ON A QUESTION OF RICKARD ON TENSOR PRODUCT OF STABLY EQUIVALENT ALGEBRAS

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Abstract. Let $r$ be a positive integer, let $p$ be a prime and $\mathbb{F}_p$ denote an algebraic closure of the prime field $\mathbb{F}_p$. After observing that the principal block $B$ of $\mathbb{F}_pPSU(3, p^r)$ is stably equivalent of Morita type to its Brauer correspondent $b$, we compute the radical series of the center $Z(b)$, and, using GAP, the radical series of $Z(B)$ in the cases $p^r \in \{3, 4, 5, 8\}$. In these cases, the dimensions of the last non zero power of the radical of $Z(b)$ and $Z(B)$ are different, and it follows that the algebra $B \otimes_{\mathbb{F}_p} \mathbb{F}_p[X]/X^p$ is not stably equivalent of Morita type to $b \otimes_{\mathbb{F}_p} \mathbb{F}_p[X]/X^p$. This yields a negative answer to a question of Rickard.

Introduction

Let $K$ be a field, and let $A$, $B$, $C$ and $D$ be finite dimensional $K$-algebras. Rickard showed in [12] that if $A$ and $B$ are derived equivalent, and if $C$ and $D$ are derived equivalent, then also $A \otimes_K C$ and $B \otimes_K D$ are derived equivalent. Rickard asks in [13, Question 3.8] if this still holds when replacing derived equivalence by stable equivalence of Morita type. It is clear that we have to suppose that all algebras involved have no semisimple direct factor. A result due to Liu [8] shows that then we may suppose that all algebras are indecomposable. In [10] Liu, Zhou and the second author showed that the question has a negative solution in case $A$, $B$, $C$ and $D$ are not necessarily selfinjective. However, a derived equivalence between selfinjective algebras $A$ and $B$ induces a stable equivalence of Morita type between $A$ and $B$. If $A$ and $B$ are not selfinjective, then this implication is not valid. Hence, the natural playground for Rickard’s question are selfinjective algebras.

The purpose of this paper is to give a counterexample to Rickard’s question. For an algebraically closed base field $K$ of characteristic $p$ we construct symmetric $K$-algebras $A$ and $B$ which are stably equivalent of Morita type, but $A \otimes_K K[X]/X^p$ and $B \otimes_K K[X]/X^p$ are not stably equivalent of Morita type.

Note that this answers the general case. Indeed, if $A \otimes_K C$ is stably equivalent of Morita type to $B \otimes_K C$ and $B \otimes_K C$ is stably equivalent of Morita type to $B \otimes_K D$ then $A \otimes_K C$ is stably equivalent of Morita type to $B \otimes_K D$. Hence, we may suppose $C = D$.

In recent years many attempts were proposed to lift a stable equivalence of Morita type between selfinjective algebras to a derived equivalence. It is known that this is not possible in general, as is seen by the mod 2 group ring of a dihedral group of order 8 and the stable equivalence induced by a uniserial endotrivial module of Loewy length 3. This was used in [10] for example. In this paper we give a new incidence of this fact. Moreover, we provide two symmetric algebras, which are stably equivalent of Morita type, and have non isomorphic centres.

Our example is the principal $p$-block of the group $PSU(3, p^r)$ and its Brauer correspondent for $p^r \in \{3, 4, 5, 8\}$.

We recall in the first section some basic facts and results which we need for our construction. In Section 2 we give our main result and its proof, and in Section 3 we display the GAP program needed for the proof. In Section 4 we determine the algebraic structure of
the centre of $KN_G(S)$ for $G = PSU(3, p^r)$ and $S$ one of its Sylow $p$-subgroups for all primes $p$ and integers $r$.

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1. Background

Recall the following

Definition 1. [2], (cf also [14, Chapter 5]) Let $A$ and $B$ be two finite dimensional algebras over a field $K$. Then $A$ and $B$ are stably equivalent of Morita type if there is an $A \otimes_K B^{op}$-module $M$ and a $B \otimes_K A^{op}$-module $N$ such that

- $M$ is projective as $A$-module, and as $B^{op}$-module
- $N$ is projective as $A^{op}$-module and as $B$-module
- there is a projective $A \otimes_K A^{op}$-module $P$ and a projective $B \otimes_K B^{op}$-module $Q$ such that $M \otimes_B N \simeq B \oplus Q$ as $B \otimes_K B^{op}$-modules and $N \otimes_A M \simeq A \oplus P$ as $A \otimes_K A^{op}$-modules.

Independently Rickard [11] as well as Keller and Vossieck [6], show that if $A$ and $B$ are derived selfinjective algebras, then $A$ and $B$ are stably equivalent of Morita type.

Broué defined $Z^{st}(A) := \text{End}_{A \otimes_K A^{op}}(A)$ and

$$Z^{pr}(A) := \ker(\text{End}_{A \otimes_K A^{op}}(A) \to \text{End}_{A \otimes_K A^{op}}(A))$$

where we denote by $\text{End}$ the endomorphisms taken in the stable module category.

The centre of an algebra is an invariant of a derived equivalence, as was shown by Rickard. The stable centre $Z^{st}(A)$ is an important invariant under stable equivalences of Morita type, as was shown by Broué.

Proposition 2. (Broué [2]; see also [14, Chapter 5]) If $A$ and $B$ are stably equivalent of Morita type, then $Z^{st}(A) \simeq Z^{st}(B)$ as algebras.

Now, Liu, Zhou and the second author give a criterion to determine the dimension of $Z^{st}(A)$.

Theorem 3. [9, Proposition 2.3 and Corollary 2.7] Let $A$ be a finite dimensional symmetric algebra over an algebraically closed field $K$ of characteristic $p > 0$. Then $\dim_K(Z^{pr}(A)) = \text{rank}_p(C_A)$ where $C_A$ is the Cartan matrix of $A$ and where $\text{rank}_p(C_A)$ denotes its rank as matrix over $K$.

Moreover, we recall a conjecture of Auslander-Reiten. In [1] Auslander and Reiten conjecture that if $A$ and $B$ are stably equivalent finite dimensional algebras, then the number of simple non-projective $A$-modules and the number of non-projective simple $B$-modules coincide. Again in [9] we show

Theorem 4. [9, Theorem 1.1] Let $K$ be an algebraically closed field and let $A$ and $B$ be two finite dimensional $K$-algebras, which are stably equivalent of Morita type and which do not have any semisimple direct factor. Then the number of isomorphism classes of non-projective simple $A$-modules is equal to the number of non-projective simple $B$-modules if and only if $\dim_K(HH_0(A)) = \dim_K(HH_0(B))$, where $HH_0$ denotes the degree 0 Hochschild homology.

In particular, if $A$ and $B$ are symmetric, then Hochschild homology and cohomology coincide, and the number of non-projective simple $A$-modules is equal to the number of non-projective simple $B$-modules if and only if the centres of $A$ and of $B$ have the same dimension.
The following lemma is well-known to the experts, but for the convenience of the reader, and since it is crucial to our arguments, we include the short proof. For an algebra $A$ denote by $J(A)$ its Jacobson radical.

**Lemma 5.** Let $K$ be a perfect field and let $A$ and $B$ be finite dimensional $K$-algebras. Then $J(A \otimes_K B) = J(A) \otimes_K B + A \otimes_K J(B)$.

**Proof.** It is clear that $J(A) \otimes_K B + A \otimes_K J(B)$ is a nilpotent ideal of $A \otimes_K B$, and therefore we get

$$J(A) \otimes_K B + A \otimes_K J(B) \subseteq J(A \otimes_K B).$$

Now, $(A \otimes_K B)/(J(A) \otimes_K B + A \otimes_K J(B)) = A/J(A) \otimes_K B/J(B)$ and both $K$-algebras $A/J(A)$ and $B/J(B)$ are semisimple. Since $K$ is perfect, every finite extension $L$ of $K$ is a separable field extension. By [3, Corollary 7.6] a finite dimensional semisimple $K$-algebra $C$ is separable if and only if the centres of each of the Wedderburn components is a separable field extension of $K$. Hence $A/J(A)$ and $B/J(B)$ are both separable $K$-algebras. By [3, Corollary 7.8] the algebra $A/J(A) \otimes_K B/J(B)$ is semisimple. Therefore

$$J(A) \otimes_K B + A \otimes_K J(B) \supseteq J(A \otimes_K B).$$

This shows the statement. □

**Remark 6.** (cf e.g. [14, Example 1.7.17]) Lemma 5 is wrong if we drop the assumption that $K$ is perfect: e.g. let $p$ be a prime, and $K = \mathbb{F}_p(U)$ be the field of rational fractions over the finite field $\mathbb{F}_p$. Let $A = K[X]/(X^p - U)$. Then $A$ is a purely inseparable extension of $K$, of dimension $p$. In particular it is a reduced (commutative) algebra, i.e. $J(A) = 0$. But $A \otimes_K A \cong K[X,Y]/(X^p - U, Y^p - U)$ contains the non zero element $X - Y$, such that $(X - Y)^p = U - U = 0$. Hence $J(A \otimes_K A) \neq 0$.

**Lemma 7.** Let $K$ be an algebraically closed field, let $\Lambda$ and $\Delta$ be finite dimensional $K$-algebras, and suppose that $\Delta$ is local. Then the projective indecomposable $\Lambda \otimes_K \Delta$-modules are precisely the modules $P \otimes_K \Delta$ for projective indecomposable $\Lambda$-modules $P$, and if $C_\Lambda$ is the Cartan matrix of $\Lambda$, then the Cartan matrix of $\Lambda \otimes \Delta$ is $C_{\Lambda \otimes \Delta} = \dim_K(\Delta) \cdot C_\Lambda$.

**Proof.** Let $P$ and $Q$ be a indecomposable projective $\Lambda$-modules. Then $P \otimes_K \Delta$ is a projective indecomposable $\Lambda \otimes_K \Delta$-module. Indeed, $\text{End}_{\Lambda \otimes \Delta}(P \otimes_K \Delta) \cong \text{End}_\Lambda(P) \otimes_K \Delta^{op}$.

Moreover, since $\Gamma := \text{End}_\Lambda(P)^{op}$ and $\Delta$ are local $K$-algebras their radical quotient are finite-dimensional skew-fields, and therefore $\Gamma/J(\Gamma) \cong K = \Delta/J(\Delta)$ since $K$ is algebraically closed. Moreover, by Lemma 5 we get $J(\Gamma \otimes \Delta) = J(\Gamma) \otimes \Delta + \Gamma \otimes_K J(\Delta)$. On the other hand,

$$(\Gamma \otimes \Delta)/(J(\Gamma) \otimes \Delta + \Gamma \otimes_K J(\Delta)) = K \otimes_K K = K$$

and hence we get $\Gamma \otimes_K \Delta$ is local, and therefore $P \otimes_K \Delta$ is indecomposable. Now,

$$\text{Hom}_{\Lambda \otimes \Delta}(P \otimes_K \Delta, Q \otimes_K \Delta) = \text{Hom}_\Lambda(P, Q) \otimes_K \Delta^{op}.$$ 

Taking $K$-dimensions proves the lemma. □

**Remark 8.** As a special case of Lemma 7 we get $C_{A \otimes_K K[X]/X^p} = p \cdot C_A$ for algebraically closed fields $K$ of characteristic $p$. Hence we get by Theorem 3 that $Z^{op}(A \otimes_K K[X]/X^p) = 0$ for algebraically closed fields $K$ of characteristic $p$ and symmetric $K$-algebras $A$.

**Lemma 9.** Let $K$ be a perfect field and let $n, m$ be positive integers. Let $A$ and $B$ be finite dimensional commutative $K$-algebras. If $J^{n+1}(A) = 0 \neq J^n(A)$ and $J^{m+1}(B) = 0 \neq J^m(B)$, then

$$J^{n+m+1}(A \otimes_K B) = 0 \neq J^{n+m}(A \otimes_K B) = J^n(A) \otimes_K J^m(B).$$

**Proof.** By Lemma 5, we have $J(A \otimes_K B) = J(A) \otimes_K B + A \otimes_K J(B)$. Therefore

$$J^{n+m+1}(A \otimes_K B) = \sum_{k=0}^{n+m+1} J^k(A) \otimes_K J^{n+m+1-k}(B) = 0.$$
Similarly
\[ J^{n+m}(A \otimes_K B) = \sum_{k=0}^{n+m} J^k(A) \otimes_K J^{n+m-k}(B) = J^n(A) \otimes_K J^m(B) \neq 0, \]
which completes the proof. \( \square \)

**Remark 10.** Let \( K \) be any field, and \( A \) be a \( K \)-algebra. We give an elementary argument to determine the centre of \( A \otimes_K K[X]/X^p \). It is clear that \( A \otimes_K K[X]/X^p \cong A[X]/X^p \).

Now, let \( a := a_0 + a_1 X + \ldots + a_{p-1} X^{p-1} \in A[X] \). Then for \( b := b_0 \in A \cdot 1 \) we get
\[ ab - ba = (a_0 b - b_0) + \cdots + (a_{p-2} b_{p-1} + a_{p-1} b_{p-1}) X^{p-1} \]
and so \( a \in Z(A) \) implies that \( a \) commutes with any \( b \in A \), and hence \( a_0, \ldots, a_{p-1} \) are all in \( Z(A) \). Conversely, it is clear that \( Z(A)[X]/X^p \subseteq Z(A[X]/X^p) \) since \( aX^n \) commutes with all elements of \( A[X]/X^p \) whenever \( a \in A \) and since sums of elements in the centre are still central.

**Lemma 11.** If \( K \) is a perfect field and \( A \) is a finite dimensional \( K \)-algebra, and if moreover \( J^n(Z(A)) \neq 0 = J^{n+1}(Z(A)) \), then
\[ 0 \neq J^{n+p-1}(Z(A)) = J^n(Z(A)) \otimes_K X^{p-1} K[X]/X^p \]
and
\[ J^{n+p}(Z(A) \otimes_K K[X]/X^p) = 0. \]

**Proof.** This is an immediate consequence of Lemma 9. \( \square \)

**Corollary 12.** Let \( K \) be an algebraically closed field of characteristic \( p > 0 \) and let \( A \) and \( B \) be two finite dimensional \( K \)-algebras and let \( n, m \in \mathbb{N} \) such that \( J^n(Z(A)) \neq 0 = J^{n+1}(Z(A)) \) and \( J^m(Z(B)) \neq 0 = J^{m+1}(Z(B)) \). If \( \dim_K(J^n(Z(A))) \neq \dim_K(J^m(Z(B))) \), then \( A \otimes_K K[X]/X^p \) and \( B \otimes_K K[X]/X^p \) are not stably equivalent of Morita type.

**Proof.** If \( n \neq m \), then \( Z(A \otimes_K K[X]/X^p) \neq Z(B \otimes_K K[X]/X^p) \) by Lemma 11 since the Loewy lengths of the centres are different. If \( n = m \), then Lemma 11 shows that the centres of \( A \otimes_K K[X]/X^p \) and of \( B \otimes_K K[X]/X^p \) are not isomorphic since the dimension of the lowest Loewy layers of the centres are not of the same dimension. Remark 8 shows that \( Z(A \otimes_K K[X]/X^p) = Z^{st}(A \otimes_K K[X]/X^p) \) and \( Z(B \otimes_K K[X]/X^p) = Z^{st}(B \otimes_K K[X]/X^p) \). Since the stable centre is invariant under stable equivalence of Morita type, we get the statement. \( \square \)

**Remark 13.** For a field \( K \) and a \( K \)-algebra \( A \) let \( n_A \) be the number of isomorphism classes of simple nonprojective \( A \)-modules. Auslander-Reiten conjecture [1, page 409, Conjecture (5)] that if \( A \) and \( B \) are stably equivalent finite dimensional \( K \)-algebras, then \( n_A = n_B \). Now [9, Theorem 1.1] shows that if \( K \) is algebraically closed and if \( A \) and \( B \) are indecomposable finite dimensional \( K \)-algebras which are stably equivalent of Morita type, then \( n_A = n_B \) is equivalent to \( \dim_K(\text{HH}_0(A)) = \dim_K(\text{HH}_0(B)) \). If \( A \) is symmetric, then there is a vector space isomorphism \( \text{HH}_0(A) \cong H^0(A) = Z(A) \), we see that the Auslander-Reiten conjecture implies that \( \dim_K(Z(A)) = \dim_K(Z(B)) \). More precisely by [9, Corollary 1.2], for two indecomposable symmetric algebras \( A \) and \( B \) over an algebraically closed field \( K \) we have \( n_A = n_B \Leftrightarrow \dim_K(Z^{pr}(A)) = \dim_K(Z^{pr}(B)) \), where by definition \( Z^{st}(A) = Z(A)/Z^{pr}(A) \). The link to our proof is now given by the fact that for every algebra the Higman ideal \( H(A) \) of \( A \) is equal to \( Z^{pr}(A) \), and for symmetric algebras \( A \) over an algebraically closed field \( K \) we have that \( \dim_K(H(A)) \) is equal to the \( p \)-rank of \( C_A \).
2. The Example

Let \( \overline{\mathbb{F}}_p \) be the algebraic closure of the prime field \( \mathbb{F}_p \) of characteristic \( p \). Let \( q = p^r \) for some positive integer \( r \).

We recall some results on the geometry of \( PSU(3, q) \) (cf e.g. [5, II Satz 10.12, page 242]). The group \( G := PSU(3, q) \) acts doubly transitively on the unitary quadric \( Q \) of cardinal \( q^3 + 1 \). Note that we use the GAP notation, not the notation used in [5, II Satz 10.12, page 242], namely, \( PSU(3, q) \) is defined over a field with \( q^2 \) elements, and is a natural quotient of a subgroup of \( SL_2(q^2) \) (and not of \( SL_2(q) \) !). The stabiliser of a point \( X \) of \( Q \) is the normaliser in \( G \) of a Sylow \( p \)-subgroup \( P \) of \( G \). Therefore two different conjugate Sylow \( p \)-subgroups \( P \) and \( qP \) of \( G \) fix two different points \( X \) and \( gX \) of \( Q \). Hence \( qP \cap P = 1 \) if \( g \notin N_G(P) \), or in other words, \( G \) has a trivial intersection Sylow \( p \)-subgroup structure. This implies that Green correspondence gives a stable equivalence of Morita type between the principal block \( B \) of \( \overline{\mathbb{F}}_p G \) and its Brauer correspondent \( b \) (cf e.g. [14, Chapter 2, Theorem 2.1.21, Proposition 2.1.23 and Proposition 5.3.17]).

The GAP [4] program in Section 3 computes the Loewy series of the ring \( Z(\mathbb{F}_2 PSU(3,4)) \) and of \( Z(\mathbb{F}_2 N_{PSU(3,4)}(S)) \) for some Sylow 2-subgroup of \( PSU(3, 4) \). Observe moreover that \( \mathbb{F}_2 PSU(3, 4) \) has two blocks, the principal one and another block of defect 0 (corresponding to the Steinberg character). Moreover, the dimensions of the Loewy series obtained over \( \mathbb{F}_2 \) also hold by extending the scalars to \( \overline{\mathbb{F}}_2 \), using Lemma 5.

We obtain that

\[
\dim_{\mathbb{F}_2}(Z(B)) = 21 = \dim_{\mathbb{F}_2}(Z(b)) \\
\dim_{\mathbb{F}_2}(J(Z(B))) = 20 = \dim_{\mathbb{F}_2}(J(Z(b))) \\
\dim_{\mathbb{F}_2}(J^2(Z(B))) = 5 \neq 4 = \dim_{\mathbb{F}_2}(J^2(Z(b))) \\
\dim_{\mathbb{F}_2}(J^3(Z(B))) = 0 = \dim_{\mathbb{F}_2}(J^3(Z(b))).
\]

Similarly we get for the centre of the principal block \( B \) of \( \mathbb{F}_2 PSU(3, 8) \) and the centre of its Brauer correspondent \( b \)

\[
\dim_{\mathbb{F}_2}(Z(B)) = 27 = \dim_{\mathbb{F}_2}(Z(b)) \\
\dim_{\mathbb{F}_2}(J(Z(B))) = 26 = \dim_{\mathbb{F}_2}(J(Z(b))) \\
\dim_{\mathbb{F}_2}(J^2(Z(B))) = 3 \neq 2 = \dim_{\mathbb{F}_2}(J^2(Z(b))) \\
\dim_{\mathbb{F}_2}(J^3(Z(B))) = 0 = \dim_{\mathbb{F}_2}(J^3(Z(b))).
\]

An immediate variant of the program shows that this is a quite general phenomenon in odd characteristic. The group \( PSU(3, 3) \) gives an example in characteristic 3 since, denoting by \( B \) the principal block of \( \mathbb{F}_3 PSU(3, 3) \) and by \( b \) its Brauer correspondent,

\[
\dim_{\mathbb{F}_3}(Z(B)) = 13 = \dim_{\mathbb{F}_3}(Z(b)) \\
\dim_{\mathbb{F}_3}(J(Z(B))) = 12 = \dim_{\mathbb{F}_3}(J(Z(b))) \\
\dim_{\mathbb{F}_3}(J^2(Z(B))) = 4 \neq 3 = \dim_{\mathbb{F}_3}(J^2(Z(b))) \\
\dim_{\mathbb{F}_3}(J^3(Z(B))) = 0 = \dim_{\mathbb{F}_3}(J^3(Z(b))).
\]

The group \( PSU(3, 5) \) gives an example in characteristic 5 since, denoting by \( B \) the principal block of \( \mathbb{F}_5 PSU(3, 5) \) and by \( b \) its Brauer correspondent,

\[
\dim_{\mathbb{F}_5}(Z(B)) = 13 = \dim_{\mathbb{F}_5}(Z(b)) \\
\dim_{\mathbb{F}_5}(J(Z(B))) = 12 = \dim_{\mathbb{F}_5}(J(Z(b))) \\
\dim_{\mathbb{F}_5}(J^2(Z(B))) = 2 \neq 1 = \dim_{\mathbb{F}_5}(J^2(Z(b))) \\
\dim_{\mathbb{F}_5}(J^3(Z(B))) = 0 = \dim_{\mathbb{F}_5}(J^3(Z(b))).
\]
Theorem 14. Let $K$ be an algebraic closure of $\mathbb{F}_p$ and let $B$ be the principal block of $K\text{PSU}(3,p^r)$. Let $b$ be the Brauer correspondent of $B$ in the group ring of the normaliser of a 2-Sylow subgroup of $PSU(3,p^r)$. Then $B$ and $b$ are stably equivalent of Morita type. If moreover $p^r \in \{3, 4, 5, 8\}$, then the square of the Jacobson radical of $Z(B)$ is of different dimension than the square of the Jacobson radical of $Z(b)$, whereas $Z(B)$ and $Z(b)$ both have Loewy length 3. In particular $B \otimes_K K[X]/X^p$ is not stably equivalent of Morita type to $b \otimes_K K[X]/X^p$.

Proof. As seen at the beginning of this section, $B$ and $b$ are stably equivalent of Morita type by Green correspondence.

The GAP [4] program in Section 3 shows that the Loewy series of the centres of $B$ and of $b$ are of the same length but the dimensions of the Loewy layers are not equal. In particular the lowest Loewy layers of the algebras $Z(B)$ and $Z(b)$ have different dimension.

Corollary 12 implies that $B \otimes_K K[X]/X^p$ is not stably equivalent of Morita type to $b \otimes_K K[X]/X^p$. \hfill $\square$

Remark 15. The above examples suggest that in general, with the notation of Theorem 14, the dimension of $J^2(Z(B))$ could always be equal to $1 + \dim_K J^2(Z(b))$. By Theorem 41, this is equal to $\frac{p^r + 1}{\gamma}$, where $\gamma$ is the greatest common divisor of $p^r + 1$ and 3.

3. The GAP program

We display here the GAP program we used.

```gap
# the characteristic p
prem:=2;
#
# The group G
g:=PSU(3,prem^2);
#
# the ground field k
corps:=GF(prem);
#
s:=SylowSubgroup(g,prem);
# the normalizer NS of a Sylow p-subgroup
ns:=Normalizer(g,s);
#
# getting a permutation representation of G of smaller degree
f:=FactorCosetAction(g,ns);
g:=Image(f);
ns:=Image(f,ns);
#
# uncomment next line to replace G by NS
#g:=ns;
#
# computing the structure constants of ZkG
c:=ConjugacyClasses(g);
rc:=List(c,Representative);
lc:=Length(c);
ci:=List([1..lc],x->First([1..lc],y->rc[x]^(-1) in c[y]));
l:=List([1..lc],x->NullMat(lc,lc,corps));
for iu in [1..lc] do
    u:=ci[iu];
    if rc[iu]=One(g) then
        for iv in [1..lc] do
            Print("\r",iu,"/",iv,"/",lc," ");
            v:=List([1..lc],x->Zero(corps));
            v[iv]:=One(corps);
            l[iu][iv]:=v;
            l[iv][iu]:=v;
```
od;
else
    for iv in [iu..lc] do
        Print("\r",iu,"":"",iv,"/",lc," ");
        w:=c[ci[iv]];
        v:=List(List(rc),x->One(corps)*Size(Intersection(u,List(w,y->x*y))));
        l[iu][iv]:=v;
        l[iv][iu]:=v;
    od;
fi;
od;
Print("\n");
za:=Algebra(corps,l);
Print("Dimension of ZkG \t= ",Dimension(za),"\n");
radza:=RadicalOfAlgebra(za);
Print("Dimension of JZkG \t= ",Dimension(radza),"\n");
bradza:=Basis(radza);
vbradza:=BasisVectors(bradza);
vbr:=vbradza;
#
# Computing the powers of the radical of the center
i:=1;
repeat
    i:=i+1;
    l:=Set(List(Cartesian(vbradza,vbr),x->x[1]*x[2]));
    r:=Ideal(za,l);
    br:=Basis(r);
    vbr:=BasisVectors(br);
    d:=Dimension(r);
    Print("Dimension of (JZkG)^",i,"\t= ",d,"\n");
until d=0;

4. The centre of the mod p group ring of the normaliser of the Sylow subgroup of \( PSU(3,p') \)

Recall that we denote by \( S \) a Sylow \( p \)-subgroup of the projective special unitary group \( G = PSU(3,q) \) over the field with \( q^2 \) elements, where \( q = p' \), and by \( N \) the normaliser of \( S \) in \( G \). In this section, we determine the ring structure of the center \( ZkN \) of the group algebra \( kN \), where \( k \) is any commutative ring.

**Notation 16.** If \( x \in N \), we denote by \( x^+ \in ZkN \) the sum of the conjugates of \( x \) in \( N \).

Then the elements \( x^+ \), for \( x \) in a set of representatives of conjugacy classes of \( N \), form a \( k \)-basis of \( ZkN \).

Let \( V \) be a three dimensional vector space over the field \( \mathbb{F}_{q^2} \), with basis \( B \). We endow \( V \) with a non degenerate hermitian product, and without loss of generality, we assume that the matrix of this product in \( B \) is equal to
\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{pmatrix}
\].

**Notation 17.** For \( x \in \mathbb{F}_{q^2} \), we set \( \overline{x} = x^q \). Then the map \( x \mapsto \overline{x} \) is the automorphism of order 2 of the extension \( \mathbb{F}_{q^2}/\mathbb{F}_q \). We also set
\[
\Psi = \{ x \in \mathbb{F}_{q^2}^* \mid x\overline{x} = 1 \}
\]

Let \( \omega \) be a non zero element of \( \mathbb{F}_{q^2} \) such that \( \omega + \overline{\omega} = 0 \), and \( \tau \) be an element of \( \mathbb{F}_{q^2} \) such that \( \tau + \overline{\tau} = -1 \).
It follows from [5, II Satz 10.12, page 242] that we can suppose that the group $N$ is equal to the image in $G$ of the group of matrices of the form

$$M(a,b,c) = \begin{pmatrix} a & b & c \\ 0 & \alpha/a & -\bar{b}/a \\ 0 & 0 & 1/\alpha \end{pmatrix},$$

where $(a,b,c)$ belongs to the set

$$\mathcal{Q} = \{(a,b,c) \in \mathbb{F}_q^3 \times (\mathbb{F}_q)^2 | b\bar{b} + a\bar{c} + c\alpha = 0\}.$$

**Lemma 18.** For $(a,b,c) \in \mathcal{Q}$, let $\hat{M}(a,b,c)$ denote the image of $M(a,b,c)$ in $N$. Then if $(a',b',c') \in \mathcal{Q}$, we have that $\hat{M}(a,b,c) = \hat{M}(a',b',c')$ if and only if there exists $\lambda \in \mathbb{F}_q$ with $\lambda^{q-2} = 1$ and $(a',b',c') = \lambda \cdot (a,b,c)$.

**Proof.** $\hat{M}(a,b,c) = \hat{M}(a',b',c')$ if and only if there exists a scalar $\lambda \in \mathbb{F}_q$ such that

$$\begin{pmatrix} a' & b' & c' \\ 0 & \alpha'/a' & -\bar{b}/a' \\ 0 & 0 & 1/\alpha' \end{pmatrix} = \lambda \begin{pmatrix} a & b & c \\ 0 & \alpha/a & -\bar{b}/a \\ 0 & 0 & 1/\alpha \end{pmatrix}.$$

Equivalently $(a',b',c') = \lambda \cdot (a,b,c)$ and $\lambda/\lambda = \lambda$, i.e. $\lambda^{q-2} = 1$. \hfill \Box

For two non-zero integers $s,t$ denote by $(s,t)$ their greatest common divisor. Observe that $(q-2,q^2-1) = (3,q+1)$, to motivate the following:

**Notation 19.** We set $\gamma = (3,q+1)$, and put

$$\Gamma = \{\lambda \in \mathbb{F}_q^* | \lambda^\gamma = 1\} \leq \Psi$$

as well as $L = \{a^\gamma | a \in \mathbb{F}_q^*\}$.

With this notation, the group $N$ has order $q^3(q^2-1)/\gamma$. It is equal to the semidirect product of the group $S$, consisting of the elements $\hat{M}(1,b,c) = \begin{pmatrix} 1 & b & c \\ 0 & 1 & -\bar{b} \\ 0 & 0 & 1 \end{pmatrix}$, where $b$ and $c$ are elements of $\mathbb{F}_q$ such that $b\bar{b} + c + \bar{c} = 0$, by the cyclic group $C$ of order $(q^2-1)/\gamma$ consisting of the elements $\hat{M}(a,0,0) = \begin{pmatrix} a & 0 & 0 \\ 0 & \alpha/a & 0 \\ 0 & 0 & 1/\alpha \end{pmatrix}$, for $a \in \mathbb{F}_q^*/\Gamma$.

**Lemma 20.**

1. Let $(a,b,c)$ and $(x,y,z)$ be elements of $\mathcal{Q}$. Then

$$M(a,b,c)M(x,y,z) = M(ax,ay + \frac{b\pi}{x},az - \frac{b\eta}{x} + \frac{c}{\pi}),$$

2. Let $(a,b,c) \in \mathcal{Q}$. Then $M(a,b,c)^{-1} = M\left(1, \frac{-b}{\alpha},\frac{-c}{\alpha}\right)$.

3. Let $(a,b,c)$ and $(x,y,z)$ be elements of $\mathcal{Q}$. Then

$$M(a,b,c)M(x,y,z)M(a,b,c)^{-1} = M\left(x, \frac{ab}{\alpha}(\frac{\pi}{x} - x) + \frac{a^2}{\alpha}y, t\right),$$

where $t = a\bar{c}x + \frac{\alpha c}{\pi} + a\bar{b}y - \frac{b\bar{\eta}}{x} + \frac{b\pi \bar{c}}{x} + a\bar{c}z$.

**Proof.** All the assertions follow from straightforward computations. \hfill \Box

**Proposition 21.**

1. Let $(x,y,z)$ and $(x',y',z')$ be elements of $\mathcal{Q}$. If $\hat{M}(x,y,z)$ and $\hat{M}(x',y',z')$ are conjugate in $N$, then $x^{-1}x' \in \Gamma$.

2. The elements $\hat{M}(x,0,0)$, for $x \in \mathbb{F}_q/\Gamma$, lie in distinct conjugacy classes of $N$.

3. Let $(x,y,z) \in \mathcal{Q}$. Then if $x \notin \Gamma$, the element $\hat{M}(x,y,z)$ of $N$ is conjugate to an element of the form $\hat{M}(x,0,ux\omega)$, for some $u \in \mathbb{F}_q$. \hfill \Box
(4) Let \( x \in \mathbb{F}_q^\times \) and \( u \in \mathbb{F}_q \). Then if \( x \overline{x} \neq 1 \), the element \( \overline{M}(x, 0, xu) \) of \( N \) is conjugate to \( \overline{M}(x, 0, 0) \). If \( x \overline{x} = 1 \), and if \( u \neq 0 \), then the element \( \overline{M}(x, 0, xu) \) is conjugate to \( \overline{M}(x, 0, x) \), and not conjugate to \( \overline{M}(x, 0, 0) \).

(5) If \((1, y, z) \in Q\), then either \( y \neq 0 \) and there exists \( u \in \mathbb{F}_q \) such that \( z = y \overline{y}(\tau + u) \), or \( y = 0 \) and there exists \( u \in \mathbb{F}_q \) such that \( z = u \overline{y} \). Moreover, if \((1, y', z') \in Q\) and \( \overline{M}(1, y', z') \) and \( \overline{M}(1, y, z) \) are conjugate in \( N \), then \( y \) and \( y' \) are both non zero, or both equal to 0.

(6) If \((1, y, z) \in Q\), and if \( y \) and \( y' \) are both non zero, \( \overline{M}(1, y', z') \) and \( \overline{M}(1, y, z) \) are conjugate in \( N \) if and only if \( y'(q^2-1)/\gamma = y(q^2-1)/\gamma \) in \( \mathbb{F}_q^\times \), i.e. if \( y'/y \in L \). In particular \( M(1, y, z) \) is conjugate to \( M(1, y, y \overline{y}) \).

**Proof.** Assertion (1) follows from Assertion (3) of Lemma 20: if

\[
\overline{M}(x', y', z') = \overline{M}\left( x; \frac{ab}{\overline{a}}\left( \frac{x'}{\overline{x}} - x \right) + \frac{a^2}{\overline{a}}y, t \right),
\]

then there exists \( \lambda \in \Gamma \) such that \( x' = \lambda x \) by Lemma 18.

 Assertion (2) is a straightforward consequence of Assertion (1).

For Assertion (3), we use Assertion (3) of Lemma 20 again: since \( x \notin \Gamma \), we have \( \frac{\overline{x}}{x} \neq x \), and we can set \( a = 1 \), \( b = -\frac{y}{\overline{x} - x} \), and \( c = b \overline{y} \). Then \((a, b, c) \in Q\) and \( M(a, b, c)M(x, y, z)M(a, b, c)^{-1} \) is of the form \( M(x, 0, t) \), for some \( t \in \mathbb{F}_q \). In particular \( (x, 0, t) \in Q \), hence \( x \overline{y} + ty = 0 \). In other words \( t = vx \) with \( v + \overline{v} = 0 \). Then \( v = \overline{w} \) and \( u = \overline{w} \), that is \( u \in \mathbb{F}_q \).

For Assertion (4), we have to decide when two elements of the form \( n = \overline{M}(x, 0, xu) \) and \( n' = \overline{M}(x', 0, xu') \) are conjugate in \( N \), where \( x, x' \notin \Gamma \), and \( u, u' \in \mathbb{F}_q \). By Assertion (1), we can assume that \( x = x' \), and then \( n \) and \( n' \) are conjugate if and only if there exists \((a, b, c) \in Q\) such that

\[
M(a, b, c)M(x, 0, xu)M(a, b, c)^{-1} = M(x, 0, xu') .
\]

By Assertion (3) of Lemma 20, we have \( \frac{ab}{\overline{a}}\left( \frac{\overline{x}}{x} - x \right) = 0 \), hence \( b = 0 \). Now \( xu'x \) is equal to the element \( t \) of Lemma 20, in the case \( y = b = 0 \) and \( z = xuw \), that is

\[
xu' = a\overline{c}x + \frac{\overline{c}x}{x} + a\overline{c}xu .
\]

Moreover \( a\overline{c} + c\overline{a} = 0 \), since \((a, 0, c) \in Q\). So there exists \( v \in \mathbb{F}_q \) such that \( c = avu \). This gives

\[
xu' = a\overline{c}x + avu + a\overline{c}xu ,
\]

or equivalently

\[
u' = a\overline{c}\left( u - v\left( 1 - \frac{1}{x} \right) \right) .
\]

Thus \( n \) and \( n' \) are conjugate in \( N \) if and only if there exist \( a \in \mathbb{F}_q^\times \) and \( v \in \mathbb{F}_q \) such that \( u' = a\overline{c}\left( u - v\left( 1 - \frac{1}{x} \right) \right) \). If \( x \overline{x} = 1 \), then we can take \( a = 1 \) and \( v = \frac{u - u'}{1 - \frac{1}{x}} \), so \( n \) and \( n' \) are conjugate. And if \( x \overline{x} = 1 \), then \( n \) and \( n' \) are conjugate if and only if there exists \( a \in \mathbb{F}_q^\times \) such that \( u' = a\overline{c}u \), or equivalently, if there exists \( \lambda \in \mathbb{F}_q^\times \) such that \( u' = \lambda u \). So either \( u = u' = 0 \), or \( u \) and \( u' \) are both non zero. This completes the proof of Assertion (4).

For Assertion (5), assume that \((1, y, z) \in Q\). Then \( y\overline{y} + z + \overline{z} = 0 \). If \( y \neq 0 \), set \( v = \frac{z}{y\overline{y}} - \tau \). Then \( v + \overline{v} = 0 \), so there exists \( u \in \mathbb{F}_q \) such that \( v = u \overline{y} \), thus \( u = y\overline{y}(\tau + \overline{y}) \). And if \( y = 0 \), then \( z + \overline{z} = 0 \), so \( z = u \overline{y} \) for some \( u \in \mathbb{F}_q \).
Now by Assertion (3) of Lemma 20, for $(1,y,z)$ and $(1,y',z')$ in $Q$, the elements $n = M(1,y,z)$ and $n' = M(1,y',z')$ are conjugate in $N$ if and only if there exists $(a,b,c) \in Q$ such that

$$
y' = \frac{a^2}{a} y \quad \text{and} \quad z' = a \bar{c} + \bar{a} c + a y b - a b y + b b + a a z,
$$

that is

$$
y' = \frac{a^2}{a} y \quad \text{and} \quad z' = a y b - a b y + a a z,
$$

In particular $y$ is non zero if and only if $y'$ is non zero. Assertion (5) follows.

Assume now that both $y$ and $y'$ are non zero. If $n$ and $n'$ are conjugate, then there exists $a \in F_q^\infty$ such that $y' = \frac{a^2}{a} y = a^2 - q y$. It follows that $y'/y$ belongs to the subgroup of $F_q^\infty$ consisting of $(q - 2)$-th powers, i.e. the unique subgroup of $\gamma$-th powers, i.e. the unique subgroup of order $(q^2 - 1)/\gamma$ of $F_q^\infty$. Equivalently $(y'/y)^{(q^2-1)/\gamma} = 1$. Conversely, suppose that there exists $a \in F_q^\infty$ such that $y' = \frac{a^2}{a} y$. There are elements $u$ and $u'$ of $F_q$ such that $z = y b (\tau + u \omega)$ and $z' = y b (\tau + u' \omega)$. If we can find $b$ and $c$ such that $(a,b,c) \in Q$ and $z' = a b y - a b y + a a z$, then $n$ and $n'$ are conjugate in $N$. This can also be written

$$
a a y b (\tau + u' \omega) = a b y - a b y + a a y b (\tau + u \omega),
$$

or equivalently

$$(*) \quad \frac{1}{a} (\frac{b}{a} - b ay) = u' - u.
$$

Now the map $b \mapsto \frac{1}{a} (\frac{b}{a} - b ay)$ is a non zero $F_q$-linear map from $F_q^\infty$ to $F_q$. Hence it is surjective, and there exists $b \in F_q^\infty$ such that $(*)$ holds. Now we set $c = \frac{b b}{a} \tau$, and then $(a,b,c) \in Q$, and the elements $n$ and $n'$ are conjugate in $N$. This proves Assertion (6), and completes the proof of Proposition 21.

**Corollary 22.** The set

$$E = \{ M(x,0,0) \mid x \in F_q^\infty / \Gamma \} \cup \{ M(x,0,x \omega) \mid x \in \Psi / \Gamma \} \cup \{ M(1,y,y \bar{y} \tau) \mid y \in F_q^\infty / L \}
$$

is a set of representatives of conjugacy classes of $N$. In particular, there are $\frac{q^2 + q}{\gamma} + \gamma$ conjugacy classes in $N$.

**Proof.** Indeed, by Proposition 21, the set $E$ is a set of representatives of conjugacy classes of $N$. Its cardinality is

$$|E| = \frac{q^2 - 1}{\gamma} + \frac{q + 1}{\gamma} + \gamma = \frac{q^2 + q}{\gamma} + \gamma.
$$

\[\square\]

**Notation 23.**

- For $x \in F_q^\infty$, we set $d_x = M(x,0,0)$ and $D_x = d_x^+ \in \mathbb{Z} k N$.
- For $x \in \Psi$, we set $t_x = M(x,0,x \omega)$ and $T_x = t_x^+$.
- For $y \in F_q^\infty$, we set $u_y = M(1,y,y \bar{y} \tau)$ and $U_y = u_y^+$.

**Proposition 24.**

(1) For $x \in F_q^\infty - \Psi$,

$$d_x^N = \{ M(x,y,z) \mid y,z \in F_q^\infty, y \bar{y} + x \bar{x} + z \bar{z} = 0 \}.\]$$

In particular $|d_x^N| = q^3$.\]
(2) For $x \in \Psi - \Gamma$,
\[
d_x^N = \{ \tilde{M}(x, y(x^2 - x), y\overline{y}(x^2 - x)) \mid y \in F_q \}.
\]
In particular $|d_x^N| = q^2$.

(3) For $x \in \Gamma$, the element $d_x$ is the identity element of $N$, and $|d_x^N| = 1$.

(4) For $x \in \Psi$, the conjugacy class of $t_x$ in $N$ has cardinality $q^2(q - 1)$ if $x \notin \Gamma$, and $q - 1$ otherwise. The conjugacy class of $T_1$ consists of the elements $\tilde{M}(1, 0, \lambda \omega)$, for $\lambda \in F_q^\times$.

(5) For $x \in F_q^\times$,
\[
u_x^N = \{ \tilde{M}(1, v, v\bar{\tau} + \lambda \omega) \mid v \in xL, \lambda \in F_q \}.
\]
In particular $|\nu_x^N| = \frac{q(q^2 - 1)}{\gamma}$.

Proof. It follows from Proposition 21 that if $(x, y, z) \in \mathcal{Q}$ and $x\bar{\tau} \neq 1$, then $\tilde{M}(x, y, z)$ is conjugate to $d_x$, and that conversely, any conjugate of $d_x$ in $N$ is of the form $\tilde{M}(x, y, z)$, for some elements $y, z \in F_q^2$ such that $(x, y, z) \in \mathcal{Q}$. This proves Assertion (1).

Now let $(a, b, c)$ and $(x, y, z)$ be elements of $\mathcal{Q}$. By Assertion (1) of Lemma 20, comparing the diagonal elements in the product in the two possible orders, the elements $\tilde{M}(a, b, c)$ and $\tilde{M}(x, y, z)$ commute if and only if
\[(**)
ay + b\tau x = xb + \frac{yn}{a} \quad \text{and} \quad az - b\overline{y} \bar{x} + c\omega = xc - \frac{yn}{a} + \frac{z}{a}.
\]

- If $y = z = 0$, this gives $b\tau x = xb$ and $c\bar{x} = xc$. If moreover $x\bar{\tau} = 1$ but $x^2 \neq \bar{x}$, then $b = 0$, but $a$ and $c$ are arbitrary, only subject to $a\bar{x} + c\bar{a} = 0$. In this case the centraliser of $d_x$ in $N$ has cardinality $q^2(q - 1)$, and the conjugacy class of $d_x$ in $N$ has cardinality $q^2$. Now Assertion (2) follows from the fact that the elements
\[
\tilde{M}(1, y, y\bar{\tau})\tilde{M}(x, 0, 0)\tilde{M}(1, y, y\bar{\tau})^{-1} = \tilde{M}(x, y(x^2 - x), y\overline{y}(x^2 - x))
\]
for $y \in F_q^\times$, are all distinct.

Finally if $x^2 = \bar{x}$, then $x \in \Gamma$, so $d_x$ is the identity element of $N$, and Assertion (3) follows.

- If $x \in \Psi$, $y = 0$ and $z = x\omega$, then the relations (***) give
\[
b\tau x = xb \quad \text{and} \quad a\omega + c\omega = xc + \frac{x\omega}{a}
\]
that is $b(x - x^2) = 0$ and $a\bar{x} = 1$. If $x \neq x^2$, i.e. if $x \notin \Gamma$, this is equivalent to $b = 0$ and $a\bar{x} = 1$. Then $c$ is arbitrary, only subject to $a\bar{x} + c\bar{a} = 0$. In this case the centraliser of $t_x$ in $N$ has cardinality $q^2(q + 1)$, and the conjugacy class of $t_x$ in $N$ has cardinality $q^2(q - 1)$. Now if $x^2 = \bar{x}$, the only condition left is $a\bar{x} = 1$, so the centraliser of $t_x$ in $N$ has cardinality $q^3$ (it is equal to $S$), and the conjugacy class of $t_x$ in $N$ has cardinality $q - 1$. Moreover, by Lemma 20, the conjugates of $t_1 = \tilde{M}(1, 0, \omega)$ are the elements $\tilde{M}(1, 0, a\omega\omega)$, for $a \in F_q^\times$. This completes the proof of Assertion (4).

- If $x = 1$, $y \in F_q^\times$, and $z = y\bar{y}\omega$, then the relations (***)
\[
a\omega = \frac{y\bar{a}}{a} \quad \text{and} \quad ay\bar{y}\omega - b\overline{y} = -\frac{y\bar{b}}{a} + \frac{y\bar{y}\omega}{a}.
\]
Since $y \neq 0$, the first relation gives $a^2 = \bar{a}$, i.e. $a \in \Gamma$, so we can assume $a = 1$ by Lemma 18. Now the second relation reads $b\overline{y} = \bar{b}$, i.e. $b = uy$, for $u \in F_q$. Since $c$ is subject to $c + \bar{x} + b\bar{x} = 0$, it follows that the centraliser of $u_y$ in $N$ has cardinality $q^2$, and the conjugacy class of $u_y$ in $N$ has cardinality $q^2(q^2 - 1)\gamma$. 

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Now Assertion (5) follows from the fact that by Proposition 21, the element $\tilde{M}(1, v, v\tau + \lambda \omega)$, for $v \in xL$ and $\lambda \in \mathbb{F}_q$, is conjugate to $u_x$, and that there are $\frac{q(q^2 - 1)}{\gamma}$ such elements in $N$.

We recall the following well known fact (cf e.g. [3, (9.28)]):

**Lemma 25.** Let $G$ be a finite group and $k$ be a commutative ring. For $x \in G$, let $x^+ \in Z_kG$ denote the sum of the elements of the conjugacy class $x^G$ of $x$ in $G$. Then for $x, y \in G$

$$x^+ \cdot y^+ = \sum_{z \in [G]} m^z_{x,y} z^+,$$

where $[G]$ denotes a set of representatives of conjugacy classes of $G$, and

$$m^z_{x,y} = |\{(x', y') \in x^G \times y^G \mid x'y' = z\}|.$$

Clearly $m^z_{x,y} = m^y_{y,x}$ and $m^{z^{-1}}_{x,y} = m^z_{x,y}$ for any $x, y, z \in G$, but since

$$m^z_{x,y}|x^G| = |\{(x', y', z') \in x^G \times y^G \times z^G \mid x'y' = z'\}|,$$

we have also $m^z_{x,y}|z^G| = m^y_{x,y} z^G = m^z_{x,y-1}|y^G|.$

Observe that $Z(kN) = k \otimes_{\mathbb{Z}} Z(\mathbb{Z}N)$ and hence we may and will suppose for the rest of this section that $k = \mathbb{Z}$, unless otherwise stated.

**Proposition 26.**

1. Let $x \in F_{q^2}^\times - \Psi$ and $y \in \mathbb{F}_{q^2}^\times$ such that $xy \notin \Psi$. Then

$$D_x D_y = \begin{cases} 
q^3 D_{xy} & \text{if } y \notin \Psi \\
q^2 D_{xy} & \text{if } y \in \Psi
\end{cases}.
$$

2. Let $x \in F_{q^2}^\times - \Psi$ and $y \in \Psi$. Then

$$D_x T_y = \begin{cases} 
q^3 (q - 1) D_{xy} & \text{if } y \notin \Gamma \\
(q - 1) D_{xy} & \text{if } y \in \Gamma
\end{cases}.
$$

3. Let $x \in F_{q^2}^\times - \Psi$ and $y \in \mathbb{F}_{q^2}^\times$. Then $D_x U_y = \frac{q(q^2 - 1)}{\gamma} D_x$.

**Proof.** The three assertions follow from the fact that the product of an element in the conjugacy class of $r = \tilde{M}(x_1, y_1, z_1)$ of $N$ and an element in the conjugacy class of $s = \tilde{M}(x_2, y_2, z_2)$ of $N$ is an element of the form $\tilde{M}(x_1 x_2, \alpha, \beta)$, for some $\alpha$ and $\beta$ in $\mathbb{F}_q^2$. In each assertion, the assumption implies that all these elements are in the conjugacy class of $t = d_{x_1 x_2}$, since $x_1 x_2 \in F_{q^2}^\times - \Psi$. It follows that there exists an integer $m$ such that

$r^+ s^+ = m D_{x_1 x_2}$.

Now the augmentation map $\epsilon : kN \rightarrow k$ restricts to a ring homomorphism $Z_kN \rightarrow k$, sending $x^+$ to $|x^G|$. Hence $|r^N ||s^N| = m |t^N|$. For the three assertions, we can assume that $r = d_x$ and $x \notin \Psi$, so $|r^N| = q^3$. Similarly $t = d_{xy}$ for Assertions (1) and (2), and $xy \notin \Psi$, so $|t^N| = q^3$. For Assertion (3), we have $t = d_x$, so $|t^N| = q^3$ again. It follows that the integer $m$ is equal to $|s^N|$, and $s = d_y$ in Assertion (1), $s = t_y$ in Assertion (2), and $s = u_y$ in Assertion (3). Now Proposition 26 follows from the values of the cardinalities $|s^N|$ given by Proposition 24.

**Proposition 27.** Let $x, y \in \mathbb{F}_{q^2} - \Psi$, such that $xy \notin \Psi - \Gamma$. Then $D_x D_y = q^3 D_{xy} + q^3 T_{xy}$.

**Proof.** Any element in the product $d_x^N \cdot d_y^N$ is of the form $\tilde{M}(x y, \alpha, \beta)$, for some $\alpha, \beta \in \mathbb{F}_q^2$. It follows that there are integers $a$ and $b$ such that $D_x D_y = a D_{xy} + b T_{xy}$. Setting $z = xy$, the integer $a$ is equal to $m^z_{d_x, d_y}$. Thus $a |d_x^N| = m^z_{d_x, d_y} |d_x^N|$, by Lemma 25. But by Proposition 26,
we have $D_zD_y^{-1} = q^2D_x$, so $m_{d_x,d_y}^{D_z} = q^2$. It follows that $a|z^N| = aq^2 = q^2q^3$, thus $a = q^3$.

Taking augmentation gives

$$a \varepsilon(D_xD_y) = q^6 = a\varepsilon(D_z) + b\varepsilon(T_z) = aq^2 + bq^2(q - 1).$$

It follows that $b = \frac{q^6 - q^5}{q^2(q - 1)} = q^3$. \hfill \qed

**Proposition 28.** Let $x \in \Psi$. Then $D_xT_1 = T_x$.

**Proof.** If $x \in \Gamma$, there is nothing to prove, because $D_x$ is equal to the identity, in this case. If $x \notin \Gamma$, then $D_xT_1$ is a sum of elements of the form $M(x, \alpha, \beta)$, so there are natural integers $a$ and $b$ such that $D_xT_1 = aD_x + bT_x$. Taking augmentation of this equality gives $q^2(q - 1) = aq^2 + bq^2(q - 1)$, that is $q - 1 = a + b(q - 1)$. Since the product $d_xT_1$ is equal to $t_x$, it follows that $b > 0$. Hence $b = 1$ and $a = 0$. \hfill \qed

**Proposition 29.** Let $x \in \Psi - \Gamma$, and $y \in \mathbb{F}_q^\times$. Then $D_xU_y = \frac{q^2 - 1}{\gamma}(D_x + T_x)$.

**Proof.** Again $D_xU_y$ is a sum of elements of $N$ of the form $M(x, \alpha, \beta)$. Hence there are natural integers $a$ and $b$ such that $D_xU_y = aD_x + bT_x$. The integer $a$ is equal to $m_{d_x,u_y}^{D_x}$, i.e.

$$a = \left| \{(d', u') \in d_x^N \times u_y^N \mid d'u' = d_x\} \right|.$$

By Proposition 24, the element $d' \in d_x^N$ is equal to $M(x, w(x^2 - x), w(\overline{w}(x^2 - x)))$, for $w \in \mathbb{F}_q^\times$, and the element $u'$ is equal to $M(1, v, v\tau + \lambda\omega)$, for $v \in xL$ and $\lambda \in \mathbb{F}_q$. Now

$$d'u' = M(x, xv + w(x^2 - x), x(v\tau + \lambda\omega) - w(x^2 - x) + w(\overline{w}(x^2 - x))).$$

This is equal to $d_x$ if and only if

$$xv + w(x^2 - x) = 0 \quad \text{and} \quad x(v\tau + \lambda\omega) - w(x^2 - x) + w(\overline{w}(x^2 - x)) = 0.$$

Since $x \notin \Gamma$, the first relation gives $w = \frac{v}{1 - \overline{v}}$. Multiplying by $\overline{v}$, the second one reads

$$v\tau + \lambda\omega - w(x^2 - 1) + w(\overline{w}(x^2 - 1)) = 0.$$

This gives

$$v\tau + \lambda\omega + v\overline{w} - \frac{v\overline{w}}{1 - x^3} = 0,$$

that is

$$\lambda = \frac{1}{\omega}\left(\overline{\tau} + \frac{1}{1 - x^3}\right).$$

This defines an element $\lambda$ of $\mathbb{F}_q$, since $\tau + \overline{\tau} = -1$ and

$$\frac{1}{1 - x^3} + \frac{1}{1 - \overline{x}^3} = \frac{2 - x^3 - \overline{x}^3}{(1 - x^3)(1 - \overline{x}^3)} = 1.$$

In other words $w$ and $\lambda$ are determined by $v \in xL$, which may be chosen arbitrarily. It follows that $a = \frac{q^2 - 1}{\gamma}$.

Now applying the augmentation to the relation $D_xU_y = aD_x + bT_x$ gives

$$q^2q(q^2 - 1) = aq^2 + bq^2(q - 1).$$

It follows that

$$q(q^2 - 1) = \frac{q^2 - 1}{\gamma} + b(q - 1),$$

hence $b = \frac{q^2 - 1}{\gamma}$. \hfill \qed
Proposition 30.

(1) Let \( x \in \Psi - \Gamma \). Then
\[
D_x D_{x^{-1}} = q^2 \text{Id} + q \sum_{y \in \mathbb{F}_q^\times / L} U_y.
\]

(2) Let \( x \in \mathbb{F}_q^\times - \Psi \). Then
\[
D_x D_{x^{-1}} = q^3 \text{Id} + q^3 T_1 + q^3 \sum_{y \in \mathbb{F}_q^\times / L} U_y.
\]

Proof. For \( x \in \mathbb{F}_q^\times \), the product \( D_x D_{x^{-1}} \) is a sum of elements of the form \( \tilde{M}(1, \alpha, \beta) \) of \( N \).

So there are integers \( a, b, c \in \mathbb{N} \), for \( y \in \mathbb{F}_q^\times / L \) such that
\[
(* *)
D_x D_{x^{-1}} = a \text{Id} + b T_1 + \sum_{y \in \mathbb{F}_q^\times / L} c_y U_y.
\]

Then \( a = m_{1d, d_{x^{-1}}} = |\{(d', d'') \in d_x^N \times d_{x^{-1}}^N \mid d'd'' = \text{Id}\}| = |d_x^N| \). Thus \( a = q^2 \) if \( x \in \Psi - \Gamma \), and \( a = q^3 \) if \( x \in \mathbb{F}_q^\times - \Psi \).

On the other hand, by Lemma 25, for \( y \in \mathbb{F}_q^\times \),
\[
c_y|u_y^N| = m_{u_y, d_y} \cdot |v_y^N| = m_{u_y, d_y} |d_y^N|.
\]

- If \( x \in \Psi - \Gamma \), then \( m_{u_y, d_y} = \frac{q^2 - 1}{\gamma} \), by Proposition 29. It follows that
\[
c_y \frac{q^2 - 1}{\gamma} = \frac{q^2 - 1}{\gamma} q^2,
\]

hence \( c_y = q \).

Applying augmentation to equation \((***)\), we get \( q^2 q^2 = a + b(q-1) + q \cdot \gamma q^2 \frac{q^2 - 1}{\gamma} \).

This gives \( b(q-1) = q^4 - q^3 - q^2(q^2 - 1) = 0 \), which proves Assertion (1).

- If \( x \in \mathbb{F}_q^\times - \Psi \), then \( m_{u_y, d_y} = \frac{q^2(q^2 - 1)}{\gamma} \) by Proposition 26. Thus \( c_y = q^3 \) in this case. Applying augmentation to equation \((***)\) gives
\[
q^3 \cdot q^3 = q^3 + b(q-1) + q^3 \gamma \cdot \frac{q^2 - 1}{\gamma},
\]

that is \( b(q-1) = q^6 - q^5 - q^4(q^2 - 1) = q^3(q-1) \), hence \( b = q^3 \), which proves Assertion (2).

\( \square \)

Proposition 31. Let \( x, y \in \Psi - \Gamma \) such that \( xy \notin \Gamma \). Then \( D_x D_y = D_{xy} + (q + 1) T_{xy} \).

Proof. The product \( D_x D_y \) is a sum of elements of \( N \) of the form \( \tilde{M}(xy, \alpha, \beta) \), so there are integers \( a \) and \( b \) such that \( D_x D_y = a D_{xy} + b T_{xy} \). The integer \( a \) is the number of pairs \((d', d'')\) in \( d_x^N \times d_y^N \) such that \( d'd'' = d_{xy} \).

By Proposition 24, the class \( d_x^N \) consists of the elements \( \tilde{M}(x, \alpha(\overline{x}^2 - x), \alpha(\overline{x}^2 - x)) \), for \( \alpha \in \mathbb{F}_q^\times \). Equivalently, in a form that will be more convenient for computation, it consists of the elements \( d' = \tilde{M}(x, u, v) \), for \( u \in \mathbb{F}_q^\times \) and \( v = \frac{u}{x^2 - x} \). Similarly, the class \( d_y^N \) consist of the elements \( d'' = \tilde{M}(y, r, s) \), for \( r \in \mathbb{F}_q^\times \) and \( s = \frac{r}{y^2 - y} \). Since \( x \overline{x} = 1 = y \overline{y} \), we have
\[
d'd'' = \begin{pmatrix} x & u & v \\ 0 & x^2 & \overline{u} \overline{x} \\ 0 & 0 & x \end{pmatrix} \begin{pmatrix} y & r & s \\ 0 & y^2 & \overline{r} \overline{y} \\ 0 & 0 & y \end{pmatrix}.
\]
The product \( d'd'' \) is equal to \( d_{xy} \) if and only if
\[
rx + uy = 0 \quad \text{and} \quad xs - uy + vy = 0.
\]
The first equation gives \( r = -uy^2 \), thus \( rT = u\bar{u} \). Now the second equation becomes
\[
\frac{xu\bar{u}}{y^2 - \bar{y}} + uyxy + \frac{yuy}{x^2 - \bar{x}} = 0.
\]
Then either \( u = 0 \), hence \( r = s = v = 0 \), or
\[
\frac{x}{y^2 - \bar{y}} + xy + \frac{y}{x^2 - \bar{x}} = 0.
\]
Equivalently \( (x^3 - 1) + (x^3 - 1)(y^3 - 1) + (y^3 - 1) = 0 \), thus \( x^3y^3 = 1 \), which doesn’t hold since \( xy \not\in \Gamma \), using the remark after Lemma 18.

So the only pair \( (d',d'') \in d_x^N \times d_y^N \) such that \( d'd'' = d_{xy} \) is the pair \((d_x,d_y)\). It follows that \( a = 1 \).

Applying augmentation to the equality \( D_xD_y = aD_{xy} + bT_{xy} \) now gives \( q^4 = q^2 + bq^2(q-1) \), hence \( b = q + 1 \)

**Proposition 32.** Let \( x \in \Psi - \Gamma \) and \( y \in \mathbb{F}_q^\times \) with \( xy \not\in \Gamma \). Then
\[
D_xT_y = (q^2 - 1)D_{xy} + (q^2 - q - 1)T_{xy}.
\]

**Proof.** The product \( D_xT_y \) is a sum of elements of the form \( \tilde{M}(xy, \alpha, \beta) \), so there are integers \( a \) and \( b \) such that \( D_xT_y = aD_{xy} + bT_{xy} \). By Lemma 25, Proposition 24, and Proposition 31, we have
\[
aq^2 = m_{d_y}^{\text{t}y} |d_{xy}| = m_{d_x,d_y}^{\text{t}y} q^2(q - 1) = q^2(q^2 - 1),
\]
hence \( a = q^2 - 1 \). Taking augmentation gives
\[
\varepsilon(D_xT_y) = q^2q^2(q - 1) = a\varepsilon(D_{xy}) + b\varepsilon(T_{xy}) = (q^2 - 1)q^2 + bq^2(q - 1),
\]
hence \( b = q^2 - q - 1 \). \( \square \)

**Proposition 33.** (1) \( T_1^2 = (q - 1)\text{Id} + (q - 2)T_1 \).

(2) If \( x \in \Psi - \Gamma \), then \( T_xT_1 = (q - 1)D_x + (q - 2)T_x \).

**Proof.** By Proposition 24, the product of any two conjugates of \( t_1 \) is either the identity, or again a conjugate of \( t_1 \). It follows that there are integers \( a \) and \( b \) such that \( T_1^2 = a\text{Id} + bT_1 \). Moreover \( a \) is equal to the cardinality of the conjugacy class of \( t_1 \), that is \( a = q - 1 \). Now taking augmentation gives \( (q - 1)^2 = a + (q - 1)b \), hence \( b = q - 2 \). Now for \( x \in \Psi - \Gamma \),
\[
T_xT_1 = D_xT_1^2 = (q - 1)D_x + (q - 2)T_x,
\]
since \( D_xT_1 = T_x \) by Proposition 28. \( \square \)

**Proposition 34.** Let \( x \in \Psi - \Gamma \). Then \( D_xT_{x-1} = q^2T_1 + q(q - 1) \sum_{y \in \mathbb{F}_{q^2}^\times /L} U_y \).

**Proof.** Again \( D_xT_{x-1} \) is a sum of elements of the form \( \tilde{M}(1, \alpha, \beta) \), so there are integers \( a, b \), and \( c_y \), for \( y \in \mathbb{F}_{q^2}^\times /L \), such that \( D_xT_{x-1} = a\text{Id} + bT_1 + \sum_{y \in \mathbb{F}_{q^2}^\times /L} c_yU_y \). Since \( t_{x-1} = t_x^{-1} \),
\[
\text{and since no conjugate of } d_x \text{ is a conjugate of } t_x, \text{ we have } a = 0. \text{ Then } b = m_{d_x,t_{x-1}}^{t_x},
\]
hence \( b(q - 1) = m_{d_x,t_{x-1}}^{t_x} q^2(q - 1) = q^2(q - 1) \), by Proposition 28. Hence \( b = q^2 \). Similarly \( c_y = m_{u_y,d_{x-1}}^{u_x} \), so \( c_y q(q^2 - 1) = m_{u_y,d_{x-1}}^{u_x} q^2(q - 1) \), hence \( c_y q(q^2 - 1) = q^2 - 1 - q^2(q - 1) \), thus \( c_y = q(q - 1) \).

\( \square \)

**Proposition 35.** Let \( x \in \Psi - \Gamma \) and \( y \in \mathbb{F}_{q^2}^\times \). Then \( T_xU_y = \frac{(q^2 - 1)(q - 1)}{q} (D_x + T_x) \).
\textit{Proof.} By Proposition 28 and Proposition 29, we have that
\begin{align*}
T_xU_y &= D_xT_1U_y = \frac{(q^2 - 1)}{\gamma}(D_x + T_x)T_1 \\
&= \frac{(q^2 - 1)}{\gamma}(T_x + (q - 1)D_x + (q - 2)T_x) \\
&= \frac{(q^2 - 1)(q - 1)}{\gamma}(D_x + T_x)
\end{align*}
\hfill \square

\textbf{Proposition 36.} Let $x \in \Psi - \Gamma$. Then
\[ T_xT_{x^{-1}} = q^2(q - 1)\text{Id} + q^2(q - 2)T_1 + q(q - 1)^2 \sum_{y \in \mathbb{F}_q^\times/L} U_y. \]

\textit{Proof.} Indeed by Proposition 30, Proposition 28, Proposition 33 and Proposition 34
\begin{align*}
T_xT_{x^{-1}} &= D_xT_1D_{x^{-1}}T_1 \\
&= D_xD_{x^{-1}}T_1^2 \\
&= D_xD_{x^{-1}}((-1)\text{Id} + (q - 2)T_1) \\
&= D_x((q - 1)D_{x^{-1}} + (q - 2)T_{x^{-1}}) \\
&= (q - 1)(q^2\text{Id} + q \sum_{y \in \mathbb{F}_q^\times/L} U_y) + (q - 2)(q^2T_1 + q(q - 1) \sum_{y \in \mathbb{F}_q^\times/L} U_y) \\
&= q^2(q - 1)\text{Id} + q^2(q - 2)T_1 + q(q - 1)^2 \sum_{y \in \mathbb{F}_q^\times/L} U_y.
\end{align*}
\hfill \square

\textbf{Proposition 37.} Let $x, y \in \Psi - \Gamma$ such that $xy \notin \Gamma$. Then
\[ T_xT_y = (q - 1)(q^2 - q - 1)D_{xy} + (q(q - 1)^2 + 1)T_{xy}. \]

\textit{Proof.} Indeed, by Proposition 31, Proposition 28 and Proposition 33
\begin{align*}
T_xT_y &= D_xT_1D_yT_1 \\
&= (D_{xy} + (q + 1)T_{xy})((q - 1)\text{Id} + (q - 2)T_1) \\
&= (q - 1)D_{xy} + (q - 2)T_{xy} + (q^2 - 1)T_{xy} + (q - 2)(q - 1)(q - 1)D_{xy} + (q - 2)T_{xy}) \\
&= (q - 1)(q^2 - q - 1)D_{xy} + (q(q - 1)^2 + 1)T_{xy}
\end{align*}
\hfill \square

\textbf{Proposition 38.} Let $x \in \mathbb{F}_q^\times$. Then $T_1U_x = (q - 1)U_x$.

\textit{Proof.} The product $T_1U_x$ is a linear combination of elements of $N$ of the form $\tilde{M}(1, \alpha, \beta)$. Hence there are integers $a, b$ and $c_y$, for $y \in \mathbb{F}_q^\times/L$, such that
\[ (\#) \quad T_1U_x = a\text{Id} + bT_1 + \sum_{y \in \mathbb{F}_q^\times/L} c_yU_y. \]

Observe now that $t_1$ and $u_x^{-1}$ are not conjugate in $N$, e.g. because the conjugacy class of $t_1$ has cardinality $q - 1$, and the conjugacy class of $u_x$ has cardinality $\frac{q(q^2 - 1)}{\gamma} \neq q - 1$. It follows that $a = 0$.

Now by Proposition 24, the conjugacy class of $T_1$ consists of the elements $\tilde{M}(1, 0, \lambda\omega)$, for $\lambda \in \mathbb{F}_q^\times$, and the conjugacy class of $u_x$ consists of the elements $\tilde{M}(1, v, v\bar{\tau} + \mu\omega)$, for $v \in xL$ and $\mu \in \mathbb{F}_q$. The product $\pi = \tilde{M}(1, 0, \lambda\omega)\tilde{M}(1, v, v\bar{\tau} + \mu\omega)$ is equal to $u_y = \tilde{M}(1, y, y\bar{\tau})$ if
and only if \( v = y \) and \( v \sigma \tau + \mu \omega + \lambda \omega = y \overline{\sigma \tau} \). It follows that \( c_y = 0 \) unless \( y \in xL \), i.e. unless \( yL = xL \). If \( yL = xL \), then \( u_y \) is conjugate to \( u_x \) in \( N \), and we can assume that \( y = x \). In this case \( \pi = u_x \) if and only if \( v = x \) and \( \mu = -\lambda \). It follows that \( c_x = q - 1 \).

Applying augmentation to Equation (\#) now gives

\[
(q - 1) \cdot \frac{q(q^2 - 1)}{\gamma} = b(q - 1) + (q - 1) \cdot \frac{q(q^2 - 1)}{\gamma},
\]

hence \( b = 0 \).

**Proposition 39.**

(1) If \( 3 \nmid q + 1 \), then \( L = F_q^\times \), and

\[
U_1^2 = q(q^2 - 1)Id + q(q^2 - 1)T_1 + q(q^2 - 2)U_1.
\]

(2) If \( 3 \mid q + 1 \), then \( F_{q^2}/L = \{ L, tL, t^2L \} \), where \( t \) is any non cube element of \( F_q^\times \). Let

\[
l = \{|v \in L | 1 - v \in L\}, \text{ and } m = \{|v \in L | t - v \in L\} \text{, and } n = \{|v \in L | t - v/t \in L\}.
\]

Then for \( x \in F_q^\times/L \),

\[
U_x^2 = \frac{q(q^2 - 1)}{\gamma}(Id + T_1) + qU_x + qm(U_{tx} + U_{t^2x})
\]

\[
U_xU_{tx} = qmU_{t^2x} + qn(U_x + U_{tx}).
\]

**Proof.** By Proposition 24, for \( x \in F_q^\times \), the conjugacy class of \( u_x \) in \( N \) consists of the elements \( \hat{M}(1,v,v\overline{\sigma \tau} + \lambda \omega) \), for \( v \in xL \) and \( \lambda \in F_q \). Since the inverse of \( u_x \) is \( \hat{M}(1,x,x\overline{\sigma \tau}) \) is \( \hat{M}(1,-x,\overline{x\sigma \tau}) \), and since \( -x \in xL \) as \( -1 = (-1)^\gamma \in L \), we have that \( u_x^{-1} \) is conjugate to \( u_x \).

For \( x,y \in F_q^\times \), the product \( U_xU_y \) is a sum of elements of the form \( \hat{M}(1,\alpha,\beta) \), hence there are integers \( a,b \) and \( c_{x,y,z}^z \), for \( z \in F_q^\times/L \), such that

\[
U_xU_y = aId + bT_1 + \sum_{z \in F_q^\times/L} c_{x,y}^z U_z.
\]

Note that for \( x,y,z \in F_q^\times \), we have

\[
c_{x,y}^z|u_z^N| = m_{u_x,u_y}^{u_z} \frac{q(q^2 - 1)}{\gamma} = m_{u_x,u_z}^{u_y} \frac{q(q^2 - 1)}{\gamma} = c_{z,x}^y \frac{q(q^2 - 1)}{\gamma} = c_{z,x}^y|u_z^N|
\]

as \( u_x^{-1} \) is conjugate to \( u_x \). So \( c_{x,y,z} \) is a symmetric function of \( x,y,z \).

If \( xL \neq yL \), then no conjugate of \( u_x^{-1} \) is conjugate to \( u_y \), so \( a = 0 \). In this case, we also have

\[
b|t_1^N| = m_{t_{1,u_x},u_y}^{u_x}(q - 1) = m_{t_{1,u_x},u_y}^{u_x}|u_y^N|,
\]

and \( m_{t_{1,u_x},u_y}^{u_x} = 0 \) by Proposition 38. It follows that \( b = 0 \) in this case.

If \( xL = yL \), i.e. \( U_x = U_y \), then clearly \( a = |u_x^N| = \frac{q(q^2 - 1)}{\gamma} \). Moreover Equation (###) gives \( b(q - 1) = (q - 1)|u_y^N| \), hence \( b = \frac{q(q^2 - 1)}{\gamma} \).

In the case \( 3 \nmid q + 1 \), we have \( \gamma = 1 \) and \( L = F_q^\times \). Then

\[
U_1^2 = q(q^2 - 1)(Id + T_1) + c_{1,1}^1 U_1.
\]

Taking augmentation gives

\[
(q(q^2 - 1))^2 = q(q^2 - 1)(1 + q - 1) + c_{1,1}^1 q(q^2 - 1),
\]

hence \( c_{1,1}^1 = q(q^2 - 2) \), which completes the proof of Assertion (1).

In the case \( 3 \nmid q + 1 \), then \( \gamma = 3 \), and \( L \) has index 3 in \( F_q^\times \), so \( F_q^\times/L = \{ 1, tL, t^2L \} \) for any non cube element \( t \) of \( F_q^\times \).
For $x, y, z \in \mathbb{F}_q^\times$, the product of the element $u' = \tilde{M}(1, v, v\bar{\tau} + \lambda \omega)$ in the conjugacy class of $u_x$ (where $v \in xL$ and $\lambda \in \mathbb{F}_q$) by the element $u'' = \tilde{M}(1, r, r\tau + \mu \omega)$ in the conjugacy class of $u_y$ (where $r \in yL$ and $\mu \in \mathbb{F}_q$) is equal to $u_z$ if and only if

$$v + r = z \quad \text{and} \quad r\tau + \mu \omega - v\tau + v\bar{\tau} + \lambda \omega = z\tau \tau.$$

The second equation determines $\mu$ once $v, r$ and $\lambda$ are known, and $\lambda$ can be chosen arbitrarily in $\mathbb{F}_q$, once $v$ and $r$ satisfy $v + r = z$. Hence in Equation $(\#\#)$, we have

$$c_{x,y}^2 = q \left| \{ v \in xL \mid z - v \in yL \} \right|.$$

In particular for any $x \in \mathbb{F}_q^\times$,

$$c_{x,x}^2 = q \left| \{ v \in xL \mid x - v \in xL \} \right| = q \left| \{ w \in L \mid x - xw \in xL \} \right| = l.$$

Similarly,

$$c_{x,x}^t = q \left| \{ v \in xL \mid xt - v \in xL \} \right| = q \left| \{ w \in L \mid xt - xw \in xL \} \right| = m.$$

Finally,

$$c_{x,x}^{2t} = q \left| \{ v \in xL \mid xt^2 - v \in xtL \} \right| = q \left| \{ w \in L \mid t^2 - w \in tL \} \right| = n.$$

This completes the proof, since $c_{x,y}^2$ is symmetric in $x, y, z$. \hfill \Box

**Remark 40.** Applying augmentation to the equations of Proposition 39 gives that $n = l + 1$ and $n + 2m = q^2 - 1 - 3$. So it suffices to know $l$, and then $m$ and $n$ can be computed.

By definition $l = \left| \{ v \in L \mid 1 - v \in L \} \right|$. Since $3 \mid q + 1 \mid q^2 - 1$, the field $\mathbb{F}_q^2$ contains all cubic roots of unity. Now clearly

$$l = \left| \{ (x, y) \in \mathbb{F}_q^\times \times \mathbb{F}_q^\times \mid x^3 + y^3 = 1 \} \right| / 9,$$

since multiplying $x$ or $y$ by any cubic root of unity doesn’t change $x^3$ nor $y^3$. It follows that $9l$ is almost equal to the number of points of the elliptic curve $x^3 + y^3 = z^3$ over $\mathbb{F}_q^2$: the difference consists of three points $(\theta, 0, 1)$ of the projective plane over $\mathbb{F}_q^2$, where $\theta$ is any cubic root of 1, three points $(0, \theta, 1)$, and three points $(\theta, -1, 0)$. It follows that $9l = N_2 - 9$, where $N_2$ is the number of points over $\mathbb{F}_q^2$ of the Fermat cubic $E$ with equation $x^3 + y^3 = z^3$.

Now this is an elliptic curve, and by [7, (2.6)], the zeta function of $E$ can be defined as

$$Z_E(u) = \exp \left( \sum_{m \geq 1} N_m \frac{u^m}{m} \right),$$

where $N_m$ is the number of points of $E$ over $\mathbb{F}_q^m$. By [7, Theorem 2.8], it has the following form

$$Z_E(u) = \frac{1 - au + q u^2}{(1 - u)(1 - qu)},$$

where $a = 1 + q - N_1$. Comparing the terms of degree 2 in $u$ in the expansion of those two expressions of $Z_E(u)$ as series in $u$ gives $N_2 = N_1(2(q + 1) - N_1)$.

Now since $3 \mid q + 1$, it follows that $3 \mid q - 1$, and $x \mapsto x^3$ is a bijection of $\mathbb{F}_q$. Hence $E$ has as many points over $\mathbb{F}_q$ as the projective line with equation $x + y = z$, that is $N_1 = q + 1$. Hence $N_2 = (q + 1)^2$, which gives the following values for $l, n$ and $m$:

$$l = \left( \frac{q + 1}{3} \right)^2 - 1, \quad m = \frac{q^2 - q - 2}{9}, \quad n = \left( \frac{q + 1}{3} \right)^2.$$

**Theorem 41.** Let $k$ be a field of characteristic $p$. Then:

1. The radical $J(ZkN)$ of the center of the group algebra $kN$ has a $k$-basis consisting of the elements $D_x$, for $x \in \mathbb{F}_q^\times / \Gamma = \{ \Gamma \}$, $T_x$, for $x \in \Psi / \Gamma = \{ \Gamma \}$, $T_1 + 1d$, and $U_x$, for $x \in \mathbb{F}_q^\times / L$. In particular, the dimension of $J(ZkN)$ is equal to $\frac{q^2 + q}{2} + \gamma - 1$. 


(2) The square $J^2(Z/kN)$ of $J(Z/kN)$ has a $k$ basis consisting of the elements $D_x + T_x$, where $x \in \Psi / \Gamma - \{\Gamma\}$. In particular, the dimension of $J^2(Z/kN)$ is equal to $q + 1 - 1$.

(3) The cube $J^3(Z/kN)$ of $J(Z/kN)$ is equal to 0.

Proof. As the group algebra $kN$ is indecomposable when $k$ is a field of characteristic $p$, the radical $J(Z/kN)$ is equal to the kernel of the augmentation $\varepsilon : Z/kN \to k$. If $X$ is the sum of the elements of a conjugacy class $C$ of $N$, then $\varepsilon(X) = |C|$, and by Proposition 24, this is a multiple of $p$, unless $C$ is the class of the identity element of $N$, or $C$ is the class of $t_1$, and $|C| = q - 1$ in this case. It follows that the elements listed in Assertion (1) generate $J(Z/kN)$. Moreover, they are obviously linearly independent, so they form a basis $B$ of $J(Z/kN)$.

Now by Proposition 28, for $x \in \Psi - \Gamma$, we have that $D_x(\text{Id} + T_1) = D_x + T_x$ in $Z/kN$, so the elements $D_x + T_x$, where $x \in \Psi / \Gamma - \{\Gamma\}$, are indeed in $J^2(Z/kN)$, and they are clearly linearly independent. Moreover, reducing mod $p$ the formulas for products stated in Propositions 26 to 39, one checks easily that any product of two elements of the basis $B$ is equal to a (possibly zero) scalar multiple of an element $D_x + T_x$, for some $x \in \Psi - \Gamma$, and that the product of any three elements of $B$ vanishes. This completes the proof of Theorem 41. □

If $k$ is a field of characteristic $p$ it is not difficult to give the explicit structure of $Z(kN)$ as a quotient of a polynomial ring in several variables.

Proposition 42. Let $\gamma$ be the greatest common divisor of 3 and $q + 1$, and let

$$
\Gamma := \{x \in \mathbb{F}_q^2 \mid x^T = 1\}, \quad \Psi := \{x \in \mathbb{F}_q^2 \mid x^{q + 1} = 1\}, \quad L := \{a^x \mid a \in \mathbb{F}_q^x\}.
$$

Let $U := \mathbb{F}_q^x / \Gamma$, let $\mathfrak{N} := \Psi / \Gamma$ and let $\mathfrak{M} := \mathbb{F}_q^x / L$. Let $k$ be a field of characteristic $p > 0$ and let $N$ be the normaliser of a Sylow $p$-subgroup of $PSU(3, q)$, where $p$ divides $q$. Then,

$$Z(kN) \simeq k[T, X_n, Y_m \mid n \in \mathfrak{M}, m \in \mathfrak{U} / \Gamma]$$

where $I$ is the ideal generated by

$$T^2, TX_{n_1}, TY_{m_1}, X_{n_1}X_{n_2}, X_{n_1}Y_{m_1}, Y_{m_1}Y_{m_2},$$

$$X_{n_1}Y_{m_2} + \frac{1}{\gamma}X_{n_1}T, Y_{m_2}Y_{m_3} - (1 - \delta_{m_2,m_3-1})X_{m_2m_3}T$$

where

$$n_1, n_2 \in \mathfrak{M}, m_1 \in \mathfrak{U} - \mathfrak{N}, m_2, m_3 \in \mathfrak{N}$$

and $\delta_{a,b}$ is the Kronecker symbol.

Proof. We have a basis of $Z(kN)$ given in Theorem 41 by the elements $D_x$, for $x \in \mathbb{F}_q^x / \Gamma$, $T_x$, for $x \in \Psi / \Gamma - \{\Gamma\}$, $T_1 + \text{Id}$, and $U_x$, for $x \in \mathbb{F}_q^x / L$. Observe that $D_1 = 1$. Moreover, by Proposition 28 we do not need to include $T_x$ as variable of the polynomial ring. This element is already the product of $T_1$ and $U_x$.

We obtain the following multiplication table.

<table>
<thead>
<tr>
<th></th>
<th>$T_1 + id$</th>
<th>$U_x$</th>
<th>$D_y \ (y \not\in \Psi)$</th>
<th>$D_y \ (y \in \Psi - \Gamma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_1 + id$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$T_y + D_y$</td>
</tr>
<tr>
<td></td>
<td>Propp. 33</td>
<td>Propp. 38</td>
<td>Propp. 26(2)</td>
<td>Propp. 28</td>
</tr>
<tr>
<td>$U_x$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-\frac{1}{2}(D_x + T_x)$</td>
</tr>
<tr>
<td></td>
<td>Propp. 38</td>
<td>Propp. 39</td>
<td>Propp. 26(3)</td>
<td>Propp. 29</td>
</tr>
<tr>
<td>$D_x \ (x \not\in \Psi)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Propp. 26(2)</td>
<td>Propp. 26(3)</td>
<td>Propp. 26(1),30</td>
<td>Propp. 26(1)</td>
</tr>
<tr>
<td>$D_x \ (x \in \Psi - \Gamma)$</td>
<td>$T_x + D_x$</td>
<td>$-\frac{1}{2}(D_x + T_x)$</td>
<td>0</td>
<td>$(1 - \delta_{xy}\Gamma)(T_{xy} + D_{xy})$</td>
</tr>
<tr>
<td></td>
<td>Propp. 28</td>
<td>Propp. 29</td>
<td>Propp. 26(1)</td>
<td>Propp. 31,30</td>
</tr>
</tbody>
</table>

Now, mapping $T$ to $T_1 + id$, $X_n$ to $U_n$ and $Y_m$ to $D_m$ gives an algebra homomorphism of the corresponding polynomial ring with kernel precisely the ideal $I$. □
References


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