Automorphisms of Green orders and their derived categories

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Abstract

In an earlier paper Raphaël Rouquier and the author introduced the group of self-equivalences of a derived category. In the case of a Brauer tree algebra we determined a non trivial homomorphism of the Artin braid group to this group of self-equivalences. The class of Brauer tree algebras include blocks of finite group rings over a large enough field with cyclic defect groups. In the present paper we give an integral version of this homomorphism Moreover, we identify some interesting arithmetic subgroups with natural groups of self-equivalences of the derived category.

Introduction

The representation theory of a group G with cyclic p-Sylow subgroup over a big enough field k of characteristic p is completely understood. It is a classical result that the indecomposable factors of the group ring are Brauer tree algebras. Brauer tree algebras are defined by means of a combinatorial object, a finite tree with some additional data, and their representations are deduced by some combinatorial process.

In order to study many of the more subtle questions for representations of kG, it has proved desirable to replace the coefficient domain k by a complete discrete valuation ring R of characteristic 0 with residue field k and field of fractions K. In order to do so we have to replace Brauer tree algebras by another object. This object is called Green order and is introduced and studied by Roggenkamp in [10]. Green orders as well are defined by a finite tree together with some additional structure. The representation theory of Green orders is quite well understood and any indecomposable factor of RG is a Green order whenever G has cyclic p-Sylow subgroup. Moreover, for any Green order Λ we have that $\Lambda \otimes_R k$ is a Brauer tree algebra and for any Brauer tree algebra A there is a Green order Λ so that $\Lambda \otimes_R k \simeq A$.

During the last decade parts of representation theory of groups were dominated by considerations of their derived categories. In an earlier work with Raphaël Rouquier the group of self-equivalences of the derived category of an algebra is studied [12]. We put special emphasis on the case of a Brauer tree algebra. There we get a close connection to Artin braid groups. In fact, let A be the Brauer tree algebra associated to a Brauer tree with n edges and no exceptional vertex. Then we prove that there is a homomorphism φ_n of the braid group B_{n+1} on $n+1$ strings to the group $TrPic_k(A)$ of self-equivalences of standard type of the derived category of A. The homomorphism φ_2 is injective and the

*2000 AMS Mathematics Subject Classification: 16G30, 20C10, 18E30 image of φ_2 is normal with cokernel of order $4 \cdot \mid Pic_k(A) \mid$. In a recent paper Khovanov and Seidel [3] announce that φ_n is injective for all n.

In the present paper we continue this research and get a very similar result replacing Brauer tree algebras by Green orders. We shall prove that there is a group homomorphism ψ_n from the Artin braid group on $n+1$ strings to the group of self-equivalences of the derived category of a Green order Λ_n. This Green order Λ_n is associated to a tree with n edges and with respect to some additional hypothesis corresponding to the absence of an exceptional vertex in the Brauer tree algebra setting. Then, $\Lambda_n \otimes_R k =: A_n$ is a Brauer tree algebra without exceptional vertex. Moreover, denoting by π_n the group homomorphism $TrPic_R(\Lambda_n) \longrightarrow TrPic_k(A_n)$ induced by $- \otimes_R k$, we get $\pi_n \circ \psi_n = \phi_n$.

We emphasize that the group ring $\hat{\mathbb{Z}}_p \mathfrak{S}_p$ of the symmetric group of degree p over the p-adic integers $\hat{\mathbb{Z}}_p$ is an example of Λ_{p-1}.

We shall study the case $n=2$ in greater detail since the braid group B_3 on 3 strings has arithmetic relations coming from $B_3/Z(B_3) \simeq PSL_2(\mathbb{Z})$. In this case we shall identify naturally defined normal subgroups $TrPicent(\Lambda)$ and $TrI_R(\Lambda)$ of $TrPic_R(\Lambda)$ with the level two congruence subgroup $\Gamma(2)$ of $PSL_2(\mathbb{Z})$ and its derived subgroup $\Gamma(2)'$. In fact, $TrPicent(\Lambda)$ is the kernel of the natural homomorphism $TrPic_R(\Lambda) \longrightarrow Aut_R(Z(\Lambda))$ induced by taking the action on degree 0 Hochschild cohomology. The group $TrI_R(\Lambda)$ is the kernel of the group homomorphism $TrPic_R(\Lambda) \longrightarrow TrPic_K(K \otimes_R \Lambda)$ induced by the functor $- \otimes_R K$. We should point out the fact that $TrI_R(\Lambda)$ is constructed using an order and is invisible on the level of the Brauer tree algebra.

The paper is organized as follows. In the sections 1.1 and 1.2 we review the basic notations and results on derived equivalences of algebras [4] and put together the results of [12] as long as they are necessary for the present paper. In section 1.3 we review in some detail Roggenkamp's theory of Green orders [10]. In section 2 we prove, under some assumptions on the combinatorial data of a Green order, that the identity is the only automorphism of a Green order fixing the indecomposable projective modules. Section 3 uses the results of section 2 to construct the homomorphism ψ_n. Finally section 4 gives the relation to the congruence subgroups.

1 Recall some facts

1.1 Review on derived equivalences

Let A and B be R-algebras over a hereditary, commutative ring R and assume that A and B are projective as modules over R. We denote by $D^b(A)$ the derived category of bounded complexes of A-modules. We follow the conventions of [4].

(1.1) Then Rickard [9], and under weaker hypotheses Keller [2], show that if there is an equivalence of triangulated categories $D^b(A) \simeq D^b(B)$ then there is a bounded complex X in $D^b(A \otimes_R B^{op})$ so that $X \otimes^{\mathbb{L}}_B -$ is an equivalence. Moreover, the inverse functor is given by left derived tensor product with a complex Y in $D^b(B \otimes_R A^{op})$. Equivalences given by tensor product by a bounded complex of bimodules are called of *standard type.* The complex X is called a *two-sided tilting complex.*

(1.2) A *one-sided tilting complex* is a bounded complex T of projective modules so that $Hom_{D^b(A)}(T, T[i]) = 0$ if $i \neq 0$ and so that the rank one free A-module is contained in the smallest triangulated category which is closed under direct sums and direct summands and which contains T. The image of the rank one free module by tensor product with X is a one-sided tilting complex, or tilting complex for short. Given a one-sided tilting complex T, then there is a two-sided tilting complex X so that T is the image of the rank one free module by tensoring with X. This complex X is unique up to an automorphism of A.

(1.3) Suppose that R is a complete discrete valuation ring with residue field k. Let A be an R-order. Suppose that $\overline{T}$ is a tilting complex in $D^b(k \otimes_R A)$. Then, (cf Rickard [8]) up to isomorphism there is a unique tilting complex T in $D^b(A)$ so that $k \otimes_R T \simeq \overline{T}$. In this case $k \otimes_R End_{D^b(A)}(T) \simeq End_{D^b(k \otimes_R A)}(\overline{T})$.

(1.4) We define [12]

$$TrPic_R(A) := \{ \text{ isomorphism classes of two-sided tilting complexes in } D^b(A \otimes_R A^{op})\}$$

It is clear by the definition that in case A and B are R-algebras which are projective as R-modules then,

$$D^b(A) \simeq D^b(B) \Longrightarrow TrPic_R(A) \simeq TrPic_R(B) \; .$$

(1.5) We recall one of the main results of [12]. Denote by

$$\begin{aligned} B_n =< s_1, s_2, \ldots, s_n \quad | \quad & s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \; ; \\ & s_i s_j = s_j s_i \forall i \in \{1, \ldots, n-1\} \forall j \in \{2, \ldots, n\} \setminus \{i+1, i-1\} > \end{aligned}$$

the Artin braid group on n strings and let $\tilde{B}_3$ be the extension of $B_3 =< s_1, s_2 >$ generated by z, s_1 and s_2 with the relations $z^4 = (s_1 s_2)^3$ and $z s_1 z^{-1} s_1^{-1} = z s_2 z^{-1} s_2^{-1} = 1$. Let $\mu_n(R)$ be the set of n-th roots of unity of a ring R and let C_m be the cyclic group of order m.

Theorem 1 *[12] Let k be a field and let A_n be a Brauer tree algebra over k associated to a tree with n edges and without exceptional vertex. Then, there is a group homomorphism*

$$\varphi_n : B_{n+1} \longrightarrow TrPic_k(A_n)$$

so that each of the generators of B_{n+1} generate an infinite cyclic group in $TrPic_k(A_n)$.
φ_2 is injective and induces an isomorphism

$$TrPic_k(A_2) \simeq \tilde{B}_3 \rtimes Pic_k(A_2)$$

where $Pic_k(A_2) \simeq C_2 \times (k^\times / \mu_n(k))$.

It is important to obtain the images of the generators of the braid group explicitly. Assume that the Brauer tree is a stem. Denote by $P_1, P_2, \ldots, P_n$ the projective indecomposable A-modules. Rearranging the numbering if necessary we may and will assume that $Hom_A(P_i, P_{i-1}) \neq 0$ for any $i \in \{2, 3, \ldots, n\}$. Then, for any $i = 1, \ldots, n$ the image of the standard braid group generator $\varphi_n(s_i)$ is isomorphic to the complex

$$\ldots \longrightarrow 0 \longrightarrow P_i \otimes_k Hom_A(P_i, A) \longrightarrow A \longrightarrow 0 \longrightarrow \ldots$$

concentrated in degree 0 and -1.

We recall the result of [12] which implies that φ_2 is injective. For any bounded complex $Y = (Y, d)$ of projective A-modules denote by Y^{mod} the underlying A-module of Y forgetting the differential and the graduation. Now, let $\alpha : B_3 \longrightarrow PSL_2(\mathbb{Z})$ and let $w \in B_3$ with $\alpha(w) = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$. It is proved in [12] that for every $i \in \{1, 2\}$ the complex of smallest dimension T_i which is homotopy equivalent to the complex $\varphi_2(w) \otimes_A P_i$ has the property $(T_i)^{mod} \simeq P_1^{|a_{1,i}|} \oplus P_2^{|a_{2,i}|}$.

1.2 Some subgroups of $TrPic$

In [12] various subgroups of $TrPic_R(A)$ are defined. In case A is an order, we shall define one more interesting subgroup.

(1.6) Assume that R is an integral domain with field of fractions K and assume that A is an R-algebra, projective as R-module. Then, the functor $K \otimes_R - : D^b(A) \longrightarrow D^b(K \otimes_R A)$ induces a group homomorphism

$$TrPic_R(A) \longrightarrow TrPic_K(K \otimes_R A) .$$

We denote by $TrI_R(A)$ the kernel of this homomorphism.

Observe that the homology of each element in $TrI_R(A)$ is R-torsion in non zero degrees. Moreover, in case A is an R-order, the homology in degree 0 contains a full $A \otimes_R A^{op}$-lattice in $K \otimes_R A$.

(1.7) J. Rickard proves in [9] that each self-equivalence of standard type of $D^b(A)$ induces a graded automorphism of the Hochschild cohomology ring $HH^*(A)$ of A. Following [12] we denote by $TrPicent(A)$ the kernel of the homomorphism

$$TrPic_R(A) \longrightarrow Aut(HH^0(A)) .$$

1.3 Review on Green Orders

For the reader's convenience we recall Roggenkamp's notion of a *'Green order'*. These orders are introduced to explain the structure of blocks with cyclic defect group. In fact, Green orders are a special class of rings with a very well prescribed ring theoretical structure. On the other hand Roggenkamp proves that every block of a group ring RG of a finite group G with cyclic defect group over a complete discrete valuation ring R of characteristic 0 is Morita equivalent to a Green order.

(1.8) Let R be a Dedekind domain with field of fractions K. Recall that an R–order is an R–algebra Λ which is finitely generated and projective as R–module and such that $K \otimes_R \Lambda$ is a semisimple K–algebra.

(1.9) For the definition of a Green order we follow a suggestion of Luis Puig. Let I be a complete set of primitive idempotents of Λ (i.e. $End_\Lambda(\oplus_{i\in I}\Lambda \cdot i)$ is Morita equivalent to Λ and the modules $\Lambda \cdot i$ for all $i \in I$ are projective indecomposable left–Λ–modules for all $i \in I$).

Definition 1.1 The indecomposable R–order Λ in the semisimple K-algebra $K \otimes_R \Lambda$ is called a *Green order* if

- there is a set E of central idempotents of $K \otimes_R \Lambda$ with $\sum_{e\in E} e = 1$ such that

  $$T := \{(i, e) \in I \times E | i \cdot e \neq 0\}$$

  is a tree (i.e. defines a connected relation on $I \times E$, $|E| = |I| + 1$ and for all $i \in I$ we get $|T \cap (\{i\} \times E)| = 2$).

- noting by $\pi : T \longrightarrow I$ and $\theta : T \longrightarrow E$ the natural projections, there are a transitive permutation ω of T and for all $t \in T$ a Λ–module homomorphism

  $$g_t : \Lambda \cdot \pi(t) \longrightarrow \Lambda \cdot \pi(\omega(t)) \text{ with } g_t(\Lambda \cdot \pi(t)) = ker(g_{\omega(t)}) \simeq \Lambda \cdot \pi(t) \cdot \theta(t) .$$

(1.10) Let R be a complete discrete valuation ring of characteristic 0 with residue field of characteristic p and with big enough field of fractions. Let G be a finite group and let $B(RG)$ be a block of RG with cyclic defect group. Roggenkamp proves in [10] that then, Λ is a Green order with the following additional property. There is at most one $e_0 \in E$ with $End_\Lambda(\Lambda\pi(t)\theta(t)) \not\simeq R$ for $\theta(t) = e_0$. This e_0 corresponds to the exceptional vertex of a Brauer tree algebra.

(1.11) Roggenkamp's (equivalent) definition of a Green order goes another way. Roggenkamp has three conditions. The third asks for the existence of what people call "Green's walk around the Brauer tree" as a projective resolution of $\Lambda \cdot \pi(t)\theta(t)$ for any t. This condition is exactly the second of our defining properties. Roggenkamp's second condition states that the edges of the underlying tree correspond to the projective indecomposable modules, a fact which we ensure by asking that the primitive idempotents are the edges of a tree. Roggenkamp's first condition asks that the vertices of the tree correspond to central idempotents in $K \otimes_R \Lambda$ which add to the identity. We just reproduced this condition.

(1.12) If we abbreviate $\Lambda \cdot t = \Lambda\pi(t)\theta(t)$ we have the following main result of [10].

Theorem 2 *(Roggenkamp [10])*

- *Let Λ be a basic Green order. Then, the tree T as above is totally ordered by ω and the choice of a first element.*

  There is an R–torsion R–algebra $\overline{\Omega_\Lambda}$ and a family $(f_t : t\Lambda t \longrightarrow \overline{\Omega_\Lambda})$ of R–algebra homomorphisms with kernel being a principal ideal $a_t t\Lambda t$ in the radical of $t\Lambda t$ such that the first element of T can be chosen such that the Pierce decomposition

$$\Lambda = \begin{pmatrix} i_1\Lambda i_1 & i_1\Lambda i_2 & \dots & i_1\Lambda i_k \\ i_2\Lambda i_1 & i_2\Lambda i_2 & \dots & i_2\Lambda i_k \\ \vdots & \vdots & & \vdots \\ i_k\Lambda i_1 & i_k\Lambda i_2 & \dots & i_k\Lambda i_k \end{pmatrix}$$

  has the following properties

  1. *For all $t < t'$ and $\theta(t) = \theta(t')$ with $t, t' \in T$ we have $t\Lambda t = \pi(t)\Lambda\pi(t') = t'\Lambda t'$.*
  2. *f_t depends only on $\theta(t)$ and we denote $f_{\theta(t)} := f_t$ and $a_{\theta(t)} := a_t$.*
  3. *For all $t > t'$ and $\theta(t) = \theta(t')$ with $t, t' \in T$ we have $a_{\theta(t)} \cdot \pi(t')\Lambda\pi(t) = \pi(t)\Lambda\pi(t')$.*
  4. *If $t \neq t' \in T$ with $\pi(t) = \pi(t')$ then $\pi(t)\Lambda\pi(t)$ is a pullback*

$$\begin{array}{ccc} \pi(t)\Lambda\pi(t) & \stackrel{\cdot\theta(t)}{\longrightarrow} & t\Lambda t \\ \downarrow \cdot\theta(t)' & & \downarrow f_t \\ t'\Lambda t' & \stackrel{f_{t'}}{\longrightarrow} & \overline{\Omega_\Lambda} \end{array}$$

- *Moreover, let Λ be an R-order. Suppose that there is a complete set of primitive idempotents I of Λ and a set of central idempotents E in $K \otimes_R \Lambda$ such that $\sum_{e \in E} e = 1$, and so that the set $T = \{(i, e) \in I \times E | i \cdot e \neq 0\}$ is a tree which may be totally ordered by the choice of a lowest element t_1 of T. If the Pierce decomposition has the properties* (1), (2), (3) *and* (4) *then Λ is a Green order.*

(1.13) One should observe that since the elements in E are central and pairwise orthogonal, most of the entries in the Pierce decomposition are zero. The matrix coefficients fall naturally into matrix rings Ae for $e \in E$. Furthermore, by the first and the third points of the theorem, the matrix rings Ae are upper triangular $n_e \times n_e$–matrix rings over $\Omega_{\theta(t)} := t\Lambda t$ which does only depend on $\theta(t)$ and the lower diagonal entries is in a principal ideal $a_t \cdot t\Lambda t$, again depending only on $\theta(t)$. Here $n_e = |\theta^{-1}(e)|$. In other words, for all $t \in T$,

$$H_{\theta(t)} := \Lambda \cdot \theta(t) = \begin{pmatrix} \Omega_{\theta(t)} & \Omega_{\theta(t)} & \dots & \dots & \Omega_{\theta(t)} \\ (a_{\theta(t)}) & \Omega_{\theta(t)} & \dots & \dots & \Omega_{\theta(t)} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ (a_{\theta(t)}) & \dots & \dots & (a_{\theta(t)}) & \Omega_{\theta(t)} \end{pmatrix}_{n_{\theta(t)}} .$$

These matrix rings are 'linked' among each other by the pullbacks with the main diagonal entries mentioned in point 4 of the theorem. This is the interpretation given by Roggenkamp in [10]. We may hence label the vertices of the tree by the pairs $(\Omega_{\theta(t)}, a_{\theta(t)})$. Mostly we only write down the labels $\Omega_{\theta(t)}$, understanding that the ideal $a_{\theta(t)}\Omega_{\theta(t)}$ is fixed once for all, if this does not cause confusion.

(1.14) We call the set $\{(\Omega_e, f_e, e) | e \in E\}$ the structure data for the Green order A.

(1.15) As is done for Brauer tree algebras by J. Rickard [7], two Green orders Λ and Γ have equivalent derived categories if there is a bijection β between the sets E_Λ and E_Γ and such that

$$\forall_{e_\Lambda \in E_\Lambda} \ \Omega_{e_\Lambda} = \Omega_{\beta(e_\Lambda)} \ , \ \overline{\Omega_\Lambda} = \overline{\Omega_\Gamma} \ \text{ and } (f_\Lambda)_{e_\Lambda} = (f_\Gamma)_{\beta(e_\Lambda)}$$

This fact was proved by K.W.Roggenkamp and the author. The result was published in [14].

2 Automorphisms of Green orders

Let R be a complete discrete valuation domain of characteristic 0.

Even though we shall restrict to very special Green orders in our main application, recent discoveries [1] of Y. Drozd on polynomial functors seem to be enough motivation to keep the discussion more general.

The following result may be of independent interest.

Theorem 3 *Let Λ be a basic Green order over R with structural data $\{(\Omega_e, f_e, e) | \ e \in E\}$. Assume that Ω_e is a commutative maximal order in a field*[1] *for all $e \in E$, and that $ker(f_e) = rad\ \Omega_e$ for all $e \in E$. Then, every ring automorphism of Λ which is linear over the centre of Λ and which maps each projective Λ–module to an isomorphic copy is inner.*

Proof. We recall that a Green order Λ with data $(f_e, \Omega_e)_{e \in E}$ can be embedded into an overorder $H := \prod_{e \in E} H_e$ with $H_e = \Lambda \cdot e$ for any $e \in E$.

$$H_e = \begin{pmatrix} \Omega_e & \Omega_e & \dots & \dots & \Omega_e \\ (a_e) & \Omega_e & \dots & \dots & \Omega_e \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ (a_e) & \dots & \dots & (a_e) & \Omega_e \end{pmatrix}_{n_e \times n_e}$$

Our assumption implies that Ω_e is a complete discrete valuation ring and that a_e generates its radical for all $e \in E$. Then, the orders H_e are hereditary for all $e \in E$ ([6, (39.14)]). By

[1] this implies that Ω_e is itself a complete discrete valuation domain

[6, Exercise 39.6], we see that the outer automorphism group of H_e is cyclic of order n_e. Conjugation by the element

$$w_e := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ a_e & 0 & \dots & \dots & 0 \end{pmatrix}_{n_e \times n_e}$$

is an outer automorphism of H_e which is easily seen to be of order n_e in $Aut(H_e)$. The conjugation by w_e is denoted by ω_e.

Let α be an automorphism of Λ which we assume to fix the central idempotents in E of $K \otimes_R \Lambda$. Then, α extends to an automorphism of

$$\Omega := \prod_{e \in E} H_e,$$

which, by abuse of notation, will be also denoted by α. By the above, there exist integers k_e; $e \in E$ with $0 \le k_e < n_e$ and inner automorphisms γ_e; $e \in E$ of H_e such that

$$\alpha = \prod_{e \in E} \omega_e^{k_e} \gamma_e$$

on Ω. We study the inner automorphisms γ_e.

Claim 2.1 *Let Λ be a Green order with respect to the set of central idempotents E with structural data (Ω_e, a_e) such that all the Ω_e are commutative. Let u_e be a unit in $H_e := \Lambda \cdot e$. Then, there is a unit u of Λ such that*

$$u^{-1} \cdot \prod_{e \in E} u_e \in \ centre(\Omega)\ .$$

Proof. We shall construct u inductively. Since

$$H_e = \begin{pmatrix} \Omega_e & \Omega_e & \dots & \dots & \Omega_e \\ (a_e) & \Omega_e & \dots & \dots & \Omega_e \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ (a_e) & \dots & \dots & (a_e) & \Omega_e \end{pmatrix}_{n_e \times n_e}$$

and (a_e) is in the Jacobson radical of Ω_e, a unit u_e in H_e has to have the property that the main diagonal entries $iu_e i$ of u_e are units in Ω_e for all $e \in E$ and $i \in I$.

Let e_0 be a vertex in E.

Suppose we already constructed units v_e in H_e for those $e \in E$ for which the distance of e to e_0 is less or equal to a fixed number d and

- so that $u_e^{-1} \cdot v_e \in centre(H_e)$
- and so that the equation $f_e(i \cdot v_e \cdot i) = f_{e'}(i \cdot v_{e'} \cdot i)$ holds for any $i \in I$ and $e' \in E$ subject to the condition $i \cdot e \neq 0 \neq i \cdot e'$ and with distance from e' to e_0 is less than d.

Let $e_n \in E$ be a vertex at distance $d+1$ from e_0. Since T is a tree, there is a unique $e_l \in E$ with distance d from e_0. Let j be the edge between e_l and e_n. Then,

$$f_{e_n}(j \cdot u_{e_n} \cdot j)^{-1} \cdot f_{e_l}(j \cdot v_{e_l} \cdot j)$$

is a unit in $\overline{\Omega_\Lambda}$. By Nakayama's lemma there is a unit $x_{e_n,i} \in \Omega_e$ so that

$$f_{e_n}(x_{e_n,i}) = f_{e_n}(j \cdot u_{e_n} \cdot j)^{-1} \cdot f_{e_l}(j \cdot v_{e_l} \cdot j) \ .$$

Now,

$$v_{e_n} := x_{e_n,i} \cdot u_{e_n}$$

has the property that

1.) $u_{e_n}^{-1} \cdot v_{e_n} \in centre(H_{e_n})$ and that
2.) $f_{e_n}(j \cdot v_{e_n} \cdot j) = f_{e_l}(j \cdot v_{e_l} \cdot j)$ in $\overline{\Omega_\Lambda}$.

By induction we define this way an element $u := \prod_{e \in E} v_e$. By the previous two properties 1.) and 2.) we get $u \in \Lambda$. Moreover, the construction of modifying successively by entral elements implies that

$$u \cdot \prod_{e \in E} u_e \in centre(\Omega) \ .$$

By the second part of Theorem 2 we see that $u \in \Lambda$. Since u is in Λ and u is a unit in H, by a lemma of Roggenkamp and Scott (cf. [11, Lemma 3]) we conclude that u is actually a unit in Λ. This proves the claim. ■

Remark 2.2 Observe that this proof works in the same way for orders which are defined similarly to Green orders, with the difference that one allows the quotient $\overline{\Omega}$ attached to the edges of T to depend on the edge we are faced with. For Green orders of course it is not necessary to use an induction. An explicit formula can easily be given.

By Claim 2.1 conjugation by u induces the automorphism $\prod_{e \in E} \gamma_e$ on $K \otimes_R \Lambda$. Hence, $\prod_{e \in E} \gamma_e$ is actually an inner automorphism of Λ.

Modifying α by $\prod_{e \in E} \gamma_e$ we may assume that $\gamma_e = 1$ for all $e \in E$.

But, α_e induces a cyclic permutation of order n_e of

$$H_e/\omega_e H_e = \prod_{j=1}^{n_e} \Omega_e/(a_e).$$

Hence, in case one α_e is not the identity, for an idempotent $i \in I$ with $i \cdot e \neq 0$, we get $\alpha(i) \notin \Lambda$. Therefore, $k_e = 0$ for all $e \in E$.

The only possibility now is an automorphism which is not the identity on the central idempotents of E in $K \otimes_R \Lambda$. This automorphism, however, necessarily moves projective indecomposable Λ–modules. This proves Theorem 3. ■

3 Consequences for the self-equivalences of the derived category of a Green order

We shall derive some consequences of the result in section 2 for the group $TrPic_R(\Lambda)$.

Proposition 3.1 *Let Λ be a basic Green order over R with structural data $\{(\Omega_e, f_e, e)| \, e \in E\}$. We assume that Ω_e is a commutative maximal order in a field for all $e \in E$, and that $ker(f_e) = rad\ \Omega_e$ for all $e \in E$. Given X and Y in $TrPicent(\Lambda)$, such that for each projective $k \otimes_R \Lambda$–module P we get $X \otimes_\Lambda P \simeq Y \otimes_\Lambda P$. Then, $X \simeq Y$.*

Proof. We form $Z := Y^{-1} \otimes_\Lambda X$. Now, $Z \otimes_\Lambda P \simeq P$. Since R is a complete discrete valuation ring, also for each projective indecomposable Λ–module Q, we get $Z \otimes_\Lambda Q \simeq Q$. Then, Z is isomorphic to an invertible bimodule which is isomorphic to Λ as left Λ-module. Now, by [12], Z is isomorphic to ${}_1\Lambda_f$ as $\Lambda \otimes_R \Lambda^{op}$-module, where the action from the right is multiplication twisted by an automorphism f fixing each projective Λ–module. Such automorphisms are inner by Theorem 3. Hence, $Z \simeq \Lambda$ as $\Lambda \otimes_R \Lambda^\circ$–modules. ■

(3.1) Hypothesis: We will assume in the rest of this section 3 that the graph T is a stem and that the Green order Λ is basic. To avoid trivial cases we assume that $|E| \geq 3$. We assume furthermore that

$$\forall_{e,e' \in E}\ \Omega_e = \Omega_{e'} \text{ is a commutative maximal order in a field and } f_e = f_{e'}\ .$$

(3.2) Let $E = \{e_1, e_2, \ldots, e_n\}$ and for any $k \in \{1, \ldots, n-1\}$ let $i_k \in I$ be the idempotent with $e_k \cdot i_k \neq 0 \neq i_k e_{k+1}$. Let P_k be the indecomposable projective left Λ-module $\Lambda \cdot i_k$ and let P_k^* be the projective right Λ-module $i_k \cdot \Lambda$. Observe that P_k is a right Ω_k-module and that P_k^* is a left Ω_k–module.

Then, we define a complex

$$S_k := (\ldots \longrightarrow 0 \longrightarrow P_k \otimes_{\Omega_k} P_k^* \xrightarrow{d_k} \Lambda \longrightarrow 0 \longrightarrow \ldots)$$

of $\Lambda \otimes_{\Omega_k} \Lambda^{op}$–modules with homology concentrated in degree -1 and 0. The differential d_k is defined by

$$d_k(\alpha \cdot i_k \otimes_{\Omega_k} i_k \cdot \alpha') = \alpha \cdot i_k \cdot \alpha' \text{ for all } \alpha, \alpha' \in \Lambda$$

Clearly, d_k is a mapping of $\Lambda \otimes_{\Omega_k} \Lambda^{op}$-modules. Define for technical reasons $\Omega_0 := \Omega_1$.

Proposition 3.2 *Suppose for a $k \in \{1, \ldots, n-1\}$ that $\Omega_{k-1} = \Omega_k = \Omega_{k+1}$. Then, the complex S_k is a two-sided tilting complex.*

Proof. We show that S_k is a two-sided tilting complex. By a lemma of Rickard [4, Lemma 6.3.15] we have to show that the restriction to the left and to the right hand side is a one-sided tilting complex with endomorphism ring Λ. We get

$$S_k \otimes_\Lambda P_l \simeq P_l\ \forall l \in \{1, \ldots, n-1\} \setminus \{k-1, k, k+1\}$$

as is immediately verified. Moreover,

$$S_k \otimes_\Lambda P_k \simeq P_k[1]$$

and

$$S_k \otimes_\Lambda P_{k-1} \simeq \ldots \longrightarrow 0 \longrightarrow P_k \longrightarrow P_{k-1} \longrightarrow 0 \longrightarrow \ldots$$

when $n-1 \geq k \geq 2$ as well as

$$S_k \otimes_\Lambda P_{k+1} \simeq \ldots \longrightarrow 0 \longrightarrow P_k \longrightarrow P_{k+1} \longrightarrow 0 \longrightarrow \ldots$$

when $1 \leq k \leq n-2$. Here the mappings are the homomorphisms with obvious maximal image. It is immediate to see that the triangulated category generated by these complexes contains the rank one free module Λ. The hypothesis on T implies that there are no morphisms between P_k and P_l for $|k-l| \geq 2$. Moreover, any morphism from P_k to P_{k-1} factors through the maximal one. There is no nonzero homomorphism from P_{k-1} to P_k composing to 0 from the left with the differential since the differential and also any of the above homomorphism is given by multiplication with elements in Ω_k, which is an integral domain. The analogous statements hold for P_{k+1}. Altogether this shows that the restriction of S_k to the left is a one-sided tilting complex. It remains to see that the endomorphism ring of this tilting complex is Λ. This can be seen as it is carried out in [14]. In case Λ is Gorenstein, a shorter argument can be applied. It is clear that it is enough to show the statement for $S_k^* = Hom_\Lambda(S_k, \Lambda)$. Applying [4, Proposition 5.1.1] to the dual S_k^*, we see that the endomorphism ring of the restriction of S_k to the left has endomorphism ring Λ.

Since the situation is symmetric with respect to the left and the right, we proved the statement. ■

(3.3) Assume $\Omega_1 = \Omega_2 = \ldots = \Omega_n$. Then, for all $k = 1, \ldots, n$, tensor product with S_k induces an element in $TrPic_R(\Lambda)$. We denote by t_k the element in $TrPic_R(\Lambda)$ which is induced by tensor product by S_k. Furthermore we denote by ω the non zero element in the Picard group $Pic_R(\Lambda)$ of Λ. Note that $Pic_R(\Lambda)$ is of order 2.

In case Ω_{k-1}, Ω_k and Ω_{k+1} are possibly different, a slight modification of S_k gives a twosided tilting complex which does not necessarily induce an *auto*-equivalence of the derived category.

Theorem 4 *Let Λ be a Green order over the complete discrete valuation domain R with structural data $(\Omega_e, f_e)_{e\in E}$ associated to a stem T and such that $\Omega_1 = \Omega_2 = \ldots = \Omega_n = R$ and $f_1 = f_2 = \ldots = f_n$ is the projection to $k := R/rad\ R$. Then, the tilting complexes $\{S_1, \ldots, S_n\}$ satisfy the braid relations*

$$S_i \otimes_\Lambda S_j \simeq S_j \otimes_\Lambda S_i \text{ for } |i-j| \geq 2;\ i,j = 1, \ldots, n.$$

$$S_i \otimes_\Lambda S_{i+1} \otimes_\Lambda S_i \simeq S_{i+1} \otimes_\Lambda S_i \otimes_\Lambda S_{i+1} \text{ for } i = 1, \ldots, n-1.$$

Proof. The statement follows from Theorem 1 and the remark following it in connection with Proposition 3.1. ■

(3.4) The assumption on the structure of the Green order has the following group theoretical interpretation. By the classical theory of blocks of group rings of finite groups with cyclic defect group, the multiplicity μ can be calculated from the number of edges e of the graph and the size p^d of the defect group.

$$\mu = \frac{p^d - 1}{e}$$

with e divides $p-1$. Hence, $\mu = 1$ is equivalent to $e = p-1$ and $d = 1$. Suppose that R is an unramified extension of the p–adic integers with residue field k and let i be an idempotent in the centre of RG. Suppose that $kG \cdot i$ is a Brauer tree algebra with exceptional multiplicity 1.

Roggenkamp proves [10] in this case that $RG{\cdot}i$ is a Green order with structure data satisfying the hypotheses of the theorem.

(3.5) Moreover, for any perfect field k and any Brauer tree algebra A over k without exceptional vertex there is a complete discrete valuation ring R with residue field k (cf. [13, II § 5 Théorème 3]) and a Green order Λ over R satisfying the assumptions in Theorem 3 such that $k \otimes_R \Lambda \simeq A$.

Recall that we denote by $\tilde{B}_3$ the following extension by the braid group on 3 strings.

$$\tilde{B}_3 := < s, t, z \mid z^4 = (st)^3;\ zs = sz;\ zt = tz;\ sts = tst >$$

Obviously, one has exact sequences of groups $1 \longrightarrow < z > \longrightarrow \tilde{B}_3 \longrightarrow PSL_2(\mathbb{Z}) \longrightarrow 1$ and $1 \longrightarrow B_3 \longrightarrow \tilde{B}_3 \longrightarrow C_4 \longrightarrow 1$ given by identifying s, t with the usual braid generators.

We get the analogous result to Theorem 1 replacing Brauer tree algebras by Green orders.

Theorem 5 *Let Λ be a Green order over the complete discrete valuation domain R such that there are exactly 2 non isomorphic simple A-modules. Suppose the structural data $(\Omega_e, f_e)_{e\in\{1,2,3\}}$ have the property $\Omega_1 = \Omega_2 = \Omega_3 = R$ and $f_1 = f_2 = f_3$ is the projection to $R/rad\ R$. Then,*

$$TrPic_R(\Lambda) \simeq \tilde{B}_3 \rtimes C_2.$$

Proof. Again, this follows immediately from Theorem 1, Theorem 3 and Proposition 3.1. ∎

(3.6) Let R be an integrally closed extension of the 3-adic integers and let $\mathfrak{S}_3$ be the symmetric group on 3 letters. Then $R\mathfrak{S}_3$ satisfies the hypotheses of Theorem 5. Y. Drozd studied in [1] right continuous quadratic functors. These can be seen as modules over an order **A**. The completion of this order **A** at the prime 2 satisfies the hypotheses of the theorem.

4 Some subgroups of $TrPic_{\hat{\mathbb{Z}}_3}(\hat{\mathbb{Z}}_3\mathfrak{S}_3)$

Let R be a complete discrete valuation domain with field of fractions K and residue field k. Let Λ be a Green order over R such that there are exactly 2 non isomorphic simple Λ-modules. Suppose Λ has that the structural data $(\Omega_e, f_e)_{e\in\{1,2,3\}}$ where $\Omega_1 = \Omega_2 = \Omega_3 = R$ and $f_1 = f_2 = f_3$ is the natural projection $R \longrightarrow k$.

Recall

$$\begin{aligned} TrPicent(\Lambda) &:= ker(TrPic_R(\Lambda) \longrightarrow Aut(Z(\Lambda))) \\ TrI_R(\Lambda) &:= ker(TrPic_R(\Lambda) \longrightarrow TrPic_K(K \otimes_R \Lambda)) \end{aligned}$$

Theorem 6 *There is an exact sequence*

$$1 \longrightarrow C_\infty \longrightarrow TrPicent(\Lambda) \longrightarrow \Gamma(2) \longrightarrow 1$$

where $\Gamma(2)$ is the congruence subgroup of level 2 of $PSL_2(\mathbb{Z})$ and C_∞ is the infinite cyclic group generated by shift in degrees $< [1] >$. Moreover, the above sequence induces an isomorphism

$$TrI_R(\Lambda) \simeq \Gamma(2)'$$

Proof. The automorphism group of the centre of Λ is a symmetric group of order 6. In fact the three irreducible characters may be permuted arbitrarily. Call the two one-dimensional characters of Λ the trivial and the sign character. Of course, this is motivated by the case of the group ring of the symmetric group on three letters.

Recall (cf for example [5]) that in $PSL_2(\mathbb{Z})$ there are two normal subgroups of index 6, the derived group $PSL_2(\mathbb{Z})'$ and the congruence subgroup $\Gamma(2)$ of level 2. The quotient by the derived subgroup is cyclic, the quotient by the congruence subgroup of level 2 is

$$PSL_2(\mathbb{Z})/\Gamma(2) \simeq PSL_2(2) = GL_2(2) \simeq \mathfrak{S}_3 \ .$$

Let us determine the image J of $TrPic_R(\Lambda)$ in $Aut_R(Z(\Lambda))$. Denote by t_1 the functor induced by tensor product with S_1 and by t_2 the functor induced by tensor product with S_2. Then, the image of the element t_1 in $Aut_R(Z(\Lambda))$ permutes the trivial and the two-dimensional character and fixes the sign character. The image of the element t_2 in $Aut_R(Z(\Lambda))$ permutes the sign and the two-dimensional character and fixes the trivial character. Hence, J is isomorphic to the symmetric group of order 6 and therefore,

$$TrPicent(\Lambda)/ < [1] > \simeq \ \Gamma(2) \ .$$

It is immediate to see that

$$TrPic_K(K \otimes_R \Lambda) \simeq C_\infty \wr \mathfrak{S}_3 \ .$$

Moreover, under this isomorphism the shift by n degrees $[n]$ corresponds to the element $(n, n, n; id) \in C_\infty \rtimes \mathfrak{S}_3$. Hence, we get a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
1 & \longrightarrow & C_\infty^2 & \longrightarrow & \frac{TrPic_K(K\Lambda)}{<[1]>} & \longrightarrow & \mathfrak{S}_3 & \longrightarrow & 1 \\
 & & \uparrow \psi & & \uparrow & & \uparrow \phi & & \\
1 & \longrightarrow & \frac{TrPicent(\Lambda)}{<[1]>} & \longrightarrow & \frac{TrPic_R(\Lambda)}{<[1]>} & \longrightarrow & \mathfrak{S}_3 & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & & & \\
1 & \longrightarrow & TrPicent(\Lambda) \cap TrI_R(\Lambda) & \longrightarrow & TrI_R(\Lambda) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

where the upper row corresponds to the isomorphism $TrPic_K(K \otimes_R \Lambda) \simeq C_\infty \wr \mathfrak{S}_3$ and the second row corresponds to the definition of $TrPicent(\Lambda)$ being the kernel of the epimorphism $TrPic(\Lambda) \longrightarrow Aut(Z(\Lambda)) \simeq \mathfrak{S}_3$.

We claim that the right vertical morphism ϕ is injective, hence even an isomorphism. In fact, any R-linear automorphism of $Z(\Lambda)$ is induced by a permutation of the characters of $K \otimes_R \Lambda$. Hence,

$$TrI_R(\Lambda) \subseteq TrPicent(\Lambda).$$

Recall that by Theorem 5 we get a monomorphism $B_3 \stackrel{\varphi_2}{\longrightarrow} TrPic_R(\Lambda)$. We call $\overline{\varphi_2}$ the isomorphism $PSL_2(\mathbb{Z}) \longrightarrow TrPic_R(\Lambda)/ < [1] >$ induced by φ_2. Recall moreover (cf for example [5]) that $\Gamma(2)$ is freely generated by the matrices $\left(\begin{array}{cc} 1 & 2 \\ 0 & 1 \end{array}\right)$ and $\left(\begin{array}{cc} 1 & 0 \\ -2 & 1 \end{array}\right)$. The homomorphism φ_2 maps t_1^2 and t_2^2 to these two matrices. It is readily verified that t_1^2 and t_2^2 are elements in $TrPicent(\Lambda)$. Moreover, these two matrices induce shifts by degree two for the sign or the trivial character respectively in $TrPic_K(K \otimes_R \Lambda)$. Therefore, $\Gamma(2)$ is mapped via $\overline{\varphi_2}^{-1}$ to a subgroup of $TrPicent(\Lambda)/ < [1] >$. Moreover, the image of ψ is a subgroup of C_∞^2 of index at most 4.

Now,

$$PSL_2(\mathbb{Z}) = C_3 * C_2$$

with a generator a of order 3 and another generator b of order 2, the image of the element t_1 in $PSL_2(\mathbb{Z})$ by $\overline{\varphi_2}$ is ab and the image of the element t_2 is ba. Therefore,

$$PSL_2(\mathbb{Z})/ < \overline{\varphi_2}^{-1}(t_1^2), \overline{\varphi_2}^{-1}(t_2^2) > \simeq < a, b|\ b^2, (ab)^2, a^3 > \simeq \mathfrak{S}_3\ .$$

Hence, $TrI_R(\Lambda)$ is normal in $TrPicent(\Lambda)/ < [1] >$ with quotient being free abelian of rank 2. This shows at once that $\overline{\varphi_2} : \Gamma(2) \simeq TrPicent(\Lambda)/ < [1] >$ and that $TrI_R(\Lambda)$ is isomorphic to $\Gamma(2)'$ the derived group of the congruence subgroup of level 2 of the modular group (see [5]). ■

(4.1) By the above discussion it is clear that the commutator c of t_1^2 with t_2^2 is in $TrI_R(\Lambda)$. One calculates immediately that $\overline{\varphi_2}(c) = \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}$. Observe that by [12] this means that the smallest complex representing the image under c of the first projective indecomposable module involves 13 copies of P_1 as well as eight copies of P_2 and the image of the second projective indecomposable involves 8 copies of P_1 as well as 5 copies of P_2. It is not difficult to write down the corresponding tilting complex explicitly.

(4.2) It is known that most normal subgroups of $PSL_2(\mathbb{Z})$ are free. In particular, since $Aut(Z(\Lambda)) \simeq \mathfrak{S}_3$, we get that $TrPicent(\Lambda)/ <$ shift $>$ is free on two generators and isomorphic to the congruence subgroup $\Gamma(2)$ of $PSL_2(\mathbb{Z})$. By the Nielsen-Schreier theorem subgroups of free groups are free. Hence $TrI_R(\Lambda)$ is free.

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