On equivalences between categories of representations

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Algebras and Groups

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- but still enough information to be interesting

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- A sequence of submodules with simple quotients is a **composition series**

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Classification problems in $A - mod$

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- determine the “mortar” to glue the bricks

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But we want to study it systematically.

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The second is more complicated even, (except in case of showing that one of the invariants in the list do not coincide. (But this is somehow absurd.))
For the first problem, we have Rickard's criterion:

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Several strategies were developed for this purpose.

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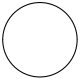
circle  $\leftrightsquigarrow$ $\boxed{X_1^2 + X_2^2 = 1}$ $\leftrightsquigarrow$ $\boxed{\mathbb{C}[X_1, X_2]/(X_1^2 + X_2^2 - 1)}$

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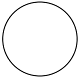
circle  $\leftrightsquigarrow \boxed{X_1^2 + X_2^2 = 1} \leftrightsquigarrow \boxed{\mathbb{C}[X_1, X_2]/(X_1^2 + X_2^2 - 1)}$

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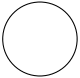
circle  $\leftrightsquigarrow$ $\boxed{X_1^2 + X_2^2 = 1}$ $\leftrightsquigarrow$ $\boxed{\mathbb{C}[X_1, X_2]/(X_1^2 + X_2^2 - 1)}$

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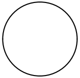
circle  $\rightsquigarrow$ $\boxed{X_1^2 + X_2^2 = 1}$ $\rightsquigarrow$ $\boxed{\mathbb{C}[X_1, X_2]/(X_1^2 + X_2^2 - 1)}$

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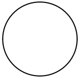
circle  $\rightsquigarrow$ $\boxed{X_1^2 + X_2^2 = 1}$ $\rightsquigarrow$ $\boxed{\mathbb{C}[X_1, X_2]/(X_1^2 + X_2^2 - 1)}$

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Morality: **Tensor product is some kind of multiplication**

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Theorem (Rickard's second main theorem 1992)

If A and B are derived equivalent algebras over a field, then there is some X in the derived category of $A - B$-bimodules such that

$$M \mapsto X \otimes_A^{\mathbb{L}} M$$

is an equivalence.

A particular case is if A, B are algebras over a field, like complex numbers.

Theorem (Rickard's second main theorem 1992)

If A and B are derived equivalent algebras over a field, then there is some X in the derived category of $A-B$-bimodules such that

$$M \mapsto X \otimes_A^{\mathbb{L}} M$$

is an equivalence.

In some sense the equivalence is

given by some kind of multiplication.

In this case say

A and B are derived equivalent of standard type

stable equivalent of Morita type

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- it has the same objects as $A-mod$
- morphism spaces are equivalence classes of module homomorphisms modulo those factoring through projective modules

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Derived, Morita, stable equivalences

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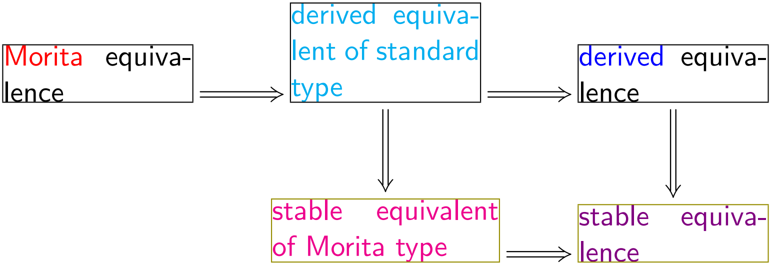

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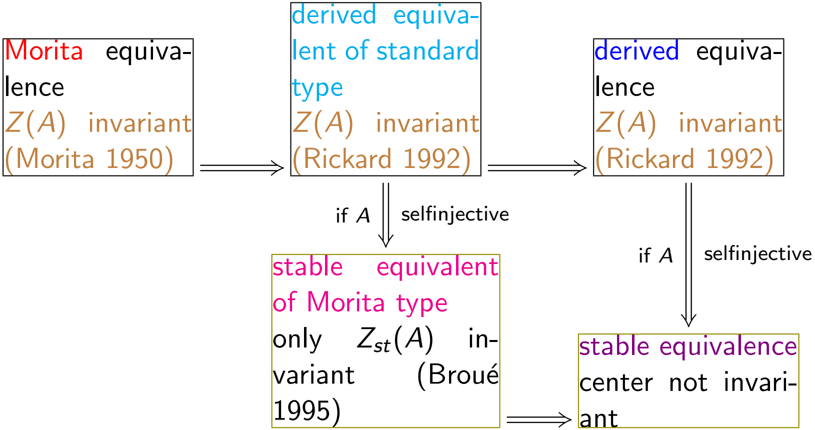

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A Question (Rickard 1998):

Is the same true for "stable equivalent of Morita type" ?

Disproving Rickard's question

Answer is NO.

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For any algebra A we get

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Lemma (Yuming Liu, Guodong Zhou, A.Z. 2017; Proceedings of the AMS)

For any algebra A over an algebraically closed field k of characteristic $p > 0$ *the Cartan matrix of* $A \otimes_k k[X]/X^p$ *has rank* 0 *over k.*

The non symmetric counterexample

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Let A and B be self-injective indecomposable finite dimensional k-algebras. Then $\begin{pmatrix} A & 0 \\ A & A \end{pmatrix}$ and $\begin{pmatrix} B & 0 \\ B & B \end{pmatrix}$ are stable equivalent of Morita type implies A and B are Morita equivalent.

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or the example below
$A = \mathbb{Z}/2\mathbb{Z}\mathfrak{A}_4$ and $B = B_0(\mathbb{Z}/2\mathbb{Z}\mathfrak{A}_5)$.

The symmetric counterexample

Let p be a prime and $k = \mathbb{Z}/p\mathbb{Z}$.

- $G(q) = PSU(3, q)$ the projective special unitary group of size 3×3 over a field with $q = p^s$ elements.
- $H(q)$ the normalizer of a Sylow p-subgroup of $G(q)$.
- C_p the cyclic group of order p.

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How big are these examples?

q	size of $G(q)$	size of $H(q)$
3	5616	27
4	20160	64
5	372000	125
7	1876896	343
8	16482816	512
9	42456960	729
11	212427600	1331

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Theorem (Serge Bouc, A.Z. 2017; Experimental Mathematics)

- *$B_0(q)$ and $H(q)$ are stably equivalent of Morita type.*
- *If $q \in \{3, 4, 5, 7, 8, 9, 11\}$ then $B_0(q) \otimes_k kC_p$ and $H(q) \otimes_k kC_p$ are* **not** *stably equivalent of Morita type.*
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Note: $kC_p \simeq k[X]/X^p$

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This is a counterexample to Rickard's question.