomorphic to the cokernel of the $k$-linear map $\eta_{E,R,X}$. Therefore this cokernel is a free $k$-module and so the image $I_k$ is a direct summand of $F_{I^{\uparrow}(E,R)}(X)$ as a $k$-module. □

We can now settle the question left open in Theorem 14.17 concerning the structure of the dual of a fundamental functor.

**18.2. Theorem.** *Let $(E,R)$ be a poset and let $T = I_{\downarrow}(E,R)$. Consider the subfunctor $\langle \gamma_T \rangle$ of $F_{T^{op}}$ generated by $\gamma_T$ (isomorphic to $\mathbb{S}_{E,R}$ by Theorem 14.16).*

(a) *For every finite set $X$, the evaluation $\langle \gamma_T \rangle(X)$ is a direct summand of $F_{T^{op}}(X)$ as a $k$-module.*

(b) *The injective morphism $\alpha_{E,R} : \mathbb{S}_{E,R} \longrightarrow \mathbb{S}^{\natural}_{E,R^{op}}$ (see Theorem 14.17) is an isomorphism. Thus $\mathbb{S}^{\natural}_{E,R} \cong \mathbb{S}_{E,R^{op}}$.*

**Proof :** (a) By Example 12.4, $F_{I_\downarrow(E,R^{op})} \cong k\mathcal{C}(-,E)R$. The proof of Theorem 14.16 shows that there is a surjective morphism

$$\overline{\xi} : F_{I_\downarrow(E,R^{op})} \cong k\mathcal{C}(-,E)R \longrightarrow \langle\gamma_T\rangle \subseteq F_{T^{op}} \cong F_T^\natural ,$$

where $T = I_\downarrow(E,R)$. Moreover $\overline{\xi}$ induces the isomorphism $\mathbb{S}_{E,R} \cong \langle\gamma_T\rangle$. Consider the canonical $k$-basis $\{\psi : X \to I_\downarrow(E,R^{op})\}$ of $F_{I_\downarrow(E,R^{op})}(X)$ and the $k$-basis

$$\{\varphi^\star \mid \varphi : X \to I_\downarrow(E,R)\}$$

of $F_{I_\downarrow(E,R)^{op}}(X)$, where $\varphi^\star$ is defined as in Notation 14.8. The matrix $M$ of the $k$-linear map

$$\overline{\xi}_X : F_{I_\downarrow(E,R^{op})}(X) \longrightarrow F_{I_\downarrow(E,R)^{op}}(X)$$

with respect to these bases is easy to compute using the arguments of the proof of Theorem 14.16. It is given by

$$M_{\varphi^\star,\psi} = \begin{cases} 1 & \text{if } \Gamma_\psi^{op}\Gamma_\varphi = R^{op} , \\ 0 & \text{otherwise} , \end{cases} \qquad \varphi : X \to I_\downarrow(E,R) ,\ \psi : X \to I_\downarrow(E,R^{op}) .$$

Since the condition $\Gamma_\psi^{op}\Gamma_\varphi = R^{op}$ is the same as the condition $\varphi \underset{E,R}{\vdash} \psi$ (see (17.11) and Lemma 13.9) and since $I^\uparrow(E,R) = I_\downarrow(E,R^{op})$, we obtain

$$M_{\varphi^\star,\psi} = \begin{cases} 1 & \text{if } \varphi \underset{E,R}{\vdash} \psi , \\ 0 & \text{otherwise} , \end{cases} \qquad \varphi : X \to I_\downarrow(E,R) ,\ \psi : X \to I^\uparrow(E,R) .$$

But this is the transpose of the matrix $N$ of the $k$-linear map

$$\eta_{E,R,X} : F_T(X) \longrightarrow F_{I^\uparrow(E,R)}(X) .$$

By Proposition 18.1, the nonzero elementary divisors of $N$ are all equal to 1. Therefore, the same holds for $M$ (because this property does not change by transposition). Equivalently, the image of the map $\overline{\xi}_X : F_{I_\downarrow(E,R^{op})}(X) \longrightarrow F_{I_\downarrow(E,R)^{op}}(X)$ is a direct summand of $F_{I_\downarrow(E,R)^{op}}(X)$. In other words, $\langle\gamma_T\rangle(X)$ is a direct summand of $F_{T^{op}}(X)$ as a $k$-module.

(b) In the proof of Theorem 14.17, we obtained the injective morphism $\alpha_{E,R} : \mathbb{S}_{E,R} \longrightarrow \mathbb{S}_{E,R^{op}}^\natural$ from the inclusion $\langle\gamma_T\rangle \subseteq \langle\gamma_T\rangle^{\perp\perp}$. Now by part (a), $\langle\gamma_T\rangle(X)$ is a direct summand of $F_{T^{op}}(X)$ as a $k$-module for every finite set $X$. Therefore $\langle\gamma_T\rangle^{\perp\perp}(X) = \langle\gamma_T\rangle(X)$ (as both are free $k$-modules with the same rank), that is, the inclusion above is an equality. It follows that $\alpha_{E,R}$ is an isomorphism. □

Our next result on fundamental functors is similar to Schur's lemma.

**18.3. Theorem.** *Let $(E,R)$ and $(E',R')$ be two finite posets.*

*(a) If $(E,R) \ncong (E',R')$, then* $\mathrm{Hom}_{\mathcal{F}_k}(\mathbb{S}_{E,R},\mathbb{S}_{E',R'}) = \{0\}$.
*(b)* $\mathrm{End}_{\mathcal{F}_k}(\mathbb{S}_{E,R}) \cong k\,\mathrm{Aut}(E,R)$ *as $k$-algebras.*

**Proof :** (a) Let $\alpha : \mathbb{S}_{E,R} \to \mathbb{S}_{E',R'}$ be a nonzero morphism of functors. Since $\mathbb{S}_{E,R}$ is generated by its evaluation $\mathbb{S}_{E,R}(E)$ (by Theorem 13.1), $\alpha_E$ cannot be zero, hence $\mathbb{S}_{E',R'}(E) \neq \{0\}$. Since $E'$ is a minimal set for $\mathbb{S}_{E',R'}$ by Lemma 3.14, we have $|E'| \leq |E|$. Passing to the dual, we have a nonzero morphism $\alpha^\natural : \mathbb{S}_{E',R'}^\natural \to \mathbb{S}_{E,R}^\natural$, hence a nonzero morphism $\mathbb{S}_{E',R'^{op}} \to \mathbb{S}_{E,R^{op}}$ by Theorem 18.2. The same argument yields $|E| \leq |E'|$, hence $|E'| = |E|$.

We can now assume that $E' = E$. By evaluation at $E$, we have a nonzero map

$$\alpha_E : \mathbb{S}_{E,R}(E) = \mathcal{P}_E f_R \longrightarrow \mathbb{S}_{E,R'}(E) = \mathcal{P}_E f_{R'} ,$$

hence a nonzero element $\alpha_E(f_R) \in f_R\mathcal{P}_E f_{R'}$ (the image of $f_R$). Since $\mathcal{P}_E f_{R'}$ has $k$-basis $\{\Delta_\sigma f_{R'} \mid \sigma \in \Sigma_E\}$ by Proposition 3.10, we must have $f_R\Delta_\sigma f_{R'} \neq 0$

for some $\sigma \in \Sigma_E$. Therefore $f_R f_{^\sigma R'} = f_R \Delta_\sigma f_{R'} \Delta_\sigma^{-1} \neq 0$ and this implies that $R = {}^\sigma R'$ because otherwise we would have orthogonal idempotents. It follows that conjugation by $\sigma$ induces an isomorphism of posets $(E,R) \cong (E',R')$.

(b) By the argument of part (a), any $\alpha \in \mathrm{End}_{\mathcal{F}_k}(\mathbb{S}_{E,R})$ yields, by evaluation at $E$, an element of $\mathrm{End}_{\mathcal{P}_E}(\mathcal{P}_E f_R) \cong f_R \mathcal{P}_E f_R$. If $\sigma \in \Sigma_E - \mathrm{Aut}(E,R)$, then $f_R \neq f_{^\sigma R}$, hence $f_R \Delta_\sigma f_R = 0$ (orthogonal idempotents). If $\sigma \in \mathrm{Aut}(E,R)$, then $f_R \Delta_\sigma f_R = \Delta_\sigma f_R$. Therefore $f_R \mathcal{P}_E f_R$ has a $k$-basis $\{\Delta_\sigma f_R \mid \sigma \in \mathrm{Aut}(E,R)\}$. It follows that $\mathrm{End}_{\mathcal{P}_E}(\mathcal{P}_E f_R) \cong k\,\mathrm{Aut}(E,R)$, by mapping $a \in k\,\mathrm{Aut}(E,R)$ to left multiplication by $a$ in $\mathcal{P}_E f_R$. Therefore, evaluation at $E$ induces a $k$-algebra map

$$\mathrm{End}_{\mathcal{F}_k}(\mathbb{S}_{E,R}) \longrightarrow \mathrm{End}_{\mathcal{P}_E}(\mathcal{P}_E f_R) \cong k\,\mathrm{Aut}(E,R)\ .$$

This map is injective because $\mathbb{S}_{E,R}$ is generated by its evaluation at $E$ (Theorem 13.1).

In order to show that this map is surjective, we construct an inverse. Let $T = I_\downarrow(E,R^{op}) = I^\uparrow(E,R)$, so that $\mathrm{Irr}(T) = (E,R^{op})$. Any element $\tau \in \mathrm{Aut}(E,R)$ induces an automorphism of $T$, hence an automorphism of $F_T$, namely, for any finite set $X$,

$$F_\tau : F_T(X) \longrightarrow F_T(X)\,, \qquad \varphi \mapsto \tau\varphi\ .$$

The surjection $\Theta_T : F_T \to \mathbb{S}_{E,R}$ is given by $\Theta_{T,X}(\varphi) = \Gamma_\varphi\,\iota$, where $\Gamma_\varphi$ is defined as in Notation 5.7 and $\iota : E \to T$ is the inclusion.

We claim that $\Gamma_{\tau\varphi} = \Gamma_\varphi \Delta_{\tau^{-1}}$. Indeed

$$\begin{aligned}\Gamma_{\tau\varphi} &= \{(x,e) \mid e \leq_T \tau\varphi(x)\} = \{(x,e) \mid \tau^{-1}(e) \leq_T \varphi(x)\}\\ &= \{(x,f) \mid f \leq_T \varphi(x)\} \cdot \{(\tau^{-1}(e),e) \mid e \in E\} = \Gamma_\varphi \Delta_{\tau^{-1}}\ .\end{aligned}$$

By Theorem 13.4, $\sum\limits_{\varphi : X \to T} \lambda_\varphi \varphi$ belongs to $\mathrm{Ker}(\Theta_{T,X})$ if and only if

$$\forall \psi : X \to I_\downarrow(E,R)\,, \qquad \sum_{\substack{\varphi \\ \Gamma_\psi^{op}\Gamma_\varphi = R}} \lambda_\varphi = 0\ .$$

Since $\Delta_\tau R \Delta_{\tau^{-1}} = R$, we have equivalences

$$\begin{aligned}\Gamma_\psi^{op}\Gamma_\varphi = R \quad &\Longleftrightarrow \quad \Delta_\tau \Gamma_\psi^{op} \Gamma_\varphi \Delta_{\tau^{-1}} = R\\ &\Longleftrightarrow \quad \Gamma_{\tau\psi}^{op}\Gamma_{\tau\varphi} = R\,,\end{aligned}$$

by the claim above and the fact that $\Delta_\tau \Gamma_\psi^{op} = \Delta_{\tau^{-1}}^{op} \Gamma_\psi^{op} = (\Gamma_\psi \Delta_{\tau^{-1}})^{op} = \Gamma_{\tau\psi}^{op}$. It follows that $\sum\limits_{\varphi : X \to T} \lambda_\varphi \varphi$ satisfies the condition for a given $\psi : X \to I_\downarrow(E,R)$ if and only if $\sum\limits_{\varphi : X \to T} \lambda_\varphi \tau\varphi$ satisfies the condition for $\tau\psi$. Therefore $\mathrm{Ker}(\Theta_{T,X})$ is invariant under the automorphism $F_\tau$ and consequently $F_\tau$ induces an automorphism

$$\overline{F_\tau} : \mathbb{S}_{E,R}(X) \longrightarrow \mathbb{S}_{E,R}(X)\ .$$

Then the map $\tau \mapsto \overline{F_\tau}$ induces a $k$-algebra homomorphism

$$k\,\mathrm{Aut}(E,R) \longrightarrow \mathrm{End}_{\mathcal{F}_k}(\mathbb{S}_{E,R})\ .$$

It is straightforward to check that this provides the required inverse. ∎

Our next result provides a description of the action of correspondences on a fundamental functor.

**18.4. Theorem.** *Let $k$ be a commutative ring. Let $(E,R)$ be a poset, let $T$ be a lattice such that $\mathrm{Irr}(T)\cong(E,R)$, and let $\Theta_T:F_T\to\mathbb{S}_{E,R^{op}}$ be the canonical surjection of Theorem 13.1. Let $G=G(T)$ be the subset defined in Notation 16.7. For a finite set $X$, let $\mathbb{S}(X)$ denote the free $k$-submodule of $F_T(X)=k(T^X)$ with basis the set $\mathcal{B}_X$ of maps $\varphi:X\to T$ such that $E\subseteq\varphi(X)\subseteq G$. Let $u_T$ be the element of $k(T^T)$ introduced in Theorem 17.7. Let $\pi_{T,X}$ be the $k$-linear idempotent endomorphism of $k(T^X)$ defined by*

$$\forall\varphi:X\to T,\ \ \pi_{T,X}(\varphi)=\begin{cases}\varphi & \text{if}\ \varphi(X)\supseteq E\ ,\\ 0 & \text{otherwise}\ .\end{cases}$$

*(a)* $k(T^X)=\mathbb{S}(X)\oplus\mathrm{Ker}\,\Theta_{T,X}$.

*(b) Let $\varphi:X\to T$. Then:*

  *(i)* $\varphi-u_T\circ\varphi\in\mathrm{Ker}\,\Theta_{T,X}$.

  *(ii) if $\varphi(X)\not\supseteq E$, then $u_T\circ\varphi\in H_T(X)$.*

  *(iii) if $\varphi(X)\subseteq G$, then $\varphi-u_T\circ\varphi\in H_T(X)$. Moreover $\pi_{T,X}(\varphi)\in\mathbb{S}(X)$.*

  *(iv)* $\pi_{T,X}(u_T\circ\varphi)\in\mathbb{S}(X)$.

*(c)* $\mathrm{Ker}\,\Theta_{T,X}=\{\varphi\in F_T(X)\mid u_T\circ\varphi\in H_T(X)\}$.

*(d) The elements $\varphi-u_T\circ\varphi$, where $\varphi$ runs through the set of maps from $X$ to $T$ such that $E\subseteq\varphi(X)\nsubseteq G$ form a $k$-basis of a complement of $H_T(X)$ in* $\mathrm{Ker}\,\Theta_{T,X}$.

*(e) For a finite set $Y$, and for $U\subseteq Y\times X$, let $\mathbb{S}(U):\mathbb{S}(X)\to\mathbb{S}(Y)$ be the $k$-linear map defined by*

$$\forall\varphi:X\to T,\ E\subseteq\varphi(X)\subseteq G,\ \ \mathbb{S}(U)(\varphi)=\pi_{T,Y}(u_T\circ U\varphi)\ .$$

*With this definition, the assignment $X\mapsto\mathbb{S}(X)$ becomes a correspondence functor, isomorphic to the fundamental functor $\mathbb{S}_{E,R^{op}}$.*

**18.5. Remark.** In general, $\mathbb{S}$ is not a subfunctor of $F_T$, since the action of a correspondence $U$ on $\varphi$ is not $U\varphi$, but $\pi_{T,Y}(u_T\circ U\varphi)$. Nevertheless, (d) provides a formula for the action of a correspondence $U$ on a fundamental functor.

**Proof :** (a) This is a straightforward consequence of Corollary 17.17.

(b) (i) By Theorem 17.5, for any $\varphi:X\to T$, the difference $\varphi-u_a\circ\varphi$ lies in $\mathrm{Ker}\,\Theta_{T,X}$, where $a\in T$ is such that $a\notin E$ and $a<r^\infty s^\infty(a)$. In other words $u_a\circ\varphi$ is equal to $\varphi$ modulo $\mathrm{Ker}\,\Theta_{T,X}$. As $u_T$ is the composition of all maps $u_a$, for $a\notin E$ and $a<r^\infty s^\infty(a)$, the same congruence holds with $u_a$ replaced by $u_T$, that is $\varphi-u_T\circ\varphi\in H_T(X)$.

(ii) If $\varphi(X)\not\supseteq E$, then $\varphi\in H_T(X)\subseteq\mathrm{Ker}\,\Theta_{T,X}$, so $u_T\circ\varphi\in\mathrm{Ker}\,\Theta_{T,X}$ by (i), hence $\pi_{T,X}(u_T\circ\varphi)\in\mathrm{Ker}\,\Theta_{T,X}$, since

$$\mathrm{Im}(\mathrm{id}-\pi_{T,X})=\mathrm{Ker}\,\pi_{T,X}=H_T(X)\subseteq\mathrm{Ker}\,\Theta_{T,X}\ .$$

But $\pi_{T,X}(u_T\circ\varphi)$ is a linear combination of elements of $\mathcal{B}_X$, thus $\pi_{T,X}(u_T\circ\varphi)=0$ by Theorem 17.10, that is $u_T\circ\varphi\in H_T(X)$.

(iii) If $\varphi(X)\subseteq G$, then $\varphi-u_T\circ\varphi$ is a linear combination of maps from $X$ to $T$ with image contained in $G$, by Theorem 17.7. Moreover $\varphi-u_T\circ\varphi$ lies in the kernel of $\Theta_{T,X}$, by (i). Then $\pi_{T,X}(\varphi-u_T\circ\varphi)$ is a linear combination of elements of $\mathcal{B}_X$, which is also in $\mathrm{Ker}\,\Theta_{T,X}$, again because $\mathrm{Im}(\mathrm{id}-\pi_{T,X})\subseteq\mathrm{Ker}\,\Theta_{T,X}$. Hence $\pi_{T,X}(\varphi-u_T\circ\varphi)=0$ by Theorem 17.10, that is $\varphi-u_T\circ\varphi\in H_T(X)$. Moreover, if $\varphi(X)\subseteq G$, then $\pi_{T,X}(\varphi)=\varphi$ if $\varphi(X)\supseteq E$, i.e. $\varphi\in\mathcal{B}_X$, and $\pi_{T,X}(\varphi)=0$ otherwise. Hence $\pi_{T,X}(\varphi)\in\mathbb{S}(X)$.

(iv) follows from (iii), since $u_T\circ\varphi$ is a linear combination of maps from $X$ to $T$ whose image is contained in $G$, by Theorem 17.7.

(c) By (b)(i), $\varphi - u_T \circ \varphi \in \mathrm{Ker}\,\Theta_{T,X}$ for $\varphi \in F_T(X)$. Hence if $u_T \circ \varphi \in H_T(X)$, then $\varphi \in \mathrm{Ker}\,\Theta_{T,X}$, as $H_T(X) \subseteq \mathrm{Ker}\,\Theta_{T,X}$. Conversely, if $\varphi \in \mathrm{Ker}\,\Theta_{T,X}$, then $u_T \circ \varphi \in \mathrm{Ker}\,\Theta_{T,X}$ also, hence $\pi_{T,X}(u_T \circ \varphi) \in \mathrm{Ker}\,\Theta_{T,X}$, again because $\mathrm{Im}(\mathrm{id} - \pi_{T,X}) \subseteq \mathrm{Ker}\,\Theta_{T,X}$. But $\pi_{T,X}(u_T \circ \varphi) \in \mathbb{S}(X)$ by (b)(iv), so $\pi_{T,X}(u_T \circ \varphi) = 0$ by (a). This means that $u_T \circ \varphi \in H_T(X)$.

(d) Since $\mathrm{Im}(\mathrm{id} - \pi_{T,X}) = \mathrm{Ker}\,\pi_{T,X} = H_T(X)$, proving (d) is equivalent to proving that the elements $\pi_{T,X}(\varphi - u_T \circ \varphi)$, for $E \subseteq \varphi(X) \nsubseteq G$, form a basis of a complement of $H_T(X)$ in $\mathrm{Ker}\,\Theta_{T,X}$.

For this, we observe that $k(T^X) = \mathrm{Ker}\,\pi_{T,X} \oplus \mathrm{Im}\,\pi_{T,X} = H_T(X) \oplus \mathrm{Im}\,\pi_{T,X}$. Since $H_T(X)$ is contained in $\mathrm{Ker}\,\Theta_{T,X}$, it follows that

$$\mathrm{Ker}\,\Theta_{T,X} = H_T(X) \oplus \big(\mathrm{Im}\,\pi_{T,X} \cap \mathrm{Ker}\,\Theta_{T,X}\big) = H_T(X) \oplus \pi_{T,X}(\mathrm{Ker}\,\Theta_{T,X}) ,$$

as $\pi_{T,X}$ is a projector. We will prove that the elements $\pi_{T,X}(\varphi - u_T \circ \varphi)$, for $E \subseteq \varphi(X) \nsubseteq G$, form a basis of $\pi_{T,X}(\mathrm{Ker}\,\Theta_{T,X})$.

Let $w = \sum_{\varphi: X \to T} \lambda_\varphi \varphi$ be an element of $\mathrm{Ker}\,\Theta_{T,X}$, where $\lambda_\varphi \in k$ for $\varphi : X \to T$. Then

$$w = \sum_{\varphi: X \to T} \lambda_\varphi(\varphi - u_T \circ \varphi) + \sum_{\varphi: X \to T} \lambda_\varphi u_T \circ \varphi .$$

By (b)(i), the sum $\sum_{\varphi: X \to T} \lambda_\varphi u_T \circ \varphi$ is in $\mathrm{Ker}\,\Theta_{T,X}$, so $\pi_{T,X}(\sum_{\varphi: X \to T} \lambda_\varphi u_T \circ \varphi)$ also. But this is a linear combination of elements of $\mathcal{B}_X$, hence it is equal to 0 by Theorem 17.10. In other words

$$\pi_{T,X}(w) = \pi_{T,X}\big(\sum_{\varphi: X \to T} \lambda_\varphi(\varphi - u_T \circ \varphi)\big) = \sum_{\varphi: X \to T} \lambda_\varphi \pi_{T,X}(\varphi - u_T \circ \varphi) .$$

Now by (b)(ii) and (b)(iii), we have $\pi_{T,X}(\varphi - u_T \circ \varphi) = 0$ if $\varphi(X) \nsupseteq E$ or $\varphi(X) \subseteq G$. It follows that

$$\pi_{T,X}(w) = \sum_{\substack{\varphi: X \to T \\ E \subseteq \varphi(X) \nsubseteq G}} \lambda_\varphi \pi_{T,X}(\varphi - u_T \circ \varphi) .$$

Hence $\pi_{T,X}(\mathrm{Ker}\,\Theta_{T,X})$ is generated by the elements $\pi_{T,X}(\varphi - u_T \circ \varphi)$, for $\varphi : X \to T$ such that $E \subseteq \varphi(X) \nsubseteq G$. Moreover, these elements are linearly independent: indeed, if

$$\sum_{\substack{\varphi: X \to T \\ E \subseteq \varphi(X) \nsubseteq G}} \lambda_\varphi \pi_{T,X}(\varphi - u_T \circ \varphi) = 0 ,$$

then as $\pi_{T,X}(\varphi) = \varphi$ if $\varphi(X) \supseteq E$, we have

$$\sum_{\substack{\varphi: X \to T \\ E \subseteq \varphi(X) \nsubseteq G}} \lambda_\varphi \varphi = \sum_{\substack{\varphi: X \to T \\ E \subseteq \varphi(X) \nsubseteq G}} \lambda_\varphi \pi_{T,X}(u_T \circ \varphi) .$$

On the right hand side, we have an element of $\mathbb{S}(X)$, by (b)(iv), but none of the maps appearing in the left hand side lies in $\mathcal{B}_X$. It follows that both sides are equal to 0, and all the coefficients $\lambda_\varphi$ are equal to 0. This completes the proof of (d).

(e) It follows from (a) that the restriction of $\Theta_{T,X}$ induces an isomorphism of $k$-modules

$$\mathbb{S}(X) \to \mathbb{S}_{E,R^{op}}(X) .$$

Now if $U \subseteq Y \times X$, we have a diagram

$$\begin{array}{ccccc}
\mathbb{S}(X) & \xhookrightarrow{i_X} & F_T(X) & \xrightarrow{\Theta_{T,X}} & \mathbb{S}_{E,R^{op}}(X) \\
\downarrow{\scriptstyle \mathbb{S}(U)} & & \downarrow{\scriptstyle F_T(U)} & & \downarrow{\scriptstyle \mathbb{S}_{E,R^{op}}(U)} \\
\mathbb{S}(Y) & \xhookrightarrow{i_Y} & F_T(Y) & \xrightarrow{\Theta_{T,Y}} & \mathbb{S}_{E,R^{op}}(Y)
\end{array}$$

(where $i_X$ is the inclusion map). This diagram is *not* commutative because the left hand square is not commutative (but the right hand square is). But for $\varphi \in \mathcal{B}_X$, we have

$$\begin{array}{rcll}
\Theta_{T,Y} \circ i_Y \circ \mathbb{S}(U)(\varphi) & = & \Theta_{T,Y}\big(\pi_{T,Y}(u_T \circ U\varphi)\big) & \\
& = & \Theta_{T,Y}(u_T \circ U\varphi) & [\text{as } \mathrm{Im}(\mathrm{id} - \pi_{T,Y}) \subseteq \mathrm{Ker}\,\Theta_{T,Y}] \\
& = & \Theta_{T,Y}(U\varphi) & [\text{by (b)(i)}] \\
& = & \Theta_{T,Y}\big(F_T(U)(\varphi)\big) & [\text{by definition of } F_T(U)(\varphi)] \\
& = & \mathbb{S}_{E,R^{op}}(U)\big(\Theta_{T,X}(\varphi)\big) & \\
& = & \mathbb{S}_{E,R^{op}}(U) \circ \Theta_{T,X} \circ i_X(\varphi)\ . &
\end{array}$$

It follows that the outer rectangle is commutative. In other words, we have a commutative diagram

$$\begin{array}{ccc}
\mathbb{S}(X) & \xrightarrow[\cong]{\Theta_{T,X} i_X} & \mathbb{S}_{E,R^{op}}(X) \\
\downarrow{\scriptstyle \mathbb{S}(U)} & & \downarrow{\scriptstyle \mathbb{S}_{E,R^{op}}(U)} \\
\mathbb{S}(Y) & \xrightarrow[\cong]{\Theta_{T,Y} i_Y} & \mathbb{S}_{E,R^{op}}(Y)
\end{array}$$

in which the horizontal maps are isomorphisms. This shows that (e) holds and the proof of Theorem 18.4 is complete. □

Our final goal in this section is to shed some light on a special case, namely when $T = \wedge E$, where $T$ is a finite lattice and $(E,R)$ is the subposet of irreducible elements in $T$. By definition of the set $G$ in Notation 16.7, we have $G = T$ (and there are no bulbs). This allows us to obtain a much finer result in that case.

**18.6. Theorem.** *Let $(E,R)$ be a finite poset and let $T$ be any lattice such that $(E,R)$ is the full subposet of irreducible elements in $T$. Assume that $T = \wedge E$.*

*(a) $F_T/H_T \cong \mathbb{S}_{E,R}$.*
*(b) If $k$ is a field and $\mathrm{Aut}(E,R)$ is the trivial group, then $F_T/H_T$ is simple, namely $F_T/H_T \cong S_{E,R,k}$.*

**Proof :** (a) The surjection $\Theta_T : F_T \to \mathbb{S}_{E,R^{op}}$ factorizes as the composition of the morphisms $\Pi : F_T \longrightarrow F_T/H_T$ and $\overline{\Theta}_T : F_T/H_T \longrightarrow \mathbb{S}_{E,R^{op}}$. For any finite set $X$, let

$$\mathcal{B}_X = \{\varphi : X \to T \mid E \subseteq \varphi(X)\}\ .$$

Then $\Pi_X(\mathcal{B}_X)$ is a $k$-basis of $(F_T/H_T)(X)$. Our assumption $T = \wedge E$ implies that $G = T$, where $G$ is, as usual, the subset defined in Notation 16.7. Therefore the set $\mathcal{B}_X$ coincides with the set defined in Notation 17.1 and used throughout Section 17. Hence $\Theta_{T,X}(\mathcal{B}_X)$ is a $k$-basis of $\mathbb{S}_{E,R^{op}}(X)$ by Corollary 17.17. It follows that $\overline{\Theta}_{T,X}$ maps a $k$-basis to a $k$-basis, hence it is an isomorphism.

(b) Our assumption on $\mathrm{Aut}(E,R)$ implies that

$$S_{E,R,k} = \mathbb{S}_{E,R} \otimes_{k\,\mathrm{Aut}(E,R)} k \cong \mathbb{S}_{E,R} \otimes_k k \cong \mathbb{S}_{E,R}\ .$$

By (a), it follows that $S_{E,R,k} \cong F_T/H_T$. □

There are many examples of the situation described in Theorem 18.6.

**18.7. Example.** Suppose that $(E, R)$ is a forest, namely a disjoint union of trees with all edges oriented from the leaves to the root, that is, in such a way that the roots of the trees are the minimal elements of $E$. Consider the lattice $T = E \sqcup \{\hat{0}, \hat{1}\}$, with $\hat{0}$ minimal and $\hat{1}$ maximal. We assume that $E$ has at least two maximal elements (i.e. $E$ is not totally ordered), so that $\hat{1}$ is not irreducible. Then $E$ is the set of irreducible elements of the lattice $T$. We claim that $G = T$, where $G$ is, as usual, the subset defined in Notation 16.7. If $E$ has at least two minimal elements (i.e. at least two connected components), then this follows from the observation that $\wedge E = T$. If $E$ has a single minimal element (i.e. $E$ is a tree), then $\wedge E = E \sqcup \{\hat{1}\}$ and $\hat{0}$ is a bulb, so $G = \wedge E \sqcup \{\hat{0}\} = T$.

In all such cases, $F_T/H_T$ is a fundamental functor. Moreover, there are many cases when $\mathrm{Aut}(E, R)$ is the trivial group (take branches of different length). In such a case, $F_T/H_T$ is simple, provided $k$ is a field.

## 19. Tensor product and internal hom

In this section, we introduce two constructions in the category $\mathcal{F}_k$: when $M$ and $M'$ are correspondence functors, we define their tensor product $M \otimes M'$ and their internal hom $\mathcal{H}(M, M')$, which are both correspondence functors. These functors are associated to the symmetric monoidal structure on $\mathcal{C}$ given by the disjoint union of finite sets: if $X, X', Y, Y'$ are finite sets, and if $U \in \mathcal{C}(X', X)$ and $V \in \mathcal{C}(Y', Y)$, then $U \sqcup V$ can be viewed as a subset of $(X' \sqcup Y') \times (X \sqcup Y)$. We will represent this correspondence in a matrix form, as follows:

$$\begin{pmatrix} U & \emptyset \\ \emptyset & V \end{pmatrix} .$$

This yields a bifunctor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, still denoted by a disjoint union symbol.

**19.1. Definition.** *Let $M$ and $M'$ be correspondence functors over a commutative ring $k$.*

*(a) The* tensor product *of $M$ and $M'$ is the correspondence $M \otimes M'$ defined for a finite set $X$ by*

$$(M \otimes M')(X) = M(X) \otimes_k M'(X) .$$

*If $Y$ is a finite set and $U \in \mathcal{C}(Y, X)$, the map*

$$(M \otimes M')(U) : (M \otimes M')(X) \to (M \otimes M')(Y)$$

*is the $k$-linear map defined by*

$$\forall m \in M(X),\ \forall m' \in M'(X),\ \ (M \otimes M')(U)(m \otimes m') = Um \otimes Um' ,$$

*where as usual $Um = M(U)(m)$.*

*(b) Let $E$ be a finite set. Let $M_E$ be the correspondence functor obtained from $M$ by precomposition with the endofunctor $t_E : X \mapsto X \sqcup E$ of $\mathcal{C}$. When $E'$ is a finite set and $V \in \mathcal{C}(E', E)$, let $M_V : M_E \to M_{E'}$ be the morphism obtained by precomposition with the natural transformation $\mathrm{id} \sqcup V : t_E \to t_{E'}$.*

*(c) Let $\mathcal{H}(M, M')$ be the correspondence functor defined on a finite set $E$ by*

$$\mathcal{H}(M, M')(E) = \mathrm{Hom}_{\mathcal{F}_k}(M, M'_E) ,$$

*and for $V \in \mathcal{C}(E', E)$, by composition with $M'_V : M'_E \to M'_{E'}$.*

**19.2. Theorem.** *Let $k$ be a commutative ring, and let $M, M'$ and $M''$ be correspondence functors over $k$.*

(a) *The assignment $(M, M') \mapsto M \otimes M'$ is a $k$-linear bifunctor $\mathcal{F}_k \times \mathcal{F}_k \to \mathcal{F}_k$, right exact in $M$ and $M'$. The assignment $(M, M') \mapsto \mathcal{H}(M, M')$ is a $k$-linear bifunctor $\mathcal{F}_k^{op} \times \mathcal{F}_k \to \mathcal{F}_k$, left exact in $M$ and $M'$.*

(b) *There are natural isomorphisms of correspondence functors*

$$\begin{aligned} M \otimes (M' \otimes M'') &\cong (M \otimes M') \otimes M'' \\ M \otimes M' &\cong M' \otimes M \\ \underline{k} \otimes M &\cong M \,, \end{aligned}$$

*where $\underline{k}$ is the constant functor introduced in Example 4.11.*

(c) *The $k$-module $\mathrm{Hom}_{\mathcal{F}_k}(M' \otimes M, M'')$ is isomorphic to the $k$-module of* bilinear pairings *$M' \times M \to M''$, i.e. the $k$-module of natural transformations of the bifunctor $(X, Y) \mapsto M'(X) \times M(Y)$ to the bifunctor $(X, Y) \mapsto M''(X \sqcup Y)$ from $\mathcal{C} \times \mathcal{C}$ to $k$-*Mod.

(d) *There are isomorphisms of $k$-modules*

$$\mathrm{Hom}_{\mathcal{F}_k}(M' \otimes M, M'') \cong \mathrm{Hom}_{\mathcal{F}_k}\big(M, \mathcal{H}(M', M'')\big)$$

*natural in $M, M', M''$. In particular, for any correspondence functor $M'$ over $k$, the endofunctor $M \mapsto M' \otimes M$ of $\mathcal{F}_k$ is left adjoint to the endofunctor $M \mapsto \mathcal{H}(M', M)$*

**Proof :** (a) The first assertion is clear, and the second assertion follows from the fact that the assignment $(M, E) \mapsto M_E$ is a functor $\mathcal{F}_k \times \mathcal{C} \to \mathcal{F}_k$, exact in $M$.

(b) For any finite set $X$, the standard $k$-linear isomorphisms

$$\begin{aligned} M(X) \otimes_k \big(M'(X) \otimes_k M''(X)\big) &\cong \big(M(X) \otimes_k M'(X)\big) \otimes_k M''(X) \\ M(X) \otimes_k M'(X) &\cong M'(X) \otimes_k M(X) \\ \underline{k}(X) \otimes_k M(X) = k \otimes_k M(X) &\cong M(X) \,, \end{aligned}$$

are compatible with the action of correspondences.

(c) Let $\psi : M' \otimes M \to M''$ be a morphism of correspondence functors. Equivalently, for any finite set $X$, let $\psi_X : M'(X) \otimes_k M(X) \to M''(X)$ be a linear map with the property that for any finite set $Z$ and any $U \in \mathcal{C}(Z, X)$, the diagram

$$\begin{array}{ccc} M'(X) \otimes_k M(X) & \xrightarrow{\psi_X} & M''(X) \\ {\scriptstyle M'(U)\otimes_k M(U)}\downarrow & & \downarrow{\scriptstyle M''(U)} \\ M'(Z) \otimes_k M(Z) & \xrightarrow{\psi_Z} & M''(Z) \end{array}$$

is commutative. If $X$ and $Y$ are finite sets, let $\binom{\Delta_X}{\emptyset} \in \mathcal{C}(X \sqcup Y, X)$ be the graph of the inclusion $i_X$ of $X$ in $X \sqcup Y$, and let $\binom{\emptyset}{\Delta_Y} \in \mathcal{C}(X \sqcup Y, Y)$ be the graph of the inclusion of $Y$ in $X \sqcup Y$. We define a map

$$\widehat{\psi}_{X,Y} : M'(X) \otimes_k M(Y) \to M''(X \sqcup Y)$$

as the following composition

$$M'(X) \otimes_k M(Y) \xrightarrow{M'\binom{\Delta_X}{\emptyset} \otimes M\binom{\emptyset}{\Delta_Y}} M'(X \sqcup Y) \otimes_k M(X \sqcup Y) \xrightarrow{\psi_{X \sqcup Y}} M''(X \sqcup Y)$$

i.e. $\widehat{\psi}_{X,Y} = \psi_{X \sqcup Y} \circ \Big(\binom{\Delta_X}{\emptyset} \otimes \binom{\emptyset}{\Delta_Y}\Big)$, with the usual abuse of notation identifying a correspondence and its action on a correspondence functor. If $X'$ and $Y'$ are finite sets, if $U \in \mathcal{C}(X', X)$ and $V \in \mathcal{C}(Y', Y)$, then we have the following commutative diagram

$$\begin{array}{ccccc}
M'(X) \otimes_k M(Y) & \xrightarrow{\binom{\Delta_X}{\emptyset}\otimes\binom{\emptyset}{\Delta_Y}} & M'(X \sqcup Y) \otimes_k M(X \sqcup Y) & \xrightarrow{\psi_{X\sqcup Y}} & M''(X \sqcup Y) \\
{\scriptstyle U} \downarrow {\scriptstyle V} & & \begin{pmatrix} U & \emptyset \\ \emptyset & V \end{pmatrix} \downarrow \begin{pmatrix} U & \emptyset \\ \emptyset & V \end{pmatrix} & & \begin{pmatrix} U & \emptyset \\ \emptyset & V \end{pmatrix} \downarrow \\
M'(X') \otimes_k M(Y') & \xrightarrow[\binom{\Delta_{X'}}{\emptyset}\otimes\binom{\emptyset}{\Delta_{Y'}}]{} & M'(X' \sqcup Y') \otimes_k M(X' \sqcup Y') & \xrightarrow[\psi_{X'\sqcup Y'}]{} & M''(X' \sqcup Y') .
\end{array}$$

The left hand side square is commutative because $\begin{pmatrix} U & \emptyset \\ \emptyset & V \end{pmatrix}\binom{\Delta_X}{\emptyset} = \binom{U}{\emptyset} = \binom{\Delta_{X'}}{\emptyset}U$, and similarly $\begin{pmatrix} U & \emptyset \\ \emptyset & V \end{pmatrix}\binom{\emptyset}{\Delta_Y} = \binom{\emptyset}{V} = \binom{\emptyset}{\Delta_{Y'}}V$. The right hand square is commutative by the defining property of the morphism $\psi : M' \otimes M \to M''$. It follows that $\begin{pmatrix} U & \emptyset \\ \emptyset & V \end{pmatrix} \circ \widehat{\psi}_{X,Y} = \widehat{\psi}_{X',Y'} \circ (U \otimes_k V)$, so the maps $\widehat{\psi}_{X,Y}$ define a natural transformation of the bifunctor $(X, Y) \mapsto M'(X) \otimes_k M(Y)$ to the bifunctor $(X, Y) \mapsto M''(X \sqcup Y)$, in other words a bilinear pairing $\widehat{\psi} : M' \times M \to M''$.

Conversely, let $\eta : M' \times M \to M''$ be a bilinear pairing. This means that for any finite sets $X, Y$, there is a map $\eta_{X,Y} : M'(X) \otimes_k M(Y) \to M''(X \sqcup Y)$, such that for any finite set $X'$ and any correspondences $U \in \mathcal{C}(X', X)$ and $V \in \mathcal{C}(Y', Y)$, the diagram

$$\begin{array}{ccc}
M'(X) \otimes_k M(Y) & \xrightarrow{\eta_{X,Y}} & M''(X \sqcup Y) \\
{\scriptstyle U} \downarrow {\scriptstyle V} & & \downarrow \begin{pmatrix} U & \emptyset \\ \emptyset & V \end{pmatrix} \\
M'(X') \otimes_k M(Y') & \xrightarrow{\eta_{X',Y'}} & M''(X' \sqcup Y')
\end{array} \tag{19.3}$$

is commutative.

In particular, for $X = Y$, we have a map $\eta_{X,X} : M'(X) \otimes_k M(X) \to M''(X \sqcup X)$, which we can compose with the map $M''(X \sqcup X) \to M''(X)$ given by the action of the “codiagonal” correspondence $(\Delta_X, \Delta_X) \in \mathcal{C}(X, X \sqcup X)$, to get a map

$$\widetilde{\eta}_X = (\Delta_X, \Delta_X) \circ \eta_{X,X} : M'(X) \otimes M(X) \to M''(X) .$$

If $Z$ is a finite set and $U \in \mathcal{C}(Z, X)$, the diagram

$$\begin{array}{ccccc}
M'(X) \otimes M(X) & \xrightarrow{\eta_{X,X}} & M''(X \sqcup X) & \xrightarrow{(\Delta_X,\Delta_X)} & M''(X) \\
{\scriptstyle U} \downarrow {\scriptstyle U} & & \downarrow \begin{pmatrix} U & \emptyset \\ \emptyset & U \end{pmatrix} & & \downarrow {\scriptstyle U} \\
M'(Z) \otimes M(Z) & \xrightarrow[\eta_{Z,Z}]{} & M''(Z \sqcup Z) & \xrightarrow[(\Delta_Z,\Delta_Z)]{} & M''(Z)
\end{array}$$

is commutative: the left hand square is commutative because $\eta$ is a bilinear pairing, and the right hand square is commutative because

$$U(\Delta_X, \Delta_X) = (U, U) = (\Delta_Z, \Delta_Z)\begin{pmatrix} U & \emptyset \\ \emptyset & U \end{pmatrix} .$$

Hence the maps $\widetilde{\eta}_X$ define a morphism of correspondence functors $\widetilde{\eta}$ from $M' \otimes M$ to $M''$.

The constructions $\psi \mapsto \widehat{\psi}$ and $\eta \mapsto \widetilde{\eta}$ are $k$-linear and inverse to each other: indeed, for any finite set $X$,

$$\begin{aligned}
\widetilde{\widehat{\psi}}_X = {}_{(\Delta_X,\Delta_X)}\widehat{\psi}_{X,X} &= {}_{(\Delta_X,\Delta_X)}\psi_{X\sqcup X}\Big(\binom{\Delta_X}{\emptyset}\otimes\binom{\emptyset}{\Delta_X}\Big) \\
&= \psi_X\Big({}_{(\Delta_X,\Delta_X)} \otimes_k {}_{(\Delta_X,\Delta_X)}\Big)\Big(\binom{\Delta_X}{\emptyset}\otimes\binom{\emptyset}{\Delta_X}\Big) \\
&= \psi_{X(\Delta_X\otimes\Delta_X)} = \psi_X ,
\end{aligned}$$

since ${}_{(\Delta_X,\Delta_X)}\binom{\Delta_X}{\emptyset} = \Delta_X$ and ${}_{(\Delta_X,\Delta_X)}\binom{\emptyset}{\Delta_X} = \Delta_X$. Similarly, for finite sets $X$ and $Y$,

$$\begin{aligned}
\widetilde{\widetilde{\eta}}_{X,Y} &= \widetilde{\eta}_{X\sqcup Y,X\sqcup Y}\Big(\tbinom{\Delta_X}{\emptyset}\otimes_k\tbinom{\emptyset}{\Delta_Y}\Big) = {}_{(\Delta_{X\sqcup Y},\Delta_{X\sqcup Y})}\eta_{X\sqcup Y,X\sqcup Y}\Big(\tbinom{\Delta_X}{\emptyset}\otimes_k\tbinom{\emptyset}{\Delta_Y}\Big)\\
&= {}_{(\Delta_{X\sqcup Y},\Delta_{X\sqcup Y})}\begin{pmatrix} \binom{\Delta_X}{\emptyset} & \emptyset \\ \emptyset & \binom{\emptyset}{\Delta_Y}\end{pmatrix}\eta_{X,Y}\\
&= \begin{pmatrix} \Delta_X & \emptyset \\ \emptyset & \Delta_Y\end{pmatrix}\eta_{X,Y} = \eta_{X,Y}\ .
\end{aligned}$$

(d) Let $\psi : M' \times M \to M''$ be a morphism of correspondence functors. By (c), for any finite sets $X$ and $Y$, we get a linear map $\widehat{\psi}_{X,Y} : M'(X) \otimes_k M(Y) \to M''(X \sqcup Y)$, or equivalently, a linear map

$$\overline{\psi}_{Y,X} : M(Y) \to \mathrm{Hom}_k\big(M'(X), M''(X \sqcup Y)\big)$$

defined by $\overline{\psi}_{Y,X}(m)(m') = \widehat{\psi}_{X,Y}(m' \otimes m)$, for $m \in M(Y)$ and $m' \in M'(X)$. Now $M''(X \sqcup Y) = M''_Y(X)$. Moreover, for any finite set $X'$ and any $U \in \mathcal{C}(X', X)$, the commutative diagram 19.3, for $Y' = Y$ and $V = \Delta_Y$, becomes

$$\begin{array}{ccc}
M'(X) \otimes_k M(Y) & \xrightarrow{\widehat{\psi}_{X,Y}} & M''(X \sqcup Y) \\
{\scriptstyle U}\big\downarrow{\scriptstyle \Delta_Y} & & \big\downarrow{\scriptstyle \begin{pmatrix} U & \emptyset \\ \emptyset & \Delta_Y\end{pmatrix}} \\
M'(X') \otimes_k M(Y) & \xrightarrow{\widehat{\psi}_{X',Y}} & M''(X' \sqcup Y)\ .
\end{array}$$

In other words $\overline{\psi}_{Y,X'}(Um') = U\overline{\psi}_{Y,X}(m')$ for any $m' \in M'(X)$. Hence for a given set $Y$, the maps $\overline{\psi}_{Y,X}$ define a morphism of correspondence functors $M' \to M''_Y$, and this gives a map $\overline{\psi}_Y : M(Y) \to \mathcal{H}(M', M'')(Y)$.

Now if $Y'$ is a finite set and $V \in \mathcal{C}(Y', Y)$, the commutative diagram 19.3, for $X' = X$ and $U = \Delta_X$, becomes

$$\begin{array}{ccc}
M'(X) \otimes_k M(Y) & \xrightarrow{\widehat{\psi}_{X,Y}} & M''(X \sqcup Y) \\
{\scriptstyle \Delta_X}\big\downarrow{\scriptstyle V} & & \big\downarrow{\scriptstyle \begin{pmatrix} \Delta_X & \emptyset \\ \emptyset & V\end{pmatrix}} \\
M'(X) \otimes_k M(Y') & \xrightarrow{\widehat{\psi}_{X,Y'}} & M''(X \sqcup Y')\ ,
\end{array}$$

and it follows that the maps $\overline{\psi}_Y$ define a morphism of correspondence functors $\overline{\psi} : M \to \mathcal{H}(M', M'')$.

Conversely, a morphism of correspondence functors $\xi : M \to \mathcal{H}(M', M'')$ is determined by maps $\xi_Y : M(Y) \to \mathcal{H}(M', M'')(Y) = \mathrm{Hom}_{\mathcal{F}_k}(M', M''_Y)$, for all finite sets $Y$. Furthermore, for $m \in M(Y)$, the map $\xi_Y(m)$ is in turn determined by maps $\xi_Y(m)_X : M'(X) \to M''(X \sqcup Y)$, for all finite sets $X$. It is easy to see that $\xi$ is a morphism of correspondence functors from $M$ to $\mathcal{H}(M', M'')$ if and only if the maps

$$\begin{aligned}
\mathring{\xi}_{X,Y} : M'(X) \otimes_k M(Y) &\longrightarrow M''(X \sqcup Y)\\
m' \otimes m &\longmapsto \xi_Y(m)(m')
\end{aligned}$$

define a bilinear pairing $\mathring{\xi} : M' \times M \to M''$, i.e. if for any finite sets $X, Y, X', Y'$, and any correspondences $U \in \mathcal{C}(X', X)$ and $V \in \mathcal{C}(Y', Y)$, the diagram

$$\begin{array}{ccc}
M'(X) \otimes_k M(Y) & \xrightarrow{\mathring{\xi}_{X,Y}} & M''(X \sqcup Y) \\
{\scriptstyle U}\ \Big\downarrow\ {\scriptstyle V} & & \Big\downarrow \begin{pmatrix} U & \emptyset \\ \emptyset & V \end{pmatrix} \\
M'(X') \otimes_k M(Y') & \xrightarrow{\mathring{\xi}_{X',Y'}} & M''(X' \sqcup Y')
\end{array}$$

is commutative. By (c), we get a morphism of correspondence functors $\check{\xi} = \widetilde{\mathring{\xi}}$ from $M' \otimes M$ to $M''$.

Now it is straightforward to check that the maps $\psi \mapsto \overline{\psi}$ and $\xi \mapsto \check{\xi}$ are inverse isomorphisms between $\mathrm{Hom}_{\mathcal{F}_k}(M' \otimes M, M'')$ and $\mathrm{Hom}_{\mathcal{F}_k}\big(M, \mathcal{H}(M', M'')\big)$. Assertion (d) follows. ∎

Now we want to apply the tensor product construction to functors of the form $F_T$, where $T$ is a finite lattice. Thus we again consider the category $k\mathcal{L}$ of lattices, introduced in Definition 11.5.

**19.4. Theorem.** *Let $k$ be a commutative ring.*

*(a) There is a well defined $k$-linear bifunctor $k\mathcal{L} \times k\mathcal{L} \to k\mathcal{L}$ sending*
- *the pair $(T, T')$ of lattices to the product lattice $T \times T'$,*
- *and the pair of morphisms $(\sum_{i=1}^{n} \lambda_i f_i, \sum_{j=1}^{m} \mu_j g_j)$, where $\lambda_i, \mu_j \in k$ and $f_i : T \to T_1$ and $g_j : T' \to T'_1$ are morphisms in $\mathcal{L}$, to the morphism $\sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \lambda_i \mu_j (f_i \times g_j) : T \times T' \to T_1 \times T'_1$.*

*(b) The bifunctors $(T, T') \mapsto F_T \otimes F_{T'}$ and $(T, T') \mapsto F_{T \times T'}$ from $k\mathcal{L} \times k\mathcal{L}$ to $\mathcal{F}_k$ are isomorphic.*

*(c) In particular, if $E$ and $F$ are finite sets, then*

$$k\mathcal{C}(-, E) \otimes k\mathcal{C}(-, F) \cong k\mathcal{C}(-, E \sqcup F) \ .$$

**Proof :** (a) If $f : T \to T_1$ and $g : T' \to T'_1$ are morphisms in $\mathcal{L}$, then $f \times g$ is a morphism in $\mathcal{L}$ from $T \times T'$ to $T_1 \times T'_1$: indeed, if $A \subseteq T \times T'$, let $B$ (resp. $B'$) denote the projection of $A$ on $T$ (resp. $T'$). Then

$$\begin{aligned}
(f \times g)\big(\bigvee_{(t,t') \in A} (t, t')\big) &= (f \times g)\Big(\bigvee_{(t,t') \in A} \big((t, \hat{0}) \vee (\hat{0}, t')\big)\Big) \\
&= (f \times g)\Big(\bigvee_{t \in B} (t, \hat{0}) \vee \bigvee_{t' \in B'} (\hat{0}, t')\Big) \\
&= (f \times g)\big(\bigvee_{t \in B} t, \bigvee_{t' \in B'} t'\big) \\
&= \Big(\bigvee_{t \in B} f(t), \bigvee_{t' \in B'} g(t')\Big)
\end{aligned}$$

On the other hand

$$\begin{aligned}\bigvee_{(t,t')\in A}\big(f(t),g(t')\big) &= \bigvee_{t\in B}\big(f(t),\hat{0}\big)\vee\bigvee_{t'\in B'}\big(\hat{0},g(t')\big)\\ &= \Big(\bigvee_{t\in B}f(t),\hat{0}\Big)\vee\Big(\hat{0},\bigvee_{t'\in B'}g(t')\Big)\\ &= \Big(\bigvee_{t\in B}f(t),\bigvee_{t'\in B'}g(t')\Big)\end{aligned}$$

It follows that $(f\times g)\big(\bigvee_{(t,t')\in A}(t,t')\big)=\bigvee_{(t,t')\in A}(f\times g)(t,t')$, as claimed. Assertion (a) follows by bilinearity.

(b) Let $T$ and $T'$ be finite lattices. Then for any finite set $X$, there is a unique isomorphism of $k$-modules

$$\tau_X:(F_T\otimes F_{T'})(X)=k(T^X)\otimes_k k(T'^X)\longrightarrow k(T\times T')^X\ ,$$

sending $\varphi\otimes\varphi'$, for $\varphi:X\to T$ and $\varphi':X\to T'$, to the map $\varphi\times\varphi':X\to T\times T'$. If $Y$ is a finite set and $U\in\mathcal{C}(Y,X)$, then for any $y\in Y$,

$$\begin{aligned}U(\varphi\times\varphi')(y) &= \bigvee_{(y,x)\in U}\big(\varphi(x),\varphi'(x)\big)\\ &= \big(\bigvee_{(y,x)\in U}\varphi(x),\bigvee_{(y,x')\in U}\varphi'(x')\big)\\ &= \big(U\varphi(y),U\varphi'(y)\big)\ .\end{aligned}$$

Thus $U\tau_X(\varphi\otimes_k\varphi')=U\varphi\times U\varphi'=\tau_Y(U\varphi\otimes_k U\varphi')$, hence $\tau:F_T\otimes F_{T'}\to F_{T\times T'}$ is an isomorphism of correspondence functors.

If $f:T\to T_1$ and $f':T'\to T_1'$ are morphisms in $\mathcal{L}$, then by (a) the map $f\times f':T\times T'\to T_1\times T_1'$ is a morphism in $\mathcal{L}$. Moreover, the diagram

$$\begin{array}{ccc}F_T\otimes F_{T'} & \xrightarrow{\ \tau\ } & F_{T\times T'}\\ {\scriptstyle F_f\otimes F_{f'}}\downarrow & & \downarrow{\scriptstyle F_{f\times f'}}\\ F_{T_1}\otimes F_{T_1'} & \xrightarrow{\ \tau_1\ } & F_{T_1\times T_1'}\end{array}$$

where $\tau_1:F_{T_1}\otimes F_{T_1'}\to F_{T_1\times T_1'}$ is the corresponding isomorphism for the lattices $T_1$ and $T_1'$, is commutative, since for any finite set $X$, any $\varphi:X\to T$ and any $\varphi':X'\to T'$

$$F_{f\times f'}\tau_X(\varphi\otimes_k\varphi')=F_{f\times f'}(\varphi\times\varphi')=(f\circ\varphi)\times(f'\circ\varphi')=\tau_{1,X}\big(F_f(\varphi)\otimes F_{f'}(\varphi')\big)\ .$$

Hence $\tau$ is an isomorphism of bifunctors.

(c) This follows from (b), applied to the lattice $T$ of subsets of $E$ and the lattice $T'$ of subsets of $F$. Then $F_T\cong k\mathcal{C}(-,E)$ and $T'\cong k\mathcal{C}(-,F)$. Moreover $T\times T'$ is isomorphic to the lattice of subsets of $E\sqcup F$. ∎

**19.5. Proposition.** *Let $T$ be a finite lattice.*

*(a) The map*

$$\upsilon:T\times T\to T\ ,\quad \upsilon(a,b)=a\vee b\ ,$$

*is a morphism in the category $\mathcal{L}$.*

*(b) The map $\upsilon$ induces a morphism $\mu:F_T\otimes F_T\to F_T$, defined for a finite set $X$ and maps $\varphi,\psi:X\to T$ by $\mu(\varphi\otimes\psi)=\varphi\vee\psi$, where $(\varphi\vee\psi)(x)=\varphi(x)\vee\psi(x)$, for $x\in X$.*

(c) *For finite sets $X$ and $Y$, let $\widehat{\mu} : F_T(X) \times F_T(Y) \to F_T(X \sqcup Y)$ be the bilinear pairing associated to $\mu$, in the sense of part (c) of Theorem 19.2. Then for $\varphi : X \to T$ and $\psi : Y \to T$, the element $\widehat{\mu}(\varphi, \psi) \in F_T(X \sqcup Y)$ is the function from $X \sqcup Y$ to $T$ equal to $\varphi$ on $X$ and to $\psi$ on $Y$.*

(d) *The triple $(F_T, \mu, \epsilon)$ is a commutative algebra in the tensor category $\mathcal{F}_k$, where $\epsilon : \underline{k} \to F_T$ is the morphism sending $1 \in \underline{k}(X) = k$ to the function $X \to T$ mapping any element of $X$ to $\hat{0}$, for any finite set $X$.*

**Proof :** (a) Clearly for $a, b, c, d \in T$, the join $(a, b) \vee (c, d) = (a \vee c, b \vee d)$ is mapped by $\upsilon$ to $(a \vee c) \vee (b \vee d) = (a \vee b) \vee (c \vee d)$, so $\upsilon\big((a, b) \vee (c, d)\big) = \upsilon(a, b) \vee \upsilon(c, d)$. Moreover the zero element of $T \times T$ is $(\hat{0}, \hat{0})$, and $\upsilon(\hat{0}, \hat{0}) = \hat{0} \vee \hat{0} = \hat{0}$.

(b) Follows from (a), by Theorem 11.7 and Theorem 19.4.

(c) One checks easily that $\binom{\Delta_X}{\emptyset}\varphi$ is the function from $X \sqcup Y$ to $T$ equal to $\varphi$ on $X$, and to $\hat{0}$ on $Y$. Similarly $\binom{\emptyset}{\Delta_Y}\psi$ is the map equal to $\hat{0}$ on $X$ and to $\psi$ on $Y$. Thus $\Big(\binom{\Delta_X}{\emptyset}\varphi\Big) \vee \Big(\binom{\emptyset}{\Delta_Y}\psi\Big)$ is the map equal to $\varphi$ on $X$ and to $\psi$ on $Y$.

(d) Clearly $\mu$ is associative, commutative, and $\epsilon$ is an identity element. □

**19.6. Theorem.** *Let $k$ be a commutative ring. Let $M$ and $N$ be correspondence functors over $k$.*

(a) *Let $E$ and $F$ be finite sets, and let $G = E \sqcup F$. Let $V$ be an $\mathcal{R}_E$-module and $W$ be an $\mathcal{R}_F$-module. Then there is an isomorphism of correspondence functors*

$$L_{E,V} \otimes L_{F,W} \cong L_{G, V\uparrow_E^G \otimes_k W\uparrow_F^G} \ ,$$

*where the $\mathcal{R}_G$-module structure on $V \uparrow_E^G \otimes_k W \uparrow_F^G$ is induced by the comultiplication $\nu : \mathcal{R}_G \to \mathcal{R}_G \otimes \mathcal{R}_G$ defined by $\nu(U) = U \otimes U$ for $U \in \mathcal{C}(G, G)$.*

(b) *If $M$ and $N$ are projective, then $M \otimes N$ is projective.*

(c) *If $E$ and $F$ are finite sets such that $M = \langle M(E) \rangle$ and $N = \langle N(F) \rangle$, then $M \otimes N = \langle (M \otimes N)(E \sqcup F) \rangle$.*

(d) *If $M$ and $N$ have bounded type, so has $M \otimes N$.*

(e) *If $M$ and $N$ are finitely generated, so is $M \otimes N$.*

**Proof :** (a) Since $(L_{E,V} \otimes L_{F,W})(G) = V \uparrow_E^G \otimes_k W \uparrow_F^G$, by adjunction, we get a morphism

$$\Phi : L_{G, V\uparrow_E^G \otimes_k W\uparrow_F^G} \to L_{E,V} \otimes L_{F,W}$$

which can be described as follows: for a finite set $X$,

$$L_{G, V\uparrow_E^G \otimes_k W\uparrow_F^G}(X) = k\mathcal{C}(X, G) \otimes_{\mathcal{R}_G} \Big(\big(k\mathcal{C}(G, E) \otimes_{\mathcal{R}_E} V\big) \otimes_k \big(k\mathcal{C}(G, F) \otimes_{\mathcal{R}_F} W\big)\Big)$$

and

$$(L_{E,V} \otimes L_{F,W})(X) = \big(k\mathcal{C}(X, E) \otimes_{\mathcal{R}_E} V\big) \otimes_k \big(k\mathcal{C}(X, F) \otimes_{\mathcal{R}_F} W\big) \ .$$

The morphism $\Phi_X$ sends the element

$$C \otimes_{\mathcal{R}_G} \big((A \otimes_{\mathcal{R}_E} v) \otimes_k (B \otimes_{\mathcal{R}_F} w)\big)$$

of $L_{G, V\uparrow_E^G \otimes_k W\uparrow_F^G}(X)$, where $C \in \mathcal{C}(X, G)$, $A \in \mathcal{C}(G, E)$, $v \in V$, $B \in \mathcal{C}(G, F)$, and $w \in W$, to the element

$$(CA \otimes_{\mathcal{R}_E} v) \otimes_k (CB \otimes_{\mathcal{R}_F} w)$$

of $(L_{E,V} \otimes L_{F,W})(X)$.

Conversely, there is a morphism $\Psi_X : (L_{E,V} \otimes L_{F,W})(X) \to L_{G,V\uparrow_E^G \otimes_k W\uparrow_F^G}(X)$ sending the element

$$(P \otimes_{\mathcal{R}_E} v) \otimes (Q \otimes_{\mathcal{R}_F} w)$$

of $(L_{E,V} \otimes L_{F,W})(X)$, where $P \in \mathcal{C}(X,E)$, $v \in V$, $Q \in \mathcal{C}(X,F)$, and $w \in W$, to the element

$$(P,Q) \otimes_{\mathcal{R}_G} \Big( (\begin{pmatrix} \Delta_E \\ \emptyset \end{pmatrix} \otimes_{\mathcal{R}_E} v) \otimes_k (\begin{pmatrix} \emptyset \\ \Delta_F \end{pmatrix} \otimes_{\mathcal{R}_F} w) \Big)$$

of $L_{G,V\uparrow_E^G \otimes_k W\uparrow_F^G}(X)$, where $(P,Q) \in \mathcal{C}(X, E \sqcup F)$, and where $\binom{\Delta_E}{\emptyset} \in \mathcal{C}(E \sqcup F, E)$ and $\binom{\emptyset}{\Delta_F} \in \mathcal{C}(E \sqcup F, F)$.

The map $\Psi_X$ is well defined, for if $R \in \mathcal{C}(E,E)$ and $S \in \mathcal{C}(F,F)$

$$\Psi_X\big((P \otimes_{\mathcal{R}_E} Rv) \otimes_k (Q \otimes_{\mathcal{R}_F} Sw)\big)$$

is equal to

$$\begin{aligned}
(P,Q) \otimes_{\mathcal{R}_G} \Big( (\begin{pmatrix} \Delta_E \\ \emptyset \end{pmatrix} \otimes_{\mathcal{R}_E} Rv) \otimes_k (\begin{pmatrix} \emptyset \\ \Delta_F \end{pmatrix} \otimes_{\mathcal{R}_F} Sw) \Big) &= \\
(P,Q) \otimes_{\mathcal{R}_G} \Big( (\begin{pmatrix} R \\ \emptyset \end{pmatrix} \otimes_{\mathcal{R}_E} v) \otimes_k (\begin{pmatrix} \emptyset \\ S \end{pmatrix} \otimes_{\mathcal{R}_F} w) \Big) &= \\
(P,Q) \otimes_{\mathcal{R}_G} \begin{pmatrix} R & \emptyset \\ \emptyset & S \end{pmatrix} \Big( (\begin{pmatrix} \Delta_E \\ \emptyset \end{pmatrix} \otimes_{\mathcal{R}_E} v) \otimes_k (\begin{pmatrix} \emptyset \\ \Delta_F \end{pmatrix} \otimes_{\mathcal{R}_F} w) \Big) &= \\
(PR, QS)\Big( (\begin{pmatrix} \Delta_E \\ \emptyset \end{pmatrix} \otimes_{\mathcal{R}_E} v) \otimes_k (\begin{pmatrix} \emptyset \\ \Delta_F \end{pmatrix} \otimes_{\mathcal{R}_F} w) \Big) &,
\end{aligned}$$

which is equal to $\Psi_X\big((PR \otimes_{\mathcal{R}_E} v) \otimes_k (QS \otimes_{\mathcal{R}_F} w)\big)$.

Moreover, if $Y$ is a finite set and $U \in \mathcal{C}(Y,X)$, then

$$\begin{aligned}
\Psi_X\Big(U\big((P \otimes_{\mathcal{R}_E} Rv) \otimes_k (Q \otimes_{\mathcal{R}_F} Sw)\big)\Big) &= \\
\Psi_X\big((UP \otimes_{\mathcal{R}_E} Rv) \otimes_k (UQ \otimes_{\mathcal{R}_F} Sw)\big) &= \\
(UP,UQ) \otimes_{\mathcal{R}_G} \Big( (\begin{pmatrix} \Delta_E \\ \emptyset \end{pmatrix} \otimes_{\mathcal{R}_E} v) \otimes_k (\begin{pmatrix} \emptyset \\ \Delta_F \end{pmatrix} \otimes_{\mathcal{R}_F} w) \Big) &= \\
U\Psi_X\big((P \otimes_{\mathcal{R}_E} Rv) \otimes_k (Q \otimes_{\mathcal{R}_F} Sw)\big) &.
\end{aligned}$$

It follows that the maps $\Psi_X$ define a morphism of correspondence functors

$$\Psi : L_{E,V} \otimes L_{F,W} \to L_{G,V\uparrow_E^G \otimes_k W\uparrow_F^G} .$$

Moreover, setting $u = (P \otimes_{\mathcal{R}_E} v) \otimes (Q \otimes_{\mathcal{R}_F} w)$

$$\begin{aligned}
\Phi_X\Psi_X(u) &= \Phi_X\bigg( ((P,Q) \otimes_{\mathcal{R}_G} \Big( (\begin{pmatrix} \Delta_E \\ \emptyset \end{pmatrix} \otimes_{\mathcal{R}_E} v) \otimes_k (\begin{pmatrix} \emptyset \\ \Delta_F \end{pmatrix} \otimes_{\mathcal{R}_F} w) \Big) \bigg) \\
&= \Big( (P,Q) \begin{pmatrix} \Delta_E \\ \emptyset \end{pmatrix} \otimes_{\mathcal{R}_E} v \Big) \otimes_k \Big( (P,Q) \begin{pmatrix} \emptyset \\ \Delta_F \end{pmatrix} \otimes_{\mathcal{R}_E} w \Big) \\
&= (P \otimes_{\mathcal{R}_E} v) \otimes (Q \otimes_{\mathcal{R}_F} w) = u ,
\end{aligned}$$

so $\Phi\Psi$ is equal to the identity morphism.

Similarly, setting $s = C \otimes_{\mathcal{R}_G} \big((A \otimes_{\mathcal{R}_E} v) \otimes_k (B \otimes_{\mathcal{R}_F} w)\big)$,

$$\begin{aligned}
\Psi_X \Phi_X(s) &= \Psi_X\big((CA \otimes_{\mathcal{R}_E} v) \otimes_k (CB \otimes_{\mathcal{R}_F} w)\big)\\
&= (CA, CB) \otimes_{\mathcal{R}_G} \Big(\big(\begin{pmatrix}\Delta_E\\ \emptyset\end{pmatrix} \otimes_{\mathcal{R}_E} v\big) \otimes_k \big(\begin{pmatrix}\emptyset\\ \Delta_F\end{pmatrix} \otimes_{\mathcal{R}_F} w\big)\Big)\\
&= C(A, B) \otimes_{\mathcal{R}_G} \Big(\big(\begin{pmatrix}\Delta_E\\ \emptyset\end{pmatrix} \otimes_{\mathcal{R}_E} v\big) \otimes_k \big(\begin{pmatrix}\emptyset\\ \Delta_F\end{pmatrix} \otimes_{\mathcal{R}_F} w\big)\Big)\\
&= C \otimes_{\mathcal{R}_G} (A, B)\Big(\big(\begin{pmatrix}\Delta_E\\ \emptyset\end{pmatrix} \otimes_{\mathcal{R}_E} v\big) \otimes_k \big(\begin{pmatrix}\emptyset\\ \Delta_F\end{pmatrix} \otimes_{\mathcal{R}_F} w\big)\Big)\\
&= C \otimes_{\mathcal{R}_G} \big((A \otimes_{\mathcal{R}_E} v) \otimes_k (B \otimes_{\mathcal{R}_F} w)\big) = s\ ,
\end{aligned}$$

so $\Psi\Phi$ is also equal to the identity morphism.

(b) It suffices to assume that $M = \bigoplus_{i \in I} k\mathcal{C}(-, E_i)$ and $N = \bigoplus_{j \in J} k\mathcal{C}(-, F_j)$, where $I$ and $J$ are sets, and $E_i$, for $i \in I$, and $F_j$, for $j \in J$, are finite sets. In this case

$$M \otimes N \cong \bigoplus_{i \in I,\, j \in J} k\mathcal{C}(-, E_i) \otimes k\mathcal{C}(-, F_j) \cong \bigoplus_{i \in I,\, j \in J} k\mathcal{C}(-, E_i \sqcup F_j)\ ,$$

so $M \otimes N$ is projective.

(c) Suppose that $M = \langle M(E) \rangle$ and $N = \langle N(F) \rangle$. Equivalently the counit morphisms $L_{E,M(E)} \to M$ and $L_{F,N(F)} \to N$ are surjective. Then $M \otimes N$ is a quotient of

$$L_{E,M(E)} \otimes L_{F,N(F)} \cong L_{E \sqcup F, M(E)\uparrow_E^{E \sqcup F} \otimes_k N(F)\uparrow_F^{E \sqcup F}}\ .$$

Since $L_{E \sqcup F, M(E)\uparrow_E^{E \sqcup F} \otimes_k N(F)\uparrow_F^{E \sqcup F}}$ is generated by its evaluation at $E \sqcup F$, so does $M \otimes N$.

(d) This follows from (c).

(e) With the assuption of (c), if moreover $M(E)$ and $N(F)$ are finitely generated $k$-modules, so is $M(E) \uparrow_E^{E \sqcup F} \otimes_k N(F) \uparrow_F^{E \sqcup F}$. Therefore, if $M$ and $N$ are finitely generated, so is $M \otimes N$. □

**19.7. Theorem.** *Let $k$ be a commutative ring. Let $M$ and $N$ be correspondence functors over $k$.*

*(a) Let $E$ and $F$ be finite sets. If $M = \langle M(E) \rangle$, then $M_F = \langle M_F(E) \rangle$. Therefore, if $M$ has bounded type, so has $M_F$. If $M$ is finitely generated, so is $M_F$.*

*(b) Let $F$ be a finite set. There is an isomorphism of correspondence functors $\mathcal{H}\big(k\mathcal{C}(-, F), N\big) \cong N_F$.*

*(c) Assume that $k$ is noetherian. If $M$ is finitely generated and if $N$ has bounded type (resp. is finitely generated), then $\mathcal{H}(M, N)$ has bounded type (resp. is finitely generated).*

**Proof :** (a) Saying that $M = \langle M(E) \rangle$ is equivalent to saying that

$$M(X) = k\mathcal{C}(X, E)M(E)$$

for each finite set $X$. Replacing $X$ by $X \sqcup F$ gives

$$M(X \sqcup F) = k\mathcal{C}(X \sqcup F, E)M(E)\ .$$

Now any $U \in \mathcal{C}(X \sqcup F, E)$ can be written in matrix form $\begin{pmatrix} V \\ W \end{pmatrix}$, where $V \in \mathcal{C}(X, E)$ and $W \in \mathcal{C}(F, E)$. Moreover for $m \in M(E)$

$$\begin{pmatrix} V \\ W \end{pmatrix}(m) = \begin{pmatrix} V & \emptyset \\ \emptyset & \Delta_F \end{pmatrix} \begin{pmatrix} \Delta_E \\ W \end{pmatrix}(m)$$

and $\begin{pmatrix} V & \emptyset \\ \emptyset & \Delta_F \end{pmatrix} \begin{pmatrix} \Delta_E \\ W \end{pmatrix}(m)$ is the image of $\begin{pmatrix} \Delta_E \\ W \end{pmatrix}(m) \in M_F(E)$ by the correspondence $V$, in the functor $M_F$. It follows that $\begin{pmatrix} V \\ W \end{pmatrix}(m) \in k\mathcal{C}(X, E)M_F(E)$, for any $m \in M(E)$. Thus $UM(E) \subseteq k\mathcal{C}(X, E)M_F(E)$, for any $U \in \mathcal{C}(X \sqcup F, E)$. Hence $M(X \sqcup F) = M_F(X) = k\mathcal{C}(X, E)M_F(E)$, as was to be shown. The last two assertions are then clear.

(b) Let $X$ be a finite set. Then

$$\begin{aligned} \mathcal{H}\big(k\mathcal{C}(-, E), N\big)(X) &= \mathrm{Hom}_{\mathcal{F}_k}\big(k\mathcal{C}(-, E), N_X\big) \\ &\cong N_X(E) = N(E \sqcup X) \cong N(X \sqcup E) = N_E(X)\ , \end{aligned}$$

and it is straightforward to check that the resulting isomorphism

$$\mathcal{H}\big(k\mathcal{C}(-, E), N\big)(X) \cong N_E(X)$$

is compatible with correspondences, i.e. that it yields an isomorphism of correspondence functors $\mathcal{H}\big(k\mathcal{C}(-, E), N\big) \to N_E$.

(c) If $M$ is finitely generated, then $M$ is a quotient of a finite direct sum $\bigoplus_{i=1}^{n} k\mathcal{C}(-, E_i)$ of representable functors. Then

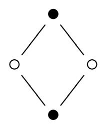
$$\mathcal{H}(M, N) \hookrightarrow \bigoplus_{i=1}^{n} \mathcal{H}\big(k\mathcal{C}(-, E_i), N\big) \cong \bigoplus_{i=1}^{n} N_{E_i}\ .$$

If $N$ has bounded type, then each $N_{E_i}$ has bounded type also, by (a), so the finite direct sum $\bigoplus_{i=1}^{n} N_{E_i}$ also has bounded type. Then $\mathcal{H}(M, N)$ is a subfunctor of a functor of bounded type. If $k$ is noetherian, this implies that $\mathcal{H}(M, N)$ has bounded type, by Corollary 10.5. The same argument with "bounded type" replaced by "finitely generated" goes through, completing the proof. ∎

## 20. Examples of decompositions

We state here without complete proofs a list of examples of decomposition of functors $F_T$ associated to some particular lattices $T$. We assume that $k$ is a field, as this allows for much simpler arguments. We emphasize however that the decompositions we obtain as direct sums of fundamental functors actually hold over an arbitrary commutative ring $k$.

**20.1. Example.** Let $T = \lozenge$ be the lozenge, in other words the lattice of subsets of a set of cardinality 2:

$\Diamond =$ 

Then we know from Theorem 15.19 that $F_\Diamond$ splits as a direct sum

$$F_\Diamond \cong \mathbb{S}_0 \oplus 3\mathbb{S}_1 \oplus 2\mathbb{S}_2 \oplus L$$

for some subfunctor $L$, and we also know that the direct sum

$$\mathbb{S}_0 \oplus 3\mathbb{S}_1 \oplus 2\mathbb{S}_2$$

corresponding to totally ordered subsets of $\Diamond$ lies in the subfunctor $H_\Diamond$, as no such subset contains the two irreducible elements of $\Diamond$ (figured with an empty circle in the above picture).

We can evaluate this at a set $X$ of cardinality $x$, and take dimensions over $k$. Using Corollary 17.17, this gives

$$4^x = 1^x + 3(2^x - 1^x) + 2(3^x - 2 \cdot 2^x + 1^x) + \dim_k L(X) \ .$$

It follows that

$$\dim_k L(X) = 4^x - 2 \cdot 3^x + 2^x \ .$$

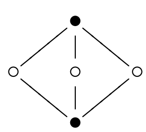
Moreover we know that $F_\Diamond / H_\Diamond$ maps surjectively on the fundamental functor $\mathbb{S}_{\circ\circ}$ associated to the (opposite) poset of irreducible elements of $\Diamond$, that is, a set of cardinality 2 ordered by the equality relation. The set $G$ is the whole of $T$ in this case, so $\dim_k \mathbb{S}_{\circ\circ}(X) = 4^x - 2 \cdot 3^x + 2^x$.

It follows that $L$ maps surjectively onto $\mathbb{S}_{\circ\circ}$, and since $\dim_k L(X) = \dim_k \mathbb{S}_{\circ\circ}(X)$ for any finite set $X$, this surjection is an isomorphism. Hence

$$F_\Diamond \cong \mathbb{S}_0 \oplus 3\mathbb{S}_1 \oplus 2\mathbb{S}_2 \oplus \mathbb{S}_{\circ\circ} \ .$$

Again, the argument given here works only over a field $k$, but one can show that this direct sum decomposition of $F_\Diamond$ actually holds over any commutative ring $k$. In particular, as $\Diamond$ is distributive, $F_\Diamond$ is projective by Theorem 11.11. It follows that $\mathbb{S}_{\circ\circ}$ is a projective object in $\mathcal{F}_k$. When $k$ is self-injective, $\mathbb{S}_{\circ\circ}$ is also injective in $\mathcal{F}_k$.

**20.2. Example.** Let $T$ be the following lattice:

$T =$ 

We know from Theorem 15.19 that $F_T$ admits a direct summand isomorphic to

$$\mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 3\mathbb{S}_2 \ .$$

Moreover, there are three obvious sublattices of $T$ isomorphic to $\Diamond$, which provide three direct summands of $F_T$ isomorphic to $\mathbb{S}_{\circ\circ}$. Thus we have a decomposition

$$F_T \cong \mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 3\mathbb{S}_2 \oplus 3\mathbb{S}_{\circ\circ} \oplus M$$

for some subfunctor $M$ of $F_T$. Taking the dimensions of the evaluations at a set $X$ of cardinality $x$ gives

$$5^x = 1^x + 4(2^x - 1^x) + 3(3^x - 2 \cdot 2^x + 1^x) + 3(4^x - 2 \cdot 3^x + 2^x) + \dim_k M(X) \ .$$

It follows that

$$\dim_k M(X) = 5^x - 3 \cdot 4^x + 3 \cdot 3^x - 2^x \ .$$

We also know that $F_T/H_T$ maps surjectively onto the fundamental functor $\mathbb{S}_{\circ\circ\circ}$ associated to the (opposite) poset of its irreducible elements, that is, a set of cardinality 3 ordered by the equality relation. The set $G$ is the whole of $T$ in this case, so

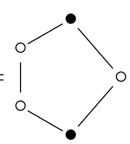
$$\dim_k \mathbb{S}_{\circ\circ\circ}(X) = 5^x - 3\cdot 4^x + 3\cdot 3^x - 2^x \ .$$

Moreover, all the direct summands of $F_T$ different from $M$ in the above decomposition are contained in $H_T$, since none of the corresponding sublattices contains all the irreducible elements of $T$. Thus $M$ maps onto $\mathbb{S}_{\circ\circ\circ}$, and since $\dim_k M(X) = \dim_k \mathbb{S}_{\circ\circ\circ}(X)$, this surjection is an isomorphism. Hence we get

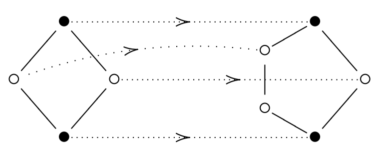
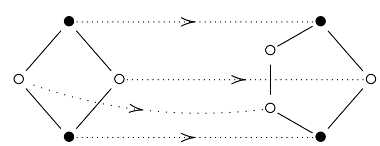
$$F_T \cong \mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 3\mathbb{S}_2 \oplus 3\mathbb{S}_{\circ\circ} \oplus \mathbb{S}_{\circ\circ\circ} \ .$$

As in the the previous example, our argument works only when $k$ is a field, but one can show that the result still holds over an arbitrary commutative ring $k$. It should be noted that all the summands in this decomposition of $F_T$, except possibly $\mathbb{S}_{\circ\circ\circ}$, are projective functors. Since the lattice $T$ is not distributive, the functor $F_T$ is not projective, thus $\mathbb{S}_{\circ\circ\circ}$ is actually not projective either.

**20.3. Example.** Let $D$ be the following lattice:

$D =$ 

Again, we know from Theorem 15.19 that $F_T$ admits a direct summand isomorphic to $\mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 4\mathbb{S}_2 \oplus \mathbb{S}_3$. Moreover, there are two inclusions

 and 

of the lattice $\lozenge$ into $D$, which yield two direct summands of $F_D$ isomorphic to $\mathbb{S}_{\circ\circ}$. So there is a decomposition

$$F_D \cong \mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 4\mathbb{S}_2 \oplus \mathbb{S}_3 \oplus 2\mathbb{S}_{\circ\circ} \oplus N$$

for a suitable subfunctor $N$ of $F_D$. As in the previous examples, the subfunctor $N$ maps surjectively onto the fundamental functor $\mathbb{S}_{\substack{\circ\\ \vdots\\ \circ}\circ}$ associated to the (opposite) poset $\substack{\circ\\ \vdots\\ \circ}\circ$ of irreducible elements of $D$. The set $G$ is the whole of $T$ in this case also. Taking dimensions of the evaluations of these functors at a set $X$ of cardinality $x$ gives

$$\begin{aligned} 5^x = 1^x &+ 4(2^x - 1^x) + 4(3^x - 2\cdot 2^x + 1^x) \\ &+ (4^x - 3\cdot 3^x + 3\cdot 2^x - 1^x) + 2(4^x - 2\cdot 3^x + 2^x) + \dim_k N(X) \ , \end{aligned}$$

hence

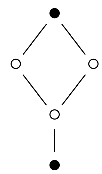
$$\dim_k N(X) = 5^x - 3\cdot 4^x + 3\cdot 3^x - 2^x \ .$$

This is also equal to $\dim_k \mathbb{S}_{\substack{\circ\\ \vdots\\ \circ}\circ}(X)$, so $N \cong \mathbb{S}_{\substack{\circ\\ \vdots\\ \circ}\circ}$, and we get

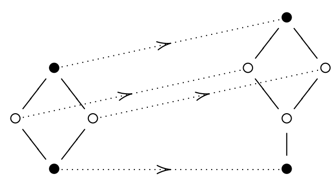
$$F_D \cong \mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 4\mathbb{S}_2 \oplus \mathbb{S}_3 \oplus 2\mathbb{S}_{\circ\circ} \oplus \mathbb{S}_{\substack{\circ\\ \vdots\\ \circ}\circ} \ .$$

Again this decomposition holds over any commutative ring $k$. As $D$ is not distributive (the lattice $D$ and the lattice $T$ of the previous example are the smallest non-distributive lattices), the functor $\mathbb{S}_{\substack{\circ\\ \vdots\\ \circ}\circ}$ is not projective either.

**20.4. Example.** Let $C$ be the following lattice:

$C =$ 

We know from Theorem 15.19 that $F_C$ admits a direct summand isomorphic to $\mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 5\mathbb{S}_2 \oplus 2\mathbb{S}_3$ and this summand is contained in $H_C$. The inclusion



of $\Diamond$ in $C$ yields a direct summand of $F_C$ isomorphic to $\mathbb{S}_{\circ\circ}$, also contained in $H_C$. So there is a decomposition

$$F_C \cong \mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 5\mathbb{S}_2 \oplus 2\mathbb{S}_3 \oplus \mathbb{S}_{\circ\circ} \oplus Q$$

for some direct summand $Q$ of $F_C$. Evaluating at a set $X$ of cardinality $x$ and taking dimensions gives

$$\begin{aligned} 5^x = 1^x + 4(2^x - 1^x) + 5(3^x - 2 \cdot 2^x + 1^x) \\ + 2(4^x - 3 \cdot 3^x + 3 \cdot 2^x - 1^x) + (4^x - 2 \cdot 3^x + 2^x) + \dim_k Q(X) \,. \end{aligned}$$

This gives

$$\dim_k Q(X) = 5^x - 3 \cdot 4^x + 3 \cdot 3^x - 2^x \,.$$

Now $F_C$ maps surjectively onto the fundamental functor $\mathbb{S}_{\substack{\circ \\ /\backslash \\ \circ\ \circ}}$ associated to the opposite poset of its irreducible elements, and the corresponding subset $G$ of $C$ is the whole of $C$. It follows that

$$\dim_k \mathbb{S}_{\substack{\circ \\ /\backslash \\ \circ\ \circ}}(X) = 5^x - 3 \cdot 4^x + 3 \cdot 3^x - 2^x \,,$$

thus $Q \cong \mathbb{S}_{\substack{\circ \\ /\backslash \\ \circ\ \circ}}$. Finally, we get a decomposition

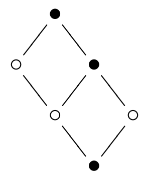
$$F_C \cong \mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 5\mathbb{S}_2 \oplus 2\mathbb{S}_3 \oplus \mathbb{S}_{\circ\circ} \oplus \mathbb{S}_{\substack{\circ \\ /\backslash \\ \circ\ \circ}} \,.$$

Again, this decomposition actually holds for an arbitrary commutative ring $k$. Since $C$ is distributive, $F_C$ is projective and we conclude that $\mathbb{S}_{\substack{\circ \\ /\backslash \\ \circ\ \circ}}$ is projective. Taking dual functors, we get a decomposition

$$F_{C^{op}} \cong \mathbb{S}_0 \oplus 4\mathbb{S}_1 \oplus 5\mathbb{S}_2 \oplus 2\mathbb{S}_3 \oplus \mathbb{S}_{\circ\circ} \oplus \mathbb{S}_{\substack{\circ\ \circ \\ \backslash/ \\ \circ}} \,,$$

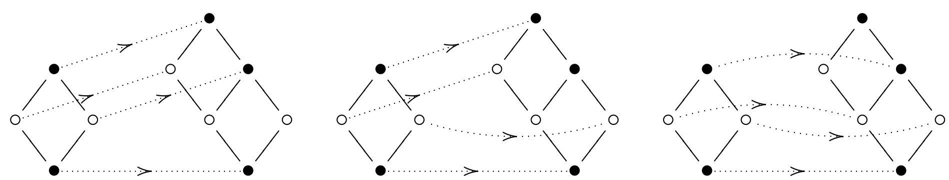
so $\mathbb{S}_{\substack{\circ\ \circ \\ \backslash/ \\ \circ}}$ is also projective.

**20.5. Example.** Let $P$ be the following lattice:

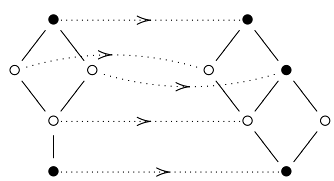
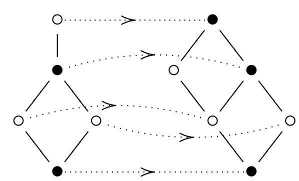
$P =$ 

that is, the direct product of a totally ordered lattice of cardinality 3 with a totally ordered lattice of cardinality 2.

We know from Theorem 15.19 that $F_P$ admits a direct summand isomorphic to $\mathbb{S}_0 \oplus 5\mathbb{S}_1 \oplus 7\mathbb{S}_2 \oplus 3\mathbb{S}_3$ and this summand is contained in $H_P$. The inclusions



of $\Diamond$ in $P$ yield 3 direct summands of $F_P$ isomorphic to $\mathbb{S}_{\circ\circ}$, and contained in $H_P$. Moreover, the inclusions

 and 

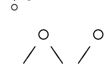
of $C$ and $C^{op}$ in $P$ yield direct summands $\mathbb{S}_{\vee}$ and $\mathbb{S}_{\wedge}$ of $F_P$, also contained in $H_P$. Hence there is a direct summand $U$ of $F_P$ such that

$$F_P \cong \mathbb{S}_0 \oplus 5\mathbb{S}_1 \oplus 7\mathbb{S}_2 \oplus 3\mathbb{S}_3 \oplus 3\mathbb{S}_{\circ\circ} \oplus \mathbb{S}_{\vee} \oplus \mathbb{S}_{\wedge} \oplus U \ .$$

Since the lattice $P$ is distributive, the functor $F_P$ is projective, hence $U$ is projective. Now $F_P$ maps surjectively onto the fundamental functor $\mathbb{S}_{\vdots\,\circ}$, and $H_P$ is contained in the kernel of this surjection. It follows that $U$ maps surjectively onto $\mathbb{S}_{\vdots\,\circ}$, which is a simple functor, as $k$ is a field and the poset $\vdots\,\circ$ has no nontrivial automorphisms.

Using the fact that $\dim \mathrm{Hom}_{\mathcal{F}_k}(M,N) = \dim \mathrm{Hom}_{\mathcal{F}_k}(N,M)$ if M is projective (by Theorem 9.7), a tedious analysis shows that $U$ is indecomposable, so $U$ is *a projective cover of the simple functor* $\mathbb{S}_{\vdots\,\circ}$. Moreover, using Theorem 18.4, one can show that the functor $U$ is uniserial, with a filtration

$$\underbrace{0 \subset W}_{\mathbb{S}_{\vdots\,\circ}} \underbrace{\subset V}_{\mathbb{S}_{\wedge\vee}} \underbrace{\subset U}_{\mathbb{S}_{\vdots\,\circ}} \ ,$$

where $W \cong U/V \cong \mathbb{S}_{\vdots\,\circ}$, and $V/W$ is isomorphic to the simple functor $\mathbb{S}_{\wedge\vee}$ associated to the poset  of cardinality 4. An easy consequence of this is that (when $k$ is a field)

$$\mathrm{Ext}^1_{\mathcal{F}_k}(\mathbb{S}_{\vdots\,\circ}, \mathbb{S}_{\wedge\vee}) \cong \mathrm{Ext}^1_{\mathcal{F}_k}(\mathbb{S}_{\wedge\vee}, \mathbb{S}_{\vdots\,\circ}) \cong k \ .$$

# Part 3

# APPENDICES

## 21. Lattices associated to posets

Posets occur in the parametrization of simple functors. In order to be able to use our results on lattices, we needed to associate a lattice $T$ to any given finite poset $(E,R)$. The easiest choice is the lattice $I_\downarrow(E,R)$. The purpose of this section is to describe the various possibilities for $T$.

We let $(E,R)$ be a finite poset. Throughout this section, we write the order relation $\leq_R$. Also, we use a subscript $E$ for intervals in $E$.

If $A\subseteq E$, we let $\mathrm{Ub}\,A$ be the set of all upper bounds of $A$ and $\mathrm{Lb}\,A$ the set of all lower bounds of $A$, that is,

$$\mathrm{Ub}\,A=\{e\in E\mid a\leq_R e,\ \forall a\in A\}\quad\text{and}\quad \mathrm{Lb}\,A=\{e\in E\mid e\leq_R a,\ \forall a\in A\}\,.$$

Clearly $\mathrm{Ub}\,A$ is an upper ideal in $E$ while $\mathrm{Lb}\,A$ is a lower ideal in $E$.

Let $I_\downarrow(E,R)$ be the lattice of all lower ideals in $E$, with respect to unions and intersections. The irreducible elements in $I_\downarrow(E,R)$ are the principal ideals $]\cdot,e]_E$, where $e\in E$. Thus the poset $E$ is isomorphic to the poset of all irreducible elements in $I_\downarrow(E,R)$ by mapping $e\in E$ to the principal ideal $]\cdot,e]_E$. Note that $I_\downarrow(E,R)$ is actually a distributive lattice. For more details, see Theorem 3.4.1 and Proposition 3.4.2 in [**St**].

**21.1. Convention.** Recall our Convention 5.6. We identify $E$ with its image in $I_\downarrow(E,R)$ via the map $e\mapsto\,]\cdot,e]_E$. Thus we view $E$ as a subposet of $I_\downarrow(E,R)$.

**21.2. Definition.** *A* closure operation *on $I_\downarrow(E,R)$ is an order-preserving map*

$$I_\downarrow(E,R)\to I_\downarrow(E,R)\,,\qquad A\mapsto\widetilde{A}\,,$$

*such that $A\subseteq\widetilde{A}$ and $\widetilde{\widetilde{A}}=\widetilde{A}$ for all $A\in I_\downarrow(E,R)$. We say that $A\in I_\downarrow(E,R)$ is* closed *if $\widetilde{A}=A$.*

We shall require that every principal ideal $]\cdot,e]_E$ is closed. In other words, the closure operation is the identity on $E$ (identified with a subposet of $I_\downarrow(E,R)$).

As before, we say that $T$ is *generated* by $E$ if any element of $T$ is a join of elements of $E$. For instance, if we view $E$ as a subposet of $I_\downarrow(E,R)$ as above, then clearly $E$ is a full subposet and $I_\downarrow(E,R)$ is generated by $E$ since any lower ideal is a union of principal ideals.

**21.3. Lemma.** *Let $(E,R)$ be a finite poset. Let $A\mapsto\widetilde{A}$ be a closure operation on $I_\downarrow(E,R)$ which is the identity on $E$. Let $T$ be the subset of $I_\downarrow(E,R)$ consisting of all closed elements of $I_\downarrow(E,R)$, viewed as a poset via inclusion.*

*(a) $T$ is a lattice. Explicitly, if $A,B\in T$, then*

$$A\wedge B=A\cap B\qquad\text{and}\qquad A\vee B=\widetilde{A\cup B}\,.$$

*Moreover, $E$ is a full subposet of $T$.*

*(b) The map $\pi=\pi_T:I_\downarrow(E,R)\to T$ defined by $\pi_T(A)=\widetilde{A}$ is a surjective map of posets which preserves joins and which is the identity on $E$ (identified with a full subposet of $I_\downarrow(E,R)$ and $T$). In other words, $\pi_T$ is a morphism in the category $\mathcal{L}$ (see Definition 11.5).*

*(c) $T$ is generated by $E$ (identified with a subposet of $T$).*

**Proof :** (a) Since closure is order-preserving, we have $\widetilde{A\cap B}\subseteq\widetilde{A}=A$ and similarly $\widetilde{A\cap B}\subseteq B$. Therefore $\widetilde{A\cap B}\subseteq A\cap B$, forcing equality $\widetilde{A\cap B}=A\cap B$.

Now if $C \in T$ satisfies $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$, hence $\widetilde{A \cup B} \subseteq \widetilde{C} = C$. This shows that $\widetilde{A \cup B}$ is the least upper bound of $A$ and $B$.

Finally, $E$ is a full subposet of $T$ because $]\cdot, e]_E \subseteq ]\cdot, f]_E$ if and only if $e \leq_R f$.

(b) Clearly $\pi$ is a map of posets because the closure operation is order-preserving. It is the identity on $E$ because so is the closure operation. If $A, B \in I_\downarrow(E,R)$, then

$$\pi(A \cup B) = \widetilde{A \cup B} = \widetilde{\widetilde{A} \cup \widetilde{B}} = \widetilde{A} \vee \widetilde{B} = \pi(A) \vee \pi(B) \ ,$$

so $\pi$ preserves joins.

(c) Since $I_\downarrow(E,R)$ is generated by $E$, so is its image $T$, because $\pi$ preserves joins and is the identity on $E$. □

We now prove the converse.

**21.4. Proposition.** *Let $(E,R)$ be a finite poset. Let $T$ be a lattice with order relation $\leq_T$, containing $E$ as a full subposet, and suppose that $T$ is generated by $E$.*

*(a) There is a unique surjective map of posets*

$$\pi = \pi_T : I_\downarrow(E,R) \longrightarrow T$$

*which preserves joins and which is the identity on $E$ (viewed as a subposet of $I_\downarrow(E,R)$).*

*(b) For every $t \in T$, the inverse image $\pi^{-1}(t)$ has a greatest element.*

*(c) If $A \in I_\downarrow(E,R)$ with $\pi(A) = t$, let $\widetilde{A}$ be the greatest element of $\pi^{-1}(t)$. Then*

$$\widetilde{A} = \{e \in E \mid e \leq_T t\} \ .$$

*(d) The map $A \mapsto \widetilde{A}$ is a closure operation on $I_\downarrow(E,R)$ which is the identity on $E$.*

*(e) The poset of closed elements in $I_\downarrow(E,R)$ is isomorphic to $T$ via $\pi$.*

**Proof :** (a) Let $A \in I_\downarrow(E,R)$ and write $A = \{e_1, \ldots, e_k\}$. Then we have $A = ]\cdot, e_1]_E \cup \ldots \cup ]\cdot, e_k]_E$ and we define

$$\pi(A) = e_1 \vee \ldots \vee e_k \in T \ .$$

It follows that $\pi(A \cup B) = \pi(A) \vee \pi(B)$, for all $A, B \in I_\downarrow(E,R)$. In the special case when $A \subseteq B$, we get $\pi(B) = \pi(A) \vee \pi(B)$, that is, $\pi(A) \leq_T \pi(B)$. Therefore $\pi$ is a map of posets. It is surjective because $T$ is generated by $E$. To prove that it is the identity on $E$, let $A = ]\cdot, e_k]_E = \{e_1, \ldots, e_k\}$. Then $e_i \leq_R e_k$ for all $i$, hence $e_i \leq_T e_k$ because $E$ is a subposet of $T$, and therefore $\pi(A) = e_1 \vee \ldots \vee e_k = e_k$. The uniqueness of $\pi$ follows from the fact that it is the identity on $E$ and that it preserves joins.

(b) If $A, B \in \pi^{-1}(t)$, then $\pi(A \cup B) = \pi(A) \vee \pi(B) = t \vee t = t$. Therefore $A \cup B \in \pi^{-1}(t)$. It follows that the union of all $A \in \pi^{-1}(t)$ is in $\pi^{-1}(t)$ and is its greatest element.

(c) Let $A \in I_\downarrow(E,R)$ with $\pi(A) = t$ and write $\breve{A} = \{e \in E \mid e \leq_T t\}$. We want to show that $\breve{A} = \widetilde{A}$. If $e \in \widetilde{A}$, then $]\cdot, e]_E \subseteq \widetilde{A}$ (because the elements of $I_\downarrow(E,R)$ are lower ideals). Since $\pi$ is a map of posets and is the identity on $E$, we obtain $e \leq_T \pi(\widetilde{A}) = t$, hence $e \in \breve{A}$. Thus $\widetilde{A} \subseteq \breve{A}$. On the other hand, $\pi(\breve{A}) = \bigvee\limits_{e \leq_T t} e = t$ because $T$ is generated by $E$. Therefore $\breve{A} \in \pi^{-1}(t)$. Since $\widetilde{A}$ is the greatest element of $\pi^{-1}(t)$, we obtain $\breve{A} \subseteq \widetilde{A}$.

(d) Since $\widetilde{A}$ is the greatest element of $\pi^{-1}(\pi(A))$, it is clear that $A \subseteq \widetilde{A}$ and $\widetilde{\widetilde{A}} = \widetilde{A}$. If $A \subseteq B$ with $\pi(A) = t$ and $\pi(B) = v$, then $t \leq_T v$ by part (a). The characterization of $\widetilde{A}$ given in (c) implies that

$$\widetilde{A} = \{e \in E \mid e \leq_T t\} \subseteq \{e \in E \mid e \leq_T v\} = \widetilde{B} \,,$$

so the closure operation is order-preserving. To show that it is the identity on $E$, let $x \in E$, which is identified with the principal ideal $A = ]\cdot, x]_E \in I_\downarrow(E, R)$. Then $\pi(A) = x$ because $\pi$ is the identity on $E$. Then $\widetilde{A} = \{e \in E \mid e \leq_T x\}$ by (c) and since $E$ is a full subposet of $T$, we obtain

$$\widetilde{A} = \{e \in E \mid e \leq_T x\} = \{e \in E \mid e \leq_R x\} = ]\cdot, x]_E = A \,,$$

showing that the closure is indeed the identity on $E$.

(e) Since $\pi$ is surjective and each fibre $\pi^{-1}(t)$ has a unique closed element (its greatest element), the restriction of $\pi$ to closed elements is a bijection. Now $\pi$ is order-preserving and we have to prove that its inverse is also order-preserving. So suppose that $A$ and $B$ are closed, with $\pi(A) \leq_T \pi(B)$. Again, the characterization of $\widetilde{A}$ given in (c) implies that $\widetilde{A} \subseteq \widetilde{B}$, that is, $A \subseteq B$. □

**21.5. Corollary.** *For any given finite poset $E$, there is a bijection between the set of isomorphism classes of lattices generated by the full subposet $E$ and the set of all closure operations on $I_\downarrow(E, R)$.*

**Proof :** This follows immediately from Lemma 21.3 and Proposition 21.4. □

Now we construct a lattice $L(E, R)$ which will turn out to be the smallest lattice generated by $E$. It will be associated with the closure operation $\mathrm{Lb}\,\mathrm{Ub}$ on $I_\downarrow(E, R)$. We first check this.

**21.6. Lemma.** *Let $E$ be a finite poset.*

*(a)* $\mathrm{Ub}$ *and* $\mathrm{Lb}$ *are order-reversing.*
*(b)* $\mathrm{Ub}\,\mathrm{Lb}\,\mathrm{Ub} = \mathrm{Ub}$ *and* $\mathrm{Lb}\,\mathrm{Ub}\,\mathrm{Lb} = \mathrm{Lb}$.
*(c) The map $A \mapsto \mathrm{Lb}\,\mathrm{Ub}\,A$ is a closure operation on $I_\downarrow(E, R)$ which is the identity on $E$.*

**Proof :** (a) follows from the definitions.

(b) Let $A$ be a lower ideal and $B$ an upper ideal of $E$. It is clear that $B \subseteq \mathrm{Ub}\,\mathrm{Lb}\,B$ and, applying this to $B = \mathrm{Ub}\,A$, we get $\mathrm{Ub}\,A \subseteq \mathrm{Ub}\,\mathrm{Lb}\,\mathrm{Ub}\,A$. Similarly $A \subseteq \mathrm{Lb}\,\mathrm{Ub}\,A$ and, applying $\mathrm{Ub}$, we get $\mathrm{Ub}\,A \supseteq \mathrm{Ub}\,\mathrm{Lb}\,\mathrm{Ub}\,A$. It follows that $\mathrm{Ub}\,A = \mathrm{Ub}\,\mathrm{Lb}\,\mathrm{Ub}\,A$. The equality $\mathrm{Lb}\,\mathrm{Ub}\,\mathrm{Lb}\,B = \mathrm{Lb}\,B$ is proved in the same manner.

(c) The inclusion $A \subseteq \mathrm{Lb}\,\mathrm{Ub}\,A$ has already been observed and the equality $\mathrm{Lb}\,\mathrm{Ub}\,\mathrm{Lb}\,\mathrm{Ub}\,A = \mathrm{Lb}\,\mathrm{Ub}\,A$ follows from (b). Now if $]\cdot, x]_E$ is a principal ideal, then $\mathrm{Ub}\,]\cdot, x]_E = [x, \cdot[_E$ and $\mathrm{Lb}\,\mathrm{Ub}\,]\cdot, x]_E = ]\cdot, x]_E$, so $\mathrm{Lb}\,\mathrm{Ub}$ is the identity on $E$. □

We define $L(E, R)$ to be the lattice associated with the closure operation $\mathrm{Lb}\,\mathrm{Ub}$ by the procedure of Lemma 21.3. For any $A \in I_\downarrow(E, R)$, we write $\overline{A} = \mathrm{Lb}\,\mathrm{Ub}\,A$. Thus $L(E, R)$ consists of all lower ideals which are closed, that is, of the form $\overline{A}$ for some $A \in I_\downarrow(E, R)$.

**21.7. Lemma.** *Let $L(E,R)$ be the lattice associated with the closure operation* $\mathrm{Lb}\,\mathrm{Ub}$ *in* $I_\downarrow(E,R)$.

(a) *$L(E,R)$ is the subset of $I_\downarrow(E,R)$ consisting of all lower ideals of the form $\mathrm{Lb}\,B$ for some upper ideal $B$ of $E$.*

(b) *$L(E,R)$ is the subset of $I_\downarrow(E,R)$ consisting of all intersections of principal ideals (with the usual convention that an empty intersection yields the whole set $E$).*

(c) *The join operation in $L(E,R)$ is described as follows. If $B_1$ and $B_2$ are two upper ideals of $E$ such that $\mathrm{Ub}\,\mathrm{Lb}\,B_1 = B_1$ and $\mathrm{Ub}\,\mathrm{Lb}\,B_2 = B_2$, then $\mathrm{Lb}\,B_1 \vee \mathrm{Lb}\,B_2 = \mathrm{Lb}(B_1 \cap B_2)$.*

**Proof :** (a) We have $\mathrm{Lb}\,B = \mathrm{Lb}\,\mathrm{Ub}\,\mathrm{Lb}\,B$ by part (b) of Lemma 21.6.

(b) $\mathrm{Lb}\,B = \bigcap\limits_{b\in B} ]\cdot, b]_E$.

(c) Note that $B_1 \cap B_2 = \mathrm{Ub}\,\mathrm{Lb}\,B_1 \cap \mathrm{Ub}\,\mathrm{Lb}\,B_2 = \mathrm{Ub}(\mathrm{Lb}\,B_1 \cup \mathrm{Lb}\,B_2)$. Therefore

$$\mathrm{Lb}(B_1 \cap B_2) = \mathrm{Lb}\,\mathrm{Ub}(\mathrm{Lb}\,B_1 \cup \mathrm{Lb}\,B_2) = \overline{\mathrm{Lb}\,B_1 \cup \mathrm{Lb}\,B_2} = \mathrm{Lb}\,B_1 \vee \mathrm{Lb}\,B_2$$

using Lemma 21.3. □

Now we prove that any lattice generated by $E$ is sandwiched between $I_\downarrow(E,R)$ and $L(E,R)$.

**21.8. Theorem.** *Let $(E,R)$ be a finite poset, let $T$ be a lattice containing $E$ as a full subposet, and suppose that $T$ is generated by $E$.*

(a) *There is a unique surjective map of posets*

$$\pi_T : I_\downarrow(E,R) \longrightarrow T$$

*which preserves joins and which is the identity on $E$ (viewed as a subposet of $I_\downarrow(E,R)$).*

(b) *There is a unique surjective map of posets*

$$\phi_T : T \longrightarrow L(E,R)$$

*which preserves joins and which is the identity on $E$ (viewed as a subposet of $L(E,R)$).*

(c) *The composite $\phi_T \circ \pi_T$ is equal to the map $\pi_{L(E,R)}$.*

**Proof :** As before, the order relation in $T$ will be written $\leq_T$ in order to avoid confusion. Similarly, we write intervals in $T$ with a subscript $T$ in order to emphasize that they are considered in the lattice $T$.

(a) follows from Proposition 21.4.

(b) We define

$$\phi_T : T \to L(E,R)\,, \qquad \phi_T(t) = \mathrm{Lb}\left([t,\hat{1}]_T \cap E\right)\,,$$

where the operator Lb is considered within the set of all subsets of $E$. Since $T$ is generated by $E$, we can write any $t \in T$ as $t = f_1 \vee \ldots \vee f_r$ where $f_1, \ldots, f_r \in E$. For any $e \in E$, there are equivalences

$$e \in [t,\hat{1}]_T \cap E \iff t \leq_T e \iff f_i \leq_T e\,, \forall i\,,$$

using the fact that $t$ is the join $t = f_1 \vee \ldots \vee f_r$. Since $E$ is a full subposet of $T$, we get further equivalences

$$f_i \leq_T e\,, \forall i \iff f_i \leq_R e\,, \forall i \iff e \in \bigcap_{i=1}^{r} [f_i, \cdot[_E\,.$$

Therefore $[t, \hat{1}]_T \cap E = \bigcap_{i=1}^{r} [f_i, \cdot[_E$ and $\phi_T(t) = \mathrm{Lb}\Big(\bigcap_{i=1}^{r} [f_i, \cdot[_E\Big)$.

Now part (c) of Lemma 21.7 applies, because $\mathrm{Ub}\,\mathrm{Lb}\ [f_i, \cdot[_E = \mathrm{Ub}\ ]\cdot, f_i]_E = [f_i, \cdot[_E$. It follows that

$$\phi_T(t) = \mathrm{Lb}\Big(\bigcap_{i=1}^{r} [f_i, \cdot[_E\Big) = \bigvee_{i=1}^{r} \mathrm{Lb}\ [f_i, \cdot[_E = \bigvee_{i=1}^{r}\ ]\cdot, f_i]_E\ .$$

It follows easily from this that $\phi_T$ preserves joins, because if $s = e_1 \vee \ldots \vee e_q$ and $t = f_1 \vee \ldots \vee f_r$ with $e_1, \ldots, e_q, f_1, \ldots, f_r \in E$, then $s \vee t = e_1 \vee \ldots \vee e_q \vee f_1 \vee \ldots \vee f_r$ and

$$\phi_T(s \vee t) = \Big(\bigvee_{j=1}^{q}\ ]\cdot, e_j]_E\Big) \bigvee \Big(\bigvee_{i=1}^{r}\ ]\cdot, f_i]_E\Big) = \phi_T(s) \vee \phi_T(t)\ .$$

In particular, $\phi_T$ is a map of posets. It is the identity on $E$ because if $e \in E$, then

$$\phi_T(e) = \mathrm{Lb}\big([e, \hat{1}]_T \cap E\big) = [\hat{0}, e]_T \cap E = \ ]\cdot, e]_E\ ,$$

the last equality using the assumption that $E$ is a full subposet of $T$. The surjectivity of $\phi_T$ follows from (c) and the surjectivity of $\pi_{L(E,R)}$.

(c) By Proposition 21.4, there is a unique surjective map

$$\pi_{L(E,R)} : I_{\downarrow}(E, R) \to L(E, R)$$

which preserves joins and is the identity on $E$. Therefore $\phi_T \circ \pi_T = \pi_{L(E,R)}$. □

**21.9. Remark.** Note that $\pi_T$ and $\phi_T$ may not preserve meets. As in Remark 11.6, we see that the relevant maps between lattices are join-preserving, but not necessarily meet-preserving.

Theorem 21.8 shows that the lattice $L(E, R)$ is the unique smallest lattice generated by the poset $E$. Its subset of irreducible elements is contained in $E$ but may not be equal to $E$. Our purpose now is to consider lattices whose subset of irreducible elements is the whole of $E$.

**21.10. Definition.** *An element $e$ of $E$ is called* reducible *in $E$ if it is the least upper bound of some subset $B$ of $]\cdot, e[_E$. In other words, $[e, \cdot[_E = \mathrm{Ub}\, B$ for some $B \subseteq\ ]\cdot, e[_E$. Note that the possibility $B = \emptyset$ occurs when $E$ has a least element $e_0$, in which case $\mathrm{Ub}\,\emptyset = E = [e_0, \cdot[$, so that $e_0$ is reducible in $E$.*

**21.11. Lemma.** *Let $(E, R)$ be a finite poset and $e \in E$.*

*(a) $e$ is reducible in $E$ if and only if there exists a nonprincipal ideal $A \in I_{\downarrow}(E, R)$ such that $\overline{A} = \ ]\cdot, e]_E$.*
*(b) $e$ is irreducible in $L(E, R)$ if and only if $e$ is not reducible in $E$.*

**Proof :** (a) If $e$ is reducible in $E$, then $[e, \cdot[_E = \mathrm{Ub}(B)$ for some $B \subseteq\ ]\cdot, e[_E$. Let $A$ be the lower ideal generated by $B$, that is, $A = \{a \in E \mid a \leq_R b \text{ for some } b \in B\}$. Note that $A \subseteq\ ]\cdot, e[_E$ again (with $A = \emptyset$ in case $B = \emptyset$). Then $\mathrm{Ub}\, A = \mathrm{Ub}\, B = [e, \cdot[_E$, hence

$$\overline{A} = \mathrm{Lb}\,\mathrm{Ub}\, A = \mathrm{Lb}\, [e, \cdot[_E = \ ]\cdot, e]_E\ .$$

We have to show that $A$ is not principal. If we had $A = \ ]\cdot, f]_E$, then we would have $\mathrm{Lb}\,\mathrm{Ub}\, A = \mathrm{Lb}\, [f, \cdot[_E = \ ]\cdot, f]_E = A$ and $[e, \cdot[_E = \mathrm{Ub}\, A = [f, \cdot[_E$, so that $e = f$. But then $A = \ ]\cdot, e]_E$, contrary to the fact that $A \subseteq\ ]\cdot, e[_E$.

Suppose conversely that $\overline{A} = \ ]\cdot, e]_E$ for some nonprincipal $A \in I_{\downarrow}(E, R)$. Then

$$\mathrm{Ub}\, A = \mathrm{Ub}\,\mathrm{Lb}\,\mathrm{Ub}\, A = \mathrm{Ub}\, \overline{A} = \mathrm{Ub}\ ]\cdot, e]_E = [e, \cdot[_E\ ,$$

showing that $e$ is the least upper bound of the subset $A$. Since $A$ is nonprincipal and contained in $]\cdot,e]_E$, we have $A \subseteq\, ]\cdot,e[_E$. This completes the proof that $e$ is reducible in $E$.

(b) If $e$ is reducible in $E$, then again $[e,\cdot[_E = \mathrm{Ub}(A)$ for some lower ideal $A \in I_\downarrow(E,R)$ such that $A \subseteq\, ]\cdot,e[_E$. Then

$$]\cdot,e]_E = \overline{A} = \overline{\bigcup_{a\in A} ]\cdot,a]_E} = \bigvee_{a\in A} ]\cdot,a]_E$$

and this shows that $]\cdot,e]_E$ is not irreducible in the lattice $L(E,R)$. (Note that this includes the case when $E$ has a least element $e_0$ and $A = \emptyset$, because in that case $L(E,R)$ has a least element $\{e_0\} = \overline{\emptyset}$, which is not irreducible by definition.)

Conversely, if $]\cdot,e]_E$ is not irreducible in the lattice $L(E,R)$, then $]\cdot,e]_E = \bigvee\limits_{a\in A} ]\cdot,a]_E$ where $A = \,]\cdot,e[_E$. Therefore

$$]\cdot,e]_E = \overline{\bigcup_{a\in A} ]\cdot,a]_E} = \overline{A}\,.$$

Moreover, $A$ is not principal, otherwise $A = \overline{A}$, hence $A = \,]\cdot,e]_E$, contrary to the fact that $A = \,]\cdot,e[_E$. By part (a), $e$ is reducible in $E$. □

We are going to construct a lattice $K(E,R)$ having $E$ as its subset of irreducible elements and minimal with this property. In order to define $K(E,R)$, we need to define a new closure operation on $I_\downarrow(E,R)$, which we write $A \mapsto \widehat{A}$ and which is defined as follows, with two cases :

(K1) If $A$ is not principal and $\overline{A} = \,]\cdot,a]_E$ , then $\widehat{A} = \,]\cdot,a[_E$.
(K2) Otherwise $\widehat{A} = \overline{A}$.

In the first case, $a$ is reducible in $E$, by Lemma 21.11. Thus we have kept the closure operation defining $L(E,R)$, except on subsets $A$ generating a reducible element $a$ of $E$, for which we distinguish $\widehat{A} = \,]\cdot,a[_E$ from $\overline{A} = ]\cdot,a]_E$.

**21.12. Lemma.** *$A \mapsto \widehat{A}$ is a closure operation on $I_\downarrow(E,R)$.*

**Proof :** Note first that $A \subseteq \widehat{A} \subseteq \overline{A}$ for all $A \in I_\downarrow(E,R)$. This is clear in case (K2), while in case (K1), $A \subseteq \,]\cdot,a[_E = \widehat{A} \subset \,]\cdot,a]_E = \overline{A}$. It follows that $\overline{\widehat{A}} = \overline{A}$ in all cases.

Now we have to prove that $\widehat{\widehat{A}} = \widehat{A}$. Suppose first that $A$ is in case (K1), so $A$ is not principal, and $\widehat{A} = \,]\cdot,a[_E$. Then $\overline{]\cdot,a[_E} = \overline{\widehat{A}} = \overline{A} = \,]\cdot,a]_E$. Moreover $]\cdot,a[_E$ is not principal otherwise $]\cdot,a[_E = \overline{]\cdot,a[_E} = \,]\cdot,a]_E$, which is a contradiction. Therefore $]\cdot,a[_E$ satisfies the conditions of case (K1) and

$$\widehat{\widehat{A}} = \widehat{]\cdot,a[_E} = \,]\cdot,a[_E = \widehat{A}\,.$$

If $A$ is in case (K2), then $\widehat{A} = \overline{A}$, which cannot be in case (K1), otherwise $\overline{A}$ would be nonprincipal and $\overline{\overline{A}}$ would be principal, contrary to the fact that $\overline{A} = \overline{\overline{A}}$. Therefore $\widehat{A} = \overline{A}$ is in case (K2) and

$$\widehat{\widehat{A}} = \overline{\widehat{A}} = \overline{\overline{A}} = \overline{A} = \widehat{A}\,.$$

So we obtain $A \subseteq \widehat{A} = \widehat{\widehat{A}}$ in all cases.

In order to show that the operation is order-preserving, we let $A, B \in I_\downarrow(E,R)$ with $A \subseteq B$. If $\widehat{B} = \overline{B}$, then

$$\widehat{A} \subseteq \overline{A} \subseteq \overline{B} = \widehat{B}\,.$$

Otherwise $\widehat{B} = ]\cdot, b[_E$ and $\overline{B} = ]\cdot, b]_E$ for some $b \in E$. Then $A \subseteq B \subseteq ]\cdot, b[_E$ and $\overline{A} \subseteq \overline{B} = ]\cdot, b]_E$. Either $\overline{A} \neq ]\cdot, b]_E$, in which case

$$\widehat{A} \subseteq \overline{A} \subseteq ]\cdot, b[_E = \widehat{B} \ ,$$

or $\overline{A} = ]\cdot, b]_E$, in which case $A$ is in case (K1) and $\widehat{A} = ]\cdot, b[_E = \widehat{B}$. □

We define $K(E, R)$ to be the lattice corresponding to the closure operation $A \mapsto \widehat{A}$ (see Lemma 21.3).

**21.13. Lemma.** *The set of irreducible elements of the lattice $K(E, R)$ is equal to $E$ (viewed as a subset of $K(E, R)$). Moreover, $E$ is a full subposet of $K(E, R)$.*

**Proof :** Since the lattice $K(E, R)$ corresponds to a closure operation, $E$ is a full subposet of $K(E, R)$ and $K(E, R)$ is generated by $E$ (see Lemma 21.3). Therefore, every irreducible element of $K(E, R)$ belongs to $E$.

Conversely, let $a \in E$ and suppose that $a$ is a join in $K(E, R)$, that is, $]\cdot, a]_E = A \vee B = \widehat{A \cup B}$, for some $\widehat{\ }$-closed subsets $A, B \in I_\downarrow(E, R)$. Note that $\overline{A \cup B} = \widehat{A \cup B}$ because

$$\overline{A \cup B} = \overline{\widehat{A \cup B}} = \overline{]\cdot, a]_E} = ]\cdot, a]_E = \widehat{A \cup B} \ .$$

Thus $A \cup B$ cannot be in case (K1) since the $\widehat{\ }$-closure and the $\overline{\ }$-closure are not equal in case (K1). As the condition $\overline{A \cup B} = ]\cdot, a]_E$ is satisfied, this means that $A \cup B$ must be principal, in particular $\overline{\ }$-closed. Therefore $A \cup B = \overline{A \cup B} = ]\cdot, a]_E$. It follows that $a \in A \cup B$. If $a \in A$, then $A = ]\cdot, a]_E$ (because $A$ is a lower ideal). Similarly, if $a \in B$, then $B = ]\cdot, a]_E$.

This completes the proof that $a$ is irreducible. □

**21.14. Remark.** The case (K1) in the construction of $K(E, R)$ shows that if an element $a \in E$ is reducible in $E$, then it is made irreducible in $K(E, R)$ by distinguishing between $]\cdot, a[_E$ and $]\cdot, a]_E$. Thus $]\cdot, a]_E$ is irreducible and $]\cdot, a[_E$ is the unique maximal element of $]\cdot, a]_E$.

We need further properties of the map $\pi_T$ of Proposition 21.4.

**21.15. Lemma.** *Let $(E, R)$ be a finite poset. Let $T$ be a lattice containing $E$ as a full subposet, and suppose that $T$ is generated by $E$. Let $\pi_T : I_\downarrow(E, R) \to T$ be the unique map of posets of Proposition 21.4.*

- *(a) If $A, B \in I_\downarrow(E, R)$ satisfy $\pi_T(A) \leq_T \pi_T(B)$, then* $\mathrm{Ub}\, A \supseteq \mathrm{Ub}\, B$. *In particular, if $\pi_T(A) = \pi_T(B)$, then* $\mathrm{Ub}\, A = \mathrm{Ub}\, B$.
- *(b) Let $e \in E$. Then $e$ is irreducible in $T$ if and only if $\pi_T^{-1}(e) = \{e\}$.*
- *(c) The set of irreducible elements of $T$ is equal to the whole of $E$ if and only if $\pi_T^{-1}(e)$ is reduced to the singleton $\{e\}$ for all $e \in E$.*

**Proof :** (a) By Proposition 21.4, $T$ corresponds to a closure operation $A \mapsto \widetilde{A}$ and if $\pi_T(A) = t$, then $\widetilde{A} = \{x \in E \mid x \leq_T t\}$. Now

$$\pi_T(A) = \pi_T\big(\bigcup_{a \in A} ]\cdot, a]_E\big) = \bigvee_{a \in A} \pi_T\big(]\cdot, a]_E\big) = \bigvee_{a \in A} a \ .$$

Then for $x \in E$, we obtain

$$x \in \mathrm{Ub}\, A \iff x \geq_E a \ , \ \forall a \in A \iff x \geq_T a \ , \ \forall a \in A \iff x \geq_T \pi_T(A)$$

because $\pi_T(A) = \bigvee_{a \in A} a$. Similarly $x \in \mathrm{Ub}\, B$ if and only if $x \geq_T \pi_T(B)$. Since $\pi_T(B) \geq_T \pi_T(A)$, we deduce that $\mathrm{Ub}\, B \subseteq \mathrm{Ub}\, A$.

(b) Suppose that $e$ is irreducible in $T$ and let $A \in \pi_T^{-1}(e)$. Since $E$ is a full subposet, we get

$$\widetilde{A} = \{x \in E \mid x \leq_T e\} = \{x \in E \mid x \leq_R e\} = ]\cdot, e]_E \; .$$

Moreover $e = \pi_T(A) = \bigvee_{a \in A} a$. Since $e$ is irreducible, we get $e = a$ for some $a \in A$, that is, $e \in A$. As $A$ is a lower ideal, we obtain

$$]\cdot, e]_E \subseteq A \subseteq \widetilde{A} = ]\cdot, e]_E \; ,$$

hence $A = \widetilde{A} = ]\cdot, e]_E$. Thus $\pi_T^{-1}(e)$ is reduced to the singleton $\{ \, ]\cdot, e]_E \, \}$, namely $\{e\}$ (in view of the usual identification), as required.

Conversely, suppose that $\pi_T^{-1}(e) = \{e\}$. Write $e = \bigvee_{i=1}^{r} f_i$ with $f_i$ irreducible for all $i$, hence $f_i \in E$ since $T$ is generated by $E$. Let $A = \bigcup_{i=1}^{r} \, ]\cdot, f_i]_E$. Then

$$\pi_T(A) = \pi_T\big( \bigcup_{i=1}^{r} \, ]\cdot, f_i]_E \big) = \bigvee_{i=1}^{r} \pi_T\big( \, ]\cdot, f_i]_E \big) = \bigvee_{i=1}^{r} f_i = e \; ,$$

that is, $A \in \pi_T^{-1}(e)$. Since this fibre is a singleton, we have

$$A = \widetilde{A} = \{x \in E \mid x \leq_T e\} \; .$$

It follows that $e \in A$, hence $e \leq_R f_i$ for some $i$. This implies $e = f_i$, showing that $e$ is irreducible in $T$.

(c) follows immediately from (b). □

We now come to a result analogous to Theorem 21.8, but for lattices whose set of irreducible elements is the whole of $E$.

**21.16. Theorem.** *Let $(E, R)$ be a finite poset, let $T$ be a lattice containing $E$ as a full subposet, and suppose that the set of irreducible elements of $T$ is equal to $E$.*

*(a) There is a unique surjective map of posets*

$$\pi_T : I_{\downarrow}(E, R) \longrightarrow T$$

*which preserves joins and which is the identity on $E$ (viewed as a subposet of $I_{\downarrow}(E, R)$).*

*(b) There is a unique surjective map of posets*

$$\psi_T : T \longrightarrow K(E, R)$$

*which preserves joins and which is the identity on $E$ (viewed as a subposet of $K(E, R)$).*

*(c) The composite $\psi_T \circ \pi_T$ is equal to the map $\pi_{K(E,R)}$.*

**Proof :** (a) follows from Proposition 21.4.

(b) We want to define $\psi_T : T \to K(E, R)$ by taking a pre-image in $I_{\downarrow}(E, R)$ and then applying $\pi_{K(E,R)} : I_{\downarrow}(E, R) \to K(E, R)$. Of course we need to see that this is well-defined. So let $A, B \in I_{\downarrow}(E, R)$ be such that $\pi_T(A) = \pi_T(B)$. We have to show that $\pi_{K(E,R)}(A) = \pi_{K(E,R)}(B)$, that is, $\widehat{A} = \widehat{B}$.

By Lemma 21.15, we have $\mathrm{Ub}\, A = \mathrm{Ub}\, B$, hence $\overline{A} = \overline{B}$. If both $A$ and $B$ are in case (K2), we get $\widehat{A} = \overline{A} = \overline{B} = \widehat{B}$. If one of them, say $A$, is in case (K1), then there exists $a \in E$ such that

$$A \subseteq \widehat{A} = ]\cdot, a[_E \subset ]\cdot, a]_E = \overline{A} \; .$$

Suppose that $a \in B$, so that $B = ]\cdot, a]_E$ because $\overline{B} = \overline{A} = ]\cdot, a]_E$. Then we have

$$a = \pi_T(\, ]\cdot, a]_E) = \pi_T(\overline{B}) = \pi_T(B) = \pi_T(A) \ ,$$

hence $A \in \pi_T^{-1}(a)$. Now part (c) of Lemma 21.15 applies, because the set of irreducible elements of $T$ is equal to $E$ by assumption. Therefore $\pi_T^{-1}(a)$ is reduced to the singleton $\{a\}$, that is, $\{\overline{A}\}$ in view of the usual identification. This is impossible because $A \in \pi_T^{-1}(a)$ and $A \neq \overline{A}$.

This contradiction shows that $a \notin B$ and therefore

$$B \subseteq ]\cdot, a[_E \subset ]\cdot, a]_E = \overline{B} \ .$$

Moreover, $B$ is not principal, otherwise $B = \overline{B}$, hence $B = ]\cdot, a]_E$, contrary to the fact that $B \subseteq ]\cdot, a[_E$. Thus we are in case (K1) and $\widehat{B} = ]\cdot, a[_E = \widehat{A}$.

We have now proved that there is a well-defined map $\psi_T : T \longrightarrow K(E, R)$, as follows. If $t \in T$, write $t = \pi_T(A)$ for some $A \in I_\downarrow(E, R)$ and set

$$\psi_T(t) = \psi_T(\pi_T(A)) = \pi_{K(E,R)}(A) \ .$$

In particular, this proves (c). The fact that $\psi_T$ preserves joins follows in a straightforward fashion :

$$\begin{aligned} \psi_T\big(\pi_T(A) \vee \pi_T(B)\big) &= \psi_T\big(\pi_T(A \cup B)\big) = \pi_{K(E,R)}(A \cup B) \\ &= \pi_{K(E,R)}(A) \vee \pi_{K(E,R)}(B) = \psi_T(\pi_T(A)) \vee \psi_T(\pi_T(B)) \ . \end{aligned}$$

The map $\psi_T$ is the identity on $E$ because both $\pi_T$ and $\pi_{K(E,R)}$ are the identity on $E$. The surjectivity of $\psi_T$ follows from the surjectivity of $\pi_{K(E,R)}$.

(c) This has been noticed above. ∎

Theorem 21.16 shows that $K(E, R)$ is the unique minimal lattice having $E$ as its set of irreducible elements. This has some importance for computational purposes, since working with $K(E, R)$ may be considerably less heavy than working with $I_\downarrow(E, R)$. This is made clear in the next example.

**21.17. Example.** Let $E$ be a nonempty finite set endowed with the equality relation. Then $I_\downarrow(E, R)$ is the set of all subsets of $E$, hence rather big. However, with the usual identification, $L(E, R) = \{\emptyset\} \sqcup E \sqcup \{E\}$, hence much smaller, and $K(E, R) = L(E, R)$ in this case.

**21.18. Example.** Let $E$ be a nonempty finite set endowed with a total order. Then $I_\downarrow(E, R) = \{\emptyset\} \sqcup E$ (with the usual identification), $L(E, R) = E$, and $K(E, R) = I_\downarrow(E, R)$.

## 22. Forests associated to lattices

Our aim in this appendix is to prove Theorem 17.9. This will follow from a much more general combinatorial result on idempotents associated to forests. We start with some idempotents associated to sequences of distinct elements in any finite set:

**22.1. Proposition.** *Let $T$ be a finite set. For a sequence $\underline{a} = (a_0, a_1, \ldots, a_n)$ of distinct elements of $T$, and for $0 \leq k \leq n$, let $[a_0, a_1, \ldots a_k]$ be the map from $T$ to itself defined by*

$$\forall t \in T, \;\; [a_0, a_1, \ldots a_k](t) = \begin{cases} a_{i+1} & \text{if } t = a_i \text{ and } i \leq k-1 \\ t & \text{if } t \notin \{a_0, \ldots a_{k-1}\} \; . \end{cases}$$

*In particular $[a_0]$ is the identity map of $T$.*

*Let $h_{\underline{a}}$ be the element of the algebra $k(T^T)$ of the monoid $T^T$ of maps from $T$ to itself defined by*

$$h_{\underline{a}} = \sum_{i=0}^{n} (-1)^i [a_0, a_1, \ldots, a_i] \; .$$

*(a) If $0 \leq j \leq i$, then $[a_0, \ldots, a_i] = [a_0, \ldots, a_j] \circ [a_j, \ldots, a_i]$.*
*(b) If $1 \leq k \leq n$ and $f : T \to T$ is a map such that $f(a_k) = f(a_{k-1})$, then $f \circ [a_0, \ldots, a_{k-1}] = f \circ [a_0, \ldots, a_k]$.*
*(c) $h_{\underline{a}}$ is an idempotent of $k(T^T)$.*

**Proof :** (a) This is straightforward.

(b) The only element $t \in T$ such that $[a_0, \ldots, a_{k-1}](t) \neq [a_0, \ldots, a_k](t)$ is $t = a_{k-1}$. Moreover

$$f \circ [a_0, \ldots, a_{k-1}](a_{k-1}) = f(a_{k-1}) = f(a_k) = f \circ [a_0, \ldots, a_k](a_{k-1}) \; .$$

(c) We have

$$\begin{aligned} h_{\underline{a}}^2 &= \sum_{0 \leq i,j \leq n} (-1)^{i+j} [a_0, \ldots, a_i] \circ [a_0, \ldots, a_j] \\ &= \sum_{0 \leq i \leq j \leq n} (-1)^{i+j} [a_0, \ldots, a_i] \circ [a_0, \ldots, a_j] + \sum_{0 \leq j < i \leq n} (-1)^{i+j} [a_0, \ldots, a_i] \circ [a_0, \ldots, a_j] \end{aligned}$$

Moreover by (a), for $j + 1 \leq i$

$$[a_0, \ldots, a_i] \circ [a_0, \ldots, a_j] = [a_0, \ldots, a_{j+1}] \circ [a_{j+1}, \ldots, a_i] \circ [a_0, \ldots, a_j] \; ,$$

and the maps $[a_{j+1}, \ldots, a_i]$ and $[a_0, \ldots, a_j]$ clearly commute. Hence

$$[a_0, \ldots, a_i] \circ [a_0, \ldots, a_j] = [a_0, \ldots, a_{j+1}] \circ [a_0, \ldots, a_j] \circ [a_{j+1}, \ldots, a_i] \; .$$

Now the map $f = [a_0, \ldots, a_{j+1}]$ is such that $f(a_j) = f(a_{j+1})$. Thus by (c)

$$[a_0, \ldots, a_{j+1}] \circ [a_0, \ldots, a_j] = [a_0, \ldots, a_{j+1}] \circ [a_0, \ldots, a_{j+1}] \; ,$$

and by (a)

$$\begin{aligned} [a_0, \ldots, a_i] \circ [a_0, \ldots, a_j] &= [a_0, \ldots, a_{j+1}] \circ [a_0, \ldots, a_{j+1}] \circ [a_{j+1}, \ldots, a_i] \\ &= [a_0, \ldots, a_{j+1}] \circ [a_0, \ldots, a_i] \; . \end{aligned}$$

Hence for all $j + 1 \leq i$, i.e. for all $j < i$

$$[a_0, \ldots, a_i] \circ [a_0, \ldots, a_j] = [a_0, \ldots, a_{j+1}] \circ [a_0, \ldots, a_i] \; .$$

This gives:

$$\begin{aligned}
h_{\underline{a}}^2 &= \sum_{0\le i\le j\le n}(-1)^{i+j}[a_0,\ldots,a_i]\circ[a_0,\ldots,a_j]+\sum_{0\le j<i\le n}(-1)^{i+j}[a_0,\ldots,a_i]\circ[a_0,\ldots,a_j]\\
&= \sum_{0\le i\le j\le n}(-1)^{i+j}[a_0,\ldots,a_i]\circ[a_0,\ldots,a_j]+\sum_{0\le j<i\le n}(-1)^{i+j}[a_0,\ldots,a_{j+1}]\circ[a_0,\ldots,a_i]\\
&= \sum_{0\le i\le j\le n}(-1)^{i+j}[a_0,\ldots,a_i]\circ[a_0,\ldots,a_j]+\sum_{1\le j\le i\le n}(-1)^{i+j-1}[a_0,\ldots,a_j]\circ[a_0,\ldots,a_i]\\
&= \sum_{0=i\le j\le n}(-1)^{i+j}[a_0,\ldots,a_i]\circ[a_0,\ldots,a_j]=\sum_{0\le j\le n}(-1)^{j}[a_0]\circ[a_0,\ldots,a_j]\\
&= h_{\underline{a}}
\end{aligned}$$

as was to be shown. □

We now introduce the terminology on forests that we will use here. Part of it may be nonstandard:

**22.2. Definition and Notation.** *A* forest *is a finite oriented graph without loops nor cycles, such that each vertex has at most one outgoing edge. A* root *of a forest is a vertex without any outgoing edge. A* leaf *of a forest is a vertex with an outgoing edge and no incoming edge. A* tree *is a connected forest.*

*When $\mathcal{G}$ is a forest, we also denote by $\mathcal{G}$ the set of its vertices. When $x\in\mathcal{G}$, and $x$ is not a root, we denote by $d(x)$ the unique vertex of $\mathcal{G}$ such that $x\longrightarrow d(x)$ is the outgoing edge at $x$. When $x$ is a root, we set $x=d(x)$. For $n\in\mathbb{N}$, we define $d^n:\mathcal{G}\to\mathcal{G}$ by $d^0(x)=x$, for any $x\in X$, and $d^n(x)=d\big(d^{n-1}(x)\big)$ for $n>0$. For $x\in\mathcal{G}$, we denote by $\delta(x)$ the smallest non negative integer $n$ such that $d^n(x)=d^{n+1}(x)$, and we set $d^\infty(x)=d^{\delta(x)}(x)$. Each connected component $C$ of $\mathcal{G}$ contains a unique root, equal to $d^\infty(x)$ for each $x\in C$.*

*A* complete geodesic *of a forest $\mathcal{G}$ is a sequence of the form $\big(x,d(x),\ldots,d^n(x)\big)$, where $x$ is a leaf of $\mathcal{G}$, and $n=\delta(x)$. A* partial geodesic *of $\mathcal{G}$ starting at some leaf $x$ is a sequence of distinct vertices of the form $\big(x,d(x),\ldots,d^l(x)\big)$, for some integer $l\ge 1$. We say that and edge $\big(y,d(y)\big)$ of $\mathcal{G}$* belongs to a partial geodesic *$\big(x,d(x),\ldots,d^l(x)\big)$ if there exists an integer $j\in\{0,\ldots,l-1\}$ such that $y=d^j(x)$.*

*When $A$ is a set of edges of a forest $\mathcal{G}$, we denote by $\tau_A$ the map from $\mathcal{G}$ to itself defined by*

$$\forall x\in\mathcal{G},\ \ \tau_A(x)=\begin{cases} d(x) & \text{if } \big(x,d(x)\big)\in A\\ x & \text{otherwise}\,.\end{cases}$$

*When $B$ is a set of leaves of a forest $\mathcal{G}$, we say that a set $A$ of edges of $\mathcal{G}$ is* a union of partial geodesics starting in $B$ *if there is a subset $C$ of $B$, and for each $x\in C$, a partial geodesic $\gamma_x$ starting at $x$, such that $A$ is the union of the sets of edges belonging to the partial geodesics $\gamma_x$, for $x\in C$. We denote by $\mathcal{S}_B$ the set of such sets of edges of $\mathcal{G}$.*

*Finally, we denote by $v_B$ the element of $k(\mathcal{G}^{\mathcal{G}})$ defined by*

$$v_B=\sum_{A\in\mathcal{S}_B}(-1)^{|A|}\tau_A\,.$$

*We will abuse notation and set $v_x=v_{\{x\}}$, for any leaf $x$ of $\mathcal{G}$.*

**22.3. Theorem.** *Let $\mathcal{G}$ be a forest.*

*(a) If $x$ is a leaf of $\mathcal{G}$, then $v_x^2=v_x$.*
*(b) If $x$ and $y$ are leaves of $\mathcal{G}$, then $v_xv_y=v_yv_x$.*
*(c) Let $B$ be a subset of the set of leaves of $\mathcal{G}$. Then $v_B$ is an idempotent of $k(\mathcal{G}^{\mathcal{G}})$, equal to the product of the (commuting) idempotents $v_x$, for $x\in B$.*

*(d) Let $B$ and $C$ be two subsets of the set of leaves of $\mathcal{G}$. Then $v_B \circ v_C = v_{B\cup C}$.*

**Proof :** (a) If there is a single leaf $x$, and if $\underline{a} = \big(x, d(x), \ldots, d^n(x)\big)$ is the complete geodesic of $\mathcal{G}$ starting at $x$, then

$$v_B = v_x = \sum_{l=0}^{n} (-1)^l [x, d(x), \ldots, d^l(x)] \ ,$$

so $v_x$ is equal to the idempotent $h_{\underline{a}}$ introduced in Proposition 22.1.

(b) Let $x$ and $y$ be distinct leaves of $\mathcal{G}$. If $d^\infty(x) \neq d^\infty(y)$, then the complete geodesics starting at $x$ and $y$ lie in different connected components of $\mathcal{G}$. It is clear in this case that $v_x$ and $v_y$ commute.

Otherwise set $r = d^\infty(x) = d^\infty(y)$. The intersection of the complete geodesics starting at $x$ and $y$ is of the form $\underline{c} = (c_0, \ldots, c_m = r)$, for some integer $m \geq 0$. The complete geodesic starting at $x$ is of the form

$$(x = a_0, \ldots, a_s, c_0, \ldots, c_m = r) \ ,$$

where $s > 0$, and the complete geodesic starting at $y$ is of the form

$$(y = b_0, \ldots, b_t, c_0, \ldots, c_m = r) \ ,$$

where $t > 0$.

Now

$$\begin{aligned} v_x &= \sum_{i=0}^{s} (-1)^i [a_0, \ldots, a_i] + (-1)^{s+1} \sum_{i=0}^{m} (-1)^i [a_0, \ldots, a_s, c_0, \ldots, c_i] \\ &= h_{\underline{a}} + (-1)^{s+1} [a_0, \ldots, a_s, c_0] h_{\underline{c}} \ , \end{aligned}$$

where $\underline{a} = [a_0, \ldots, a_s]$. Similarly $v_y = h_{\underline{b}} + (-1)^{t+1}[b_0, \ldots, b_t, c_0]h_{\underline{c}}$, where $\underline{b} = (b_0, \ldots, b_t)$. Clearly $h_{\underline{a}}$ commutes with $h_{\underline{b}}$, $[b_0, \ldots, b_t, c_0]$, and $h_{\underline{c}}$, hence it commutes with $v_y$. Similarly $h_{\underline{b}}$ commutes with $v_x$. So proving that that $v_x$ and $v_y$ commute is equivalent to proving that $[a_0, \ldots, a_s, c_0]h_{\underline{c}}$ and $[b_0, \ldots, b_t, c_0]h_{\underline{c}}$ commute.

Setting $f = [b_0, \ldots, b_t, c_0]$, we have

$$\begin{aligned} h_{\underline{c}} f h_{\underline{c}} &= \sum_{0 \leq i,j \leq m} (-1)^{i+j} [c_0, \ldots, c_i] f [c_0, \ldots, c_j] \\ &= \sum_{0 \leq i \leq j \leq m} (-1)^{i+j} [c_0, \ldots, c_i] f [c_0, \ldots, c_j] + \sum_{0 \leq j < i \leq m} (-1)^{i+j} [c_0, \ldots, c_i] f [c_0, \ldots, c_j] \end{aligned}$$

Now for $0 \leq j < i$, by Proposition 22.1,

$$\begin{aligned} [c_0, \ldots, c_i] f [c_0, \ldots, c_j] &= [c_0, \ldots, c_{j+1}][c_{j+1}, \ldots, c_i] f [c_0, \ldots, c_j] \\ &= [c_0, \ldots, c_{j+1}] f [c_0, \ldots, c_j][c_{j+1}, \ldots, c_i] \ , \end{aligned}$$

since $[c_{j+1}, \ldots, c_i]$ clearly commutes with both $f$ and $[c_0, \ldots, c_j]$. Now setting $g = [c_0, \ldots, c_{j+1}]f$, we have that $g(c_j) = c_{j+1} = g(c_{j+1})$, hence $g[c_0, \ldots, c_j] = g[c_0, \ldots, c_{j+1}]$, by Proposition 22.1. It follows that

$$\begin{aligned} [c_0, \ldots, c_i] f [c_0, \ldots, c_j] &= g[c_0, \ldots, c_j][c_{j+1}, \ldots, c_i] \\ &= g[c_0, \ldots, c_{j+1}][c_{j+1}, \ldots, c_i] \\ &= g[c_0, \ldots, c_i] \ , \end{aligned}$$

thus $[c_0,\ldots,c_i]f[c_0,\ldots,c_j]=[c_0,\ldots,c_{j+1}]f[c_0,\ldots,c_i]$ for $j<i$. This gives

$$\begin{aligned}
h_{\underline{c}}fh_{\underline{c}} &= \sum_{0\le i\le j\le m}(-1)^{i+j}[c_0,\ldots,c_i]f[c_0,\ldots,c_j]+\sum_{0\le j<i\le m}(-1)^{i+j}[c_0,\ldots,c_i]f[c_0,\ldots,c_j]\\
&= \sum_{0\le i\le j\le m}(-1)^{i+j}[c_0,\ldots,c_i]f[c_0,\ldots,c_j]+\sum_{0\le j<i\le m}(-1)^{i+j}[c_0,\ldots,c_{j+1}]f[c_0,\ldots,c_i]\\
&= \sum_{0\le i\le j\le m}(-1)^{i+j}[c_0,\ldots,c_i]f[c_0,\ldots,c_j]+\sum_{1\le j\le i\le m}(-1)^{i+j-1}[c_0,\ldots,c_j]f[c_0,\ldots,c_i]\\
&= \sum_{0=i\le j\le m}(-1)^{i+j}[c_0,\ldots,c_i]f[c_0,\ldots,c_j]=\sum_{j=0}^{n}(-1)^{j}f[c_0,\ldots,c_j]\\
&= fh_{\underline{c}}\ .
\end{aligned}$$

It follows that

$$[a_0,\ldots,a_s,c_0]h_{\underline{c}}[b_0,\ldots,b_t,c_0]h_{\underline{c}}=[a_0,\ldots,a_s,c_0][b_0,\ldots,b_t,c_0]h_{\underline{c}}\ .$$

Similarly

$$[b_0,\ldots,b_t,c_0]h_{\underline{c}}[a_0,\ldots,a_s,c_0]h_{\underline{c}}=[b_0,\ldots,b_t,c_0][a_0,\ldots,a_s,c_0]h_{\underline{c}}\ .$$

Now $[a_0,\ldots,a_s,c_0]$ and $[b_0,\ldots,b_t,c_0]$ clearly commute, so $[a_0,\ldots,a_s,c_0]h_{\underline{c}}$ and $[b_0,\ldots,b_t,c_0]h_{\underline{c}}$ also commute. This completes the proof of (b).

(c) We will prove that $v_B=\prod_{x\in B}v_x$ by induction on the cardinality of $B$. If $B=\emptyset$, then $v_B=\tau_\emptyset=\mathrm{id}_{\mathcal{G}}$, which is also equal to an empty product of $v_x$. If $B=\{x\}$, then by definition $v_B=v_x$. Assume now that (c) holds for any set of leaves of cardinality smaller than $|B|$ and $|B|\ge 2$.

We can assume that $B$ is the set of all leaves of $\mathcal{G}$: otherwise we replace $\mathcal{G}$ by the full subgraph $\mathcal{G}'$ consisting of the elements $d^j(x)$, for $x\in B$ and $j\ge 0$. The operation $f\mapsto\widetilde{f}$ extending any function from $\mathcal{G}'$ to itself to a function from $\mathcal{G}$ to itself by $\widetilde{f}(y)=y$ if $y\in\mathcal{G}-\mathcal{G}'$ is an injective morphism of monoids from $\mathcal{G}'^{\mathcal{G}'}$ to $\mathcal{G}^{\mathcal{G}}$, which sends the element $v_B$ computed for the forest $\mathcal{G}'$ to $v_B$.

We can now assume that $\mathcal{G}$ is a tree: indeed, if $\mathcal{G}$ is a disjoint union of nonempty subforests $\mathcal{G}_1$ and $\mathcal{G}_2$, then its set of leaves $B$ splits as $B=B_1\sqcup B_2$, where $B_i$ is a set of leaves of $\mathcal{G}_i$, for $i=1,2$. Similarly, any set of edges $A\in\mathcal{S}_B$ splits as $A=A_1\sqcup A_2$, where $A_i\in\mathcal{S}_{B_i}$, for $i=1,2$. Conversely, if $A_i\in\mathcal{S}_{B_i}$, for $i=1,2$, then the set $A=A_1\sqcup A_2$ belongs to $\mathcal{S}_B$. Moreover the maps $\tau_{A_1}$ and $\tau_{A_2}$ clearly commute, and $\tau_{A_1}\tau_{A_2}=\tau_A$. It follows that

$$v_B=\sum_{\substack{A_1\in\mathcal{S}_{B_1}\\A_2\in\mathcal{S}_{B_2}}}(-1)^{|A_1|+|A_2|}\tau_{A_1}\tau_{A_2}=v_{B_1}v_{B_2}=v_{B_2}v_{B_1}\ .$$

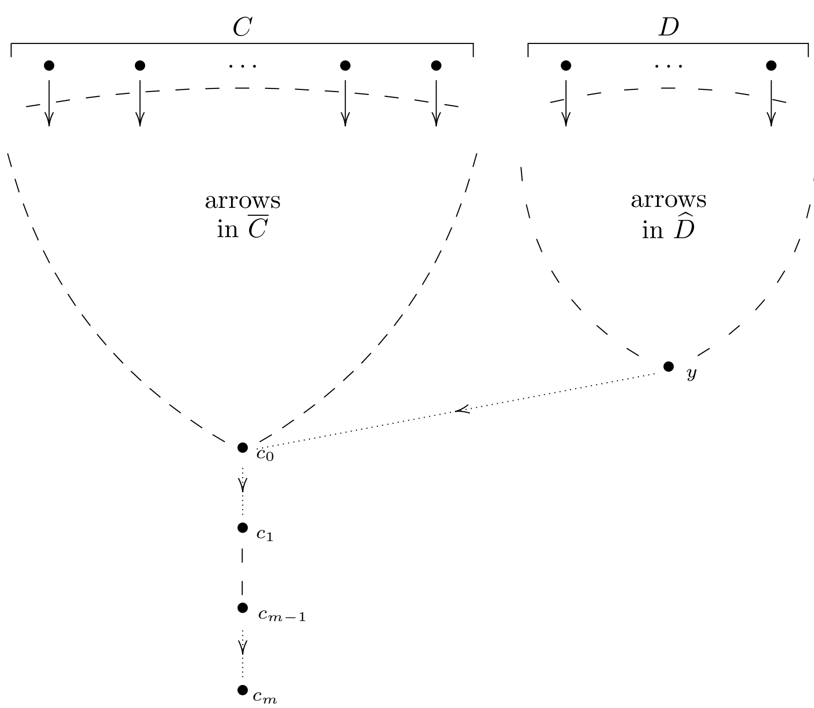
By induction hypothesis (c) holds for $B_1$ and $B_2$, thus $v_{B_1}=\prod_{x\in B_1}v_x$ and $v_{B_2}=\prod_{x\in B_2}v_x$, then $v_B=\prod_{x\in B_1\sqcup B_2}v_x$, so (c) holds for $B$.

Let $r$ be the root of $\mathcal{G}$, and let $\underline{c}=(c_0,\ldots,c_m=r)$ be the intersection of the complete geodesics of $\mathcal{G}$ (possibly $m=0$ and $\underline{c}=(r)$). If $c_0$ is a leaf, then $\mathcal{G}$ is an interval $(c_0,\ldots,c_m=r)$, but this contradicts $|B|\ge 2$. Thus the element $c_0$ is not a leaf, so the set $d^{-1}(c_0)=\{y\in\mathcal{G}\mid d(y)=c_0\}$ is nonempty, and actually of cardinality at least 2: if $d^{-1}(c_0)=\{y\}$, then $y$ belongs to the intersection of the complete geodesics of $\mathcal{G}$, contradicting the definition of $\underline{c}$.

Fix $y\in d^{-1}(c_0)$, and let $D=\{x\in B\mid \exists j\ge 0,\ \ d^j(x)=y\}$. Then $\emptyset\ne D\ne B$, and we set $C=B-D$. We denote by $C^+$ (resp. $D^+$) the set of edges of $\mathcal{G}$ which belong to a complete geodesic starting in $C$ (resp $D$). We also set

$$\begin{aligned} \overline{C} &= C^+ - \{(c_0, c_1), (c_1, c_2), \ldots, (c_{m-1}, c_m)\} \\ \widehat{D} &= D^+ - \{(y, c_0), (c_0, c_1), (c_1, c_2), \ldots, (c_{m-1}, c_m)\} . \end{aligned}$$

We illustrate these definitions with the following schematic diagram representing $\mathcal{G}$, in which all edges are oriented downwards:




Let $A \in \mathcal{S}_B$. There are two cases:

(1) either $(y, c_0) \notin A$: in this case $A$ splits as the disjoint union of a set $U \in \mathcal{S}_C$ and a set $V \in \mathcal{S}_D$ such that $(y, c_0) \notin V$, and this decomposition is unique. Observe that the conditions $V \in \mathcal{S}_D$ and $(y, c_0) \notin V$ are equivalent to the conditions $V \in \mathcal{S}_D$ and $V \subseteq \widehat{D}$. Conversely, if $U \in \mathcal{S}_C$ and $V \in \mathcal{S}_D$ with $V \subseteq \widehat{D}$, then $A = U \sqcup V \in \mathcal{S}_B$, and $(y, c_0) \notin A$. Moreover $\tau_A = \tau_U \tau_V$.

(2) or $(y, c_0) \in A$: then $A$ splits a the disjoint union of a set $U \in \mathcal{S}_C$ contained in $\overline{C}$ and a set $V \in \mathcal{S}_D$ such that $(y, c_0) \in V$, and this decomposition is again unique. Conversely, if $U \in \mathcal{S}_C$ and $U \subseteq \overline{C}$, and $V \in \mathcal{S}_D$, then $A = U \sqcup V \in \mathcal{S}_B$ and $(y, c_0) \in A$. Moreover $\tau_A = \tau_U \tau_V$ in this case also.

We now compute the product $v_C v_D$:

$$\begin{aligned}
v_C v_D &= \Big(\sum_{U\in\mathcal{S}_C}(-1)^{|U|}\tau_U\Big)\Big(\sum_{V\in\mathcal{S}_D}(-1)^{|V|}\tau_V\Big)\\
&= \sum_{\substack{U\in\mathcal{S}_C\\ V\in\mathcal{S}_D\\ V\subseteq\widehat{D}}}(-1)^{|U|+|V|}\tau_U\tau_V + \sum_{\substack{U\in\mathcal{S}_C\\ V\in\mathcal{S}_D\\ (y,c_0)\in V}}(-1)^{|U|+|V|}\tau_U\tau_V\\
&= \sum_{\substack{U\in\mathcal{S}_C\\ V\in\mathcal{S}_D\\ V\subseteq\widehat{D}}}(-1)^{|U\sqcup V|}\tau_{U\sqcup V} + \sum_{\substack{U\in\mathcal{S}_C\\ U\subseteq\overline{C}\\ (y,c_0)\in V\in\mathcal{S}_D}}(-1)^{|U|+|V|}\tau_U\tau_V + \sum_{\substack{U\in\mathcal{S}_C\\ U\not\subseteq\overline{C}\\ (y,c_0)\in V\in\mathcal{S}_D}}(-1)^{|U|+|V|}\tau_U\tau_V\\
&= \sum_{\substack{A\in\mathcal{S}_B\\ (y,c_0)\notin A}}(-1)^{|A|}\tau_A + \sum_{\substack{U\in\mathcal{S}_C\\ U\subseteq\overline{C}\\ (y,c_0)\in V\in\mathcal{S}_D}}(-1)^{|U\sqcup V|}\tau_{U\sqcup V} + \sum_{\substack{U\in\mathcal{S}_C\\ U\not\subseteq\overline{C}\\ (y,c_0)\in V\in\mathcal{S}_D}}(-1)^{|U|+|V|}\tau_U\tau_V\\
&= \sum_{\substack{A\in\mathcal{S}_B\\ (y,c_0)\notin A}}(-1)^{|A|}\tau_A + \sum_{\substack{A\in\mathcal{S}_B\\ (y,c_0)\in A}}(-1)^{|A|}\tau_A + \sum_{\substack{U\in\mathcal{S}_C\\ U\not\subseteq\overline{C}\\ (y,c_0)\in V\in\mathcal{S}_D}}(-1)^{|U|+|V|}\tau_U\tau_V\\
&= v_B + \sum_{\substack{U\in\mathcal{S}_C\\ U\not\subseteq\overline{C}\\ (y,c_0)\in V\in\mathcal{S}_D}}(-1)^{|U|+|V|}\tau_U\tau_V
\end{aligned}$$

We claim that the sum

$$(\mathbf{22.4}) \qquad \Sigma = \sum_{\substack{U\in\mathcal{S}_C\\ U\not\subseteq\overline{C}\\ (y,c_0)\in V\in\mathcal{S}_D}}(-1)^{|U|+|V|}\tau_U\tau_V$$

is equal to 0. This claim gives $v_C v_D = v_B$. As $C$ and $D$ are both strictly smaller than $B$, we may apply the induction hypothesis to both, and this gives

$$v_B = v_C v_D = \Big(\prod_{x\in C} v_x\Big)\Big(\prod_{x\in D} v_x\Big) = \prod_{x\in C\sqcup D} v_x = \prod_{x\in B} v_x \ ,$$

which completes the induction step in the proof of (c).

So it remains to prove the above claim that the sum $\Sigma$ defined in 22.4 is equal to 0. If $m = 0$, the claim is trivial, because the sum $\Sigma$ is empty. So we assume $m \geq 1$. We define two sets

$$\begin{aligned}
\mathcal{U} &= \{U\in\mathcal{S}_C \mid U\not\subseteq\overline{C}\} = \{U\in\mathcal{S}_C \mid (c_0,c_1)\in U\}\\
\mathcal{V} &= \{V\in\mathcal{S}_D \mid (y,c_0)\in V\}\ .
\end{aligned}$$

With this notation,

$$\Sigma = \sum_{(U,V)\in\mathcal{U}\times\mathcal{V}}(-1)^{|U|+|V|}\tau_U\tau_V\ .$$

Let $U\in\mathcal{U}$. If $(c_{m-1},c_m)\in U$, we set $l = m$. Otherwise there is a well defined integer $l \geq 1$ such that $(c_{l-1},c_l)\in U$, but $(c_l,c_{l+1})\notin U$. Moreover, the set $U' = U\cap\overline{C}$ belongs to the set $\mathcal{S}_C^\sharp$ of elements of $\mathcal{S}_C$ contained in $\overline{C}$ and containing at least one edge of the form $(z,c_0)$, and

$$U = U'\sqcup\{(c_0,c_1),\ldots,(c_{l-1},c_l)\}\ .$$

Moreover $\tau_U = \tau_{U'}[c_0, c_1, \ldots, c_l]$. Conversely, if $U' \in \mathcal{S}_C^\sharp$ and $l \in \{1, \ldots, m\}$, then the set $U = U' \sqcup \{(c_0, c_1), \ldots, (c_{l-1}, c_l)\}$ belongs to $\mathcal{U}$. In this way we get a bijection

$$\begin{aligned} \alpha : \mathcal{S}_C^\sharp \times \{1, \ldots, m\} &\longrightarrow \mathcal{U} \\ (U', l) &\longmapsto U = U' \sqcup \{(c_0, c_1), \ldots, (c_{l-1}, c_l)\} \end{aligned}$$

with the property that $\tau_{\alpha(U',l)} = \tau_{U'}[c_0, c_1, \ldots, c_l]$.

Similarly, let $V \in \mathcal{V}$. If $(c_{m-1}, c_m) \in V$, we set $q = m$. Otherwise, setting $c_{-1} = y$, there is a well defined integer $q \geq 0$ such that $(c_{q-1}, c_q) \in V$ but $(c_q, c_{q+1}) \notin V$. Moreover, the set $V' = V \cap \widehat{D}$ belongs to the set $\mathcal{S}_D^\flat$ of elements of $\mathcal{S}_D$ contained in $\widehat{D}$ and containing at least one edge of the form $(t, y)$, and

$$V = V' \sqcup \{(y, c_0), \ldots, (c_{q-1}, c_q)\} \ .$$

Moreover $\tau_V = \tau_{V'}[y, c_0, \ldots, c_l]$. Conversely, if $V' \in \mathcal{S}_D^\flat$ and $q \in \{0, \ldots, m\}$, then $V = V' \sqcup \{(y, c_0), \ldots, (c_{q-1}, c_q)\}$ belongs to $\mathcal{V}$. In this way we get a bijection

$$\begin{aligned} \beta : \mathcal{S}_D^\flat \times \{0, \ldots, m\} &\longrightarrow \mathcal{V} \\ (V', q) &\longmapsto V = V' \sqcup \{(y, c_0), \ldots, (c_{q-1}, c_q)\} \end{aligned}$$

with the property that $\tau_{\beta(V',q)} = \tau_{V'}[y, c_0, \ldots, c_q]$.

We can now compute $\Sigma$:

$$\begin{aligned} \Sigma &= \sum_{(U,V) \in \mathcal{U} \times \mathcal{V}} (-1)^{|U|+|V|} \tau_U \tau_V \\ &= \sum_{\substack{U' \in \mathcal{S}_C^\sharp \\ V' \in \mathcal{S}_D^\flat}} \sum_{\substack{1 \leq l \leq m \\ 0 \leq q \leq m}} (-1)^{|U'|+l+|V'|+q+1} \, \tau_{U'}[c_0, c_1, \ldots, c_l] \tau_{V'}[y, c_0, \ldots, c_q] \\ &= \sum_{\substack{U' \in \mathcal{S}_C^\sharp \\ V' \in \mathcal{S}_D^\flat}} (-1)^{|U'|+|V'|+1} \, \tau_{U'} \, \Sigma(V') \ , \end{aligned}$$

where, for a fixed $V' \in \mathcal{S}_D^\flat$,

$$\Sigma(V') = \sum_{\substack{1 \leq l \leq m \\ 0 \leq q \leq m}} (-1)^{l+q} [c_0, c_1, \ldots, c_l] \tau_{V'}[y, c_0, \ldots, c_q] \ .$$

Now $\Sigma(V') = \Sigma(V')_{\leq} + \Sigma(V')_{>}$, where

$$\begin{aligned} \Sigma(V')_{\leq} &= \sum_{1 \leq l \leq q \leq m} (-1)^{l+q} [c_0, c_1, \ldots, c_l] \tau_{V'}[y, c_0, \ldots, c_q] \\ \Sigma(V')_{>} &= \sum_{0 \leq q < l \leq m} (-1)^{l+q} [c_0, c_1, \ldots, c_l] \tau_{V'}[y, c_0, \ldots, c_q] \end{aligned}$$

For $0 \leq q < l \leq m$, by Proposition 22.1, we have

$$[c_0, c_1, \ldots, c_l] \tau_{V'}[y, c_0, \ldots, c_q] = [c_0, c_1, \ldots, c_{q+1}][c_{q+1}, \ldots, c_l] \tau_{V'}[y, c_0, \ldots, c_q] \ .$$

Moreover $[c_{q+1}, \ldots, c_l]$ clearly commutes with $\tau_{V'}$, since $V' \subseteq \widehat{D}$, and it also commutes with $[y, c_0, \ldots, c_q]$. Thus

$$[c_0, c_1, \ldots, c_l] \tau_{V'}[y, c_0, \ldots, c_q] = [c_0, c_1, \ldots, c_{q+1}] \tau_{V'}[y, c_0, \ldots, c_q][c_{q+1}, \ldots, c_l] \ .$$

Now the map $f = [c_0, c_1, \ldots, c_{q+1}]\tau_{V'}$ has the property that $f(c_q) = c_{q+1} = f(c_{q+1})$, so $f[y, c_0, \ldots, c_q] = f[y, c_0, \ldots, c_{q+1}]$, by Proposition 22.1 again. Thus

$$\begin{aligned}[c_0, c_1, \ldots, c_l]\tau_{V'}[y, c_0, \ldots, c_q] &= [c_0, c_1, \ldots, c_{q+1}]\tau_{V'}[y, c_0, \ldots, c_{q+1}][c_{q+1}, \ldots, c_l] \\ &= [c_0, c_1, \ldots, c_{q+1}]\tau_{V'}[y, c_0, \ldots, c_l] \ .\end{aligned}$$

This gives

$$\begin{aligned}\Sigma(V')_> &= \sum_{0 \le q < l \le m} (-1)^{l+q}[c_0, c_1, \ldots, c_{q+1}]\tau_{V'}[y, c_0, \ldots, c_l] \\ &= \sum_{1 \le q \le l \le m} (-1)^{l+q-1}[c_0, c_1, \ldots, c_q]\tau_{V'}[y, c_0, \ldots, c_l] \\ &= \sum_{1 \le l \le q \le m} (-1)^{l+q-1}[c_0, c_1, \ldots, c_l]\tau_{V'}[y, c_0, \ldots, c_q] \\ &= -\big(\Sigma(V')_\le\big) \ ,\end{aligned}$$

and it follows that $\Sigma(V') = 0$, for any $V' \in \mathcal{S}_D^\flat$. Our claim follows, and this completes the proof of Assertion (c).

(d) This follows from (a), (b), (c). □

**22.5. Corollary.** *Let $\mathcal{G}$ be a forest, and let $B$ be a set of leaves of $\mathcal{G}$. We say that a set $A$ of edges of $\mathcal{G}$ is* a union of partial geodesics with support $B$ *if $A$ is a union of partial geodesics starting in $B$ and if, moreover, for any $x \in B$, $(x, d(x)) \in A$. We denote by $\mathring{\mathcal{S}}_B$ the set of such sets of edges of $\mathcal{G}$.*

*We define the element $u_B$ of $k(\mathcal{G}^\mathcal{G})$ by*

$$u_B = \sum_{A \in \mathring{\mathcal{S}}_B} (-1)^{|B|+|A|}\tau_A \ .$$

*If $B$ contains a single element $x$, we set $u_x = u_B$.*

*(a) Let $x$ be a leaf of $\mathcal{G}$. Then*

$$u_x = \sum_{l=1}^{\delta(x)} (-1)^{l-1}[x, d(x), \ldots, d^l(x)] \ .$$

*Thus $u_x = \mathrm{id}_\mathcal{G} - v_x$. In particular $u_x^2 = u_x$.*

*(b) If $x$ and $y$ are leaves of $\mathcal{G}$, then $u_x u_y = u_y u_x$.*

*(c) Let $B$ be a set of leaves of $\mathcal{G}$. Then $u_B$ is an idempotent, equal to the product of the (commuting) idempotents $u_x$, for $x \in B$.*

**Proof :** (a) This is clear from the definition.

(b) This follows from (a), since $v_x$ and $v_y$ commute.

(c) This follows from the observation that $\mathcal{S}_B = \bigsqcup_{C \subseteq B} \mathring{\mathcal{S}}_C$. Thus

$$v_B = \sum_{C \subseteq B} w_C \ ,$$

where $w_C = \sum_{A\in\mathring{\mathcal{S}}_C} (-1)^{|A|}\tau_A$. By Möbius inversion in the poset of subsets of $B$, this gives

$$\begin{aligned} w_B &= \sum_{C\subseteq B} (-1)^{|B-C|} v_C \\ &= \sum_{C\subseteq B} (-1)^{|B-C|} \prod_{x\in C} (\mathrm{id}_{\mathcal{G}} - u_x) \\ &= \sum_{C\subseteq B} (-1)^{|B-C|} \sum_{D\subseteq C} (-1)^{|D|} \prod_{x\in D} u_x \\ &= \sum_{D\subseteq B} (-1)^{|D|} \prod_{x\in D} u_x \Big( \sum_{D\subseteq C\subseteq B} (-1)^{|B-C|} \Big) \\ &= (-1)^{|B|} \prod_{x\in B} u_x \ , \end{aligned}$$

becasue the inner sum is zero if $D \subset B$ with $D \neq B$. Hence $u_B = (-1)^{|B|} w_B = \prod_{x\in B} u_x$, as was to be shown. □

We conclude this section with a proof of Theorem 17.9.

**22.6. Theorem.** *Let $T$ be a finite lattice, let $(E,R)$ the full subposet of its irreducible elements, let $\Gamma = \{a \in T \mid a \notin E,\ a < r^\infty s^\infty(a)\}$, and let $\mathcal{G}(T)$ be the graph structure on $T$ introduced in Definition 17.8. For $a \in \Gamma$, let $u_a$ be the element of $k(T^T)$ introduced in Notation 17.4, and let $u_T$ denote the composition of all the elements $u_a$, for $a \in \Gamma$.*

*(a) The graph $\mathcal{G}(T)$ is a forest, and $\Gamma$ is the set of its leaves.*
*(b) For $a \in \Gamma$, the element $u_a$ is an idempotent of $k(T^T)$.*
*(c) $u_a \circ u_b = u_b \circ u_a$ for any $a, b \in \Gamma$.*
*(d) The element $u_T$ is an idempotent of $k(T^T)$. It is equal to*

$$u_T = \sum_{A\in\mathring{\mathcal{S}}_\Gamma} (-1)^{|\Gamma|+|A|}\tau_A \ ,$$

*where $\mathring{\mathcal{S}}_\Gamma$ is the set of sets of edges of $\mathcal{G}(T)$ which are union of partial geodesics with support $\Gamma$, introduced in Corollary 22.5.*

**Proof :** (a) If $x \twoheadrightarrow y$ is an edge in $\mathcal{G}(T)$, then $x <_T y$. So $\mathcal{G}(T)$ has no loops and no oriented cycles. Moreover, for each $x \in T$, there exists at most one edge $x \twoheadrightarrow y$ in $\mathcal{G}(T)$, so $\mathcal{G}(T)$ is a forest. By Definition 17.8, the set of leaves of $\mathcal{G}(T)$ is exactly $\Gamma$.

(b), (c), and (d) follow from the fact that if $a \in \Gamma$, then the complete geodesic of $\mathcal{G}(T)$ is the reduction sequence associated to $a$. Hence the element $u_a$ introduced in Notation 17.4 coincides with the element with the same name introduced in Corollary 22.5. □

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