Degeneration in Triangulated Categories

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The origin: Gabriel et al

Let k be an algebraically closed field and A a finite dimensional k-algebra.

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- Generators of A: $\{a_1, \ldots, a_n\}$
- relations: $\{\rho_1(a_1, \ldots, a_n), \ldots, \rho_m(a_1, \ldots, a_n)\}$

where ρ_i are polynomials in non-commuting variables (free algebra).

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A representation of A of dimension d is given by

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The *module variety* $mod(A,d)$ is the Zariski closed subset of the nd^2-dimensional affine space defined by the above polynomial equations.

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In particular $mod(A,d)$ is an affine variety.

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$$M \simeq N \Leftrightarrow G \cdot m = G \cdot n$$

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- d-dimensional A-modules correspond to points in $mod(A, d)$.
- Isomorphism classes of d-dimensional A-modules correspond to G-orbits in $mod(A, d)$.
- In general orbits are not Zariski closed. Examples will follow.

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Main classical definition

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$$n \in \overline{G \cdot m} \Leftrightarrow M \leq_{\text{deg}} N$$

The origin: Gabriel et al

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  and $t \neq 0$ is isomorphic to original action;
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- semisimple modules are the maximal objects.
- $Ext^1_A(M, M) = 0 \Rightarrow M$ is minimal.
  (Voigt's lemma; open orbit in this case)

Zwara's breakthrough

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If there is an A-module Z and a ses:

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Theorem: (Riedtmann 1986, Zwara 2000)

If there is an A-module Z and a ses:

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then $M \leq_{\text{deg}} N$ and conversely degeneration implies the existence of Z and a ses as above.

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- Z and Z' may be quite different in general
- and difficult to construct.

Where did I learn all this ?

Reading group first semester 2003/4
module varieties; around 10 lectures

Classical Theory: Drawbacks

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- algebraically closed fields

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- concept that works for general rings
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- or even triangulated categories

Examples limiting our ambition

Recall an example due to Swan (1962):

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$$\begin{array}{rcl} \mathfrak{a} & \not\simeq & \mathbb{Z}Q_{32} \\ \mathbb{Z}Q_{32} \oplus \mathfrak{a} & \simeq & \mathbb{Z}Q_{32} \oplus \mathbb{Z}Q_{32} \end{array}$$

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$$0 \to \overbrace{\mathbb{Z}Q_{32}}^{Z} \to \underbrace{\overbrace{\mathbb{Z}Q_{32}}^{Z} \oplus \mathbb{Z}Q_{32}}_{\mathbb{Z}Q_{32} \oplus \mathfrak{a}} \to \mathfrak{a} \to 0$$

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What do we learn?

There are problems without Krull-Schmidt.

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Be careful with embedding $Z \to Z \oplus M$

The algebraic degeneration in a triangulated setting

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$M \leq_\Delta N :\Leftrightarrow \exists Z$ and distinguished triangle $Z \to Z \oplus M \to N \to Z[1]$

$\leq_\Delta$ is called the triangle degeneration.

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The proof is a construction of iterated cones, then using Fitting's lemma in some sophisticated way (artinian endomorphism algebra).

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This uses the *Hom*-order and Bongartz' proof for the module case.
What is the *Hom*-order?

$$M \leq_{\text{deg}} N \Rightarrow \dim_k(Hom_A(U,M)) \leq \dim_k(Hom_A(U,N)) \forall U$$

Call this last property *Hom*-order $\leq_{Hom}$.

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Auslander (1982), then Bongartz (1989) show that

$$\dim_k(Hom_A(U, M)) = \dim_k(Hom_A(U, N)) \forall U \Rightarrow M \simeq N$$

The algebraic degeneration in a triangulated setting

Get easily if $\mathcal{T}$ is triangulated R-linear category, R a commutative ring, and if $Hom_{\mathcal{T}}(X, Z)$ is of finite R-length for all X, Z, then

$$M \leq_{\Delta} N \Rightarrow M \leq_{Hom} N.$$

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$$M \leq_{Hom} N \leq_{Hom} M \Rightarrow M \simeq N.$$

Where did I learn all this ?

Reading group February/March 2010
partial orders on isomorphism classes of modules

Generic points, abstract description

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Definition (Yoshino)

Let A be a k-algebra. Then M degenerates to N along a dvr if there is a discrete valuation k-algebra V, maximal ideal $\wp = tV$ and $k = V/\wp$, and a V-flat $V \otimes_k A$-module Q such that $Q/tQ \simeq N$ and $Q[\frac{1}{t}] \simeq M \otimes_k V[\frac{1}{t}]$ as $A \otimes_k V[\frac{1}{t}]$-modules. Write $M \leq_{\mathrm{dvr}} N$.

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- and for triangulated categories $M \leq_{\Delta + \text{ nil}} N :\Leftrightarrow \exists Z$ and distinguished triangle $Z \xrightarrow{\binom{\phi}{\psi}} M \oplus Z \to N \to Z[1]$ with ψ nilpotent

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- for the stable category of maximal Cohen-Macaulay modules over a local commutative Gorenstein k-algebra A we get $M \leq_{\Delta\,+\,\text{nil}} N \Rightarrow M \leq_{\text{d}vr} N$. If A is in addition artinian, then we get $\Leftrightarrow$.
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Maximal elements do not exist in the triangulated setting:
$X \to 0 \to X[1] \stackrel{id}{\to} X[1]$ is distinguished triangle. Hence
$X \stackrel{0}{\to} U \oplus X \to U \oplus X \oplus X[1] \to X[1]$ is distinguished triangle

Generic points, abstract description

Yoshino shows (2004, 2011) that

- if A is any k-algebra then $M \leq_{\text{Zwara+nil}} N \Leftrightarrow M \leq_{\text{dvr}} N$
- for the stable category of maximal Cohen-Macaulay modules over a local commutative Gorenstein k-algebra A we get $M \leq_{\Delta + \text{nil}} N \Rightarrow M \leq_{\text{dvr}} N$. If A is in addition artinian, then we get $\Leftrightarrow$.
- partial order properties for $\leq_{\text{dvr}}$.

Maximal elements do not exist in the triangulated setting: $X \to 0 \to X[1] \stackrel{id}{\to} X[1]$ is distinguished triangle. Hence $X \stackrel{0}{\to} U \oplus X \to U \oplus X \oplus X[1] \to X[1]$ is distinguished triangle and therefore $U \leq_{\Delta + \text{nil}} U \oplus X \oplus X[1]$

A line in a triangulated category, jt with Saorin

We model Yoshino's setting in a general triangulated k-category $\mathcal{C}_k^\circ$ for a commutative ring k.

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- If $\mathcal{C}$ is triangulated, $\mathcal{C}[t^{-1}]$ is as well.
- Comes with a triangle functor $\mathcal{C} \to \mathcal{C}[t^{-1}]$.

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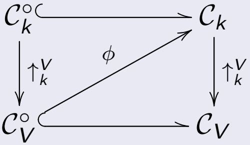
- triangle functors 

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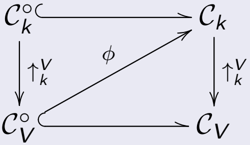

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$$\begin{array}{ccc} \mathcal{C}_k^\circ & \hookrightarrow & \mathcal{C}_k \\ \downarrow \uparrow_k^V & \overset{\phi}{\nearrow} & \downarrow \uparrow_k^V \\ \mathcal{C}_V^\circ & \hookrightarrow & \mathcal{C}_V \end{array}$$

- $t \in Z(\mathcal{C}_V)$
- such that $\forall M \in \mathcal{C}_k^\circ : \phi(t_{M\uparrow_k^V})$ is split mono and $\phi(cone(t_{M\uparrow_k^V})) \simeq M$

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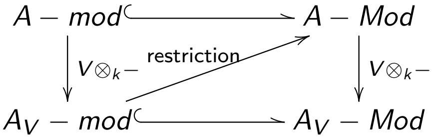	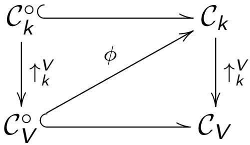
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Degeneration in a triangulated category, jt with Saorin

Recall that if $t \in Z(\mathcal{C}_V)$, then there is a triangle functor

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This specialises to Yoshino in the obvious sense in his setting.

The result, jt with Saorin

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Let k be a commutative ring and let $\mathcal{C}_k^\circ$ be a triangulated k-category with split idempotents. Let $M, N \in ob(\mathcal{C}_k^\circ)$.

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Second part uses Keller's characterisation of those categories as derived categories of dg-categories.

Where did I learn all this ?

Happy birthday Serge !